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Statistical parameters and grading curves

Paramètres statistiques et courbes de classement

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ABSTRACT: The two grading entropy coordinate pairs (base entropy and entropy increment) are compared with the statistical central moments and the usual grading curve parameters through some theoretical fractal grading curve series and through real data used for the elaboration of a soil parameter – GSD relationships. It is shown how some proper grading curve series can be selected from the real data set using the central moments and that the grading entropy parameters can be some new grading curve descriptors.

RÉSUMÉ : Les deux paires de coordonnées d'entropie de gradation (entropie de base et incrément d'entropie), proposées par Lőrincz, sont comparées aux moments centraux statistiques et aux paramètres de courbe de gradation habituels à travers des séries de courbes de gradation fractales théoriques et à travers des données réelles utilisées pour l'élaboration d'un paramètre de sol – Relations GSD. Il est montré comment certaines séries de courbes de classement appropriées peuvent être sélectionnées à partir de l'ensemble de données réelles en utilisant les moments centraux et que les paramètres d'entropie de classement peuvent être de nouveaux descripteurs de courbes de classement.

KEYWORDS: grading curve, central moments, uniformity coefficient, dominant grain size, grading entropy, finite fractal distribution

1 INTRODUCTION

In order to classify a soil on the basis of its particle sizes, it is necessary to quantify the sizes of the particles present in a soil with the grading curve. For geotechnical correlations diameter values such as d_{10}, d_{30}, d_{50} and d_{60} and or their ratios are used. Another statistical way is discussed here. The aim of this paper is to study the mathematical and physical meaning of the classical grading entropy parameters to extend their use as grading curve descriptors in regression forms. The relations of the two of grading entropy parameter pairs, and the four central moments and the traditional parameters are compared using fractal grading curves (which are some kind of mean grading curves). Then these relationships are used for general grading curves.

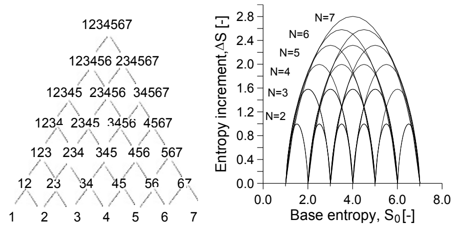
2 REVIEW

2.1 *Grading curve characterization by statistical entropy*

The statistical entropy (the entropy of a finite, discrete distribution function) is presented in many textbooks (e.g. Korn and Korn 1975) and is modified for the GSD (Lőrincz 1986). A double cell system is used, the so called fractions and the refinement of the fractions are used for the entropy derivation.

Table 1. Definition of fractions and abstract diameter.

j	1	...	23	24
Limits, d [mm]	d_0 to 2 d_0	...	2^{22} d_0 to 2^{23} d_0	2^{23} d_0 to 2^{24} d_0
S_{0j} [-] or D_j	1	...	23	24



(a) (b)

Figure 1 (a), (b) Simplex with N=7, lattice of continuous sub-simplexes (integers: fractions) and its image (or the image of the optimal lines of the sub-simplexes) in the non-normalised entropy diagram.

2.1.1 *Space of grading curve, primary cells*

The grading curve is the finite, discrete distribution of the diameter of grains by dry weight. In the grading curve measurement the sieve hole diameters, the fraction limits are doubled. An abstract fraction system is defined. The diameter range for fraction j (j =1, 2...j see Table 1, [18]):

$$2^j d_0 \geq d > 2^{j-1} d_0, \quad (1)$$

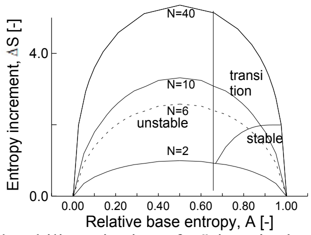


Figure 2. Internal stability criterion of Lőrincz in the partly normalized diagram

where d_0 is the smallest diameter which may be equal to the height of the SiO_4 tetrahedron (d_0=2^{-22} mm). The relative frequencies of the fractions x_i (i = 1, 2, 3...*N*) for each grading curve fulfil the following equation:

$$\sum_{i=1}^{N} x_i = 1, \; x_i \geq 0, \; N \geq 1 \quad (2)$$

where the integer variable *N* – the number of the fractions between the finest and coarsest non-zero fractions – is used. The relative frequencies x_i can be identified with the barycentre coordinates of the points of an *N*-1 dimensional, closed simplex (which is the *N*-1 dimensional analogy of the triangle or tetrahedron, the 2 and 3 dimensional instances) and, the space of the grading curves with *N* fractions can be identified with the *N*-1 dimensional, closed simplex. The vertices of the simplex represent the fractions, the 2 dimensional edges are related to the two-mixtures etc. The sub-simplexes of a simplex surface are partly continuous, partly gap-graded. The continuous sub-simplexes have a lattice structure, as is indicated in Fig 1. In the statistical entropy concept, the width of the statistical cells is uniform. Therefore, the fractions are "embedded" into an elementary cell system. The width of the elementary cell is equal to d_0.

2.1.2 *Entropy parameters*

The grading entropy *S* is a statistical entropy, modified for the unequal cells (fractions are doubled, (Lőrincz 1986)). It can be separated into the sum of two parts:

$$S = S_0 + \Delta S \quad (2)$$

where S_0 is base entropy and ΔS is entropy increment. The base entropy is It is a kind of mean of the log diameter:

$$S_0 = \sum x_i S_{0i} = \sum x_i i \quad (3)$$

where S_{0k} is the *k*-th fraction entropy (Table 1). The relative base entropy *A*:

$$A = \frac{S_0 - S_{0min}}{S_{0max} - S_{0min}} \quad (4)$$

where S_{0max} and S_{0min} are the entropies of the largest and the smallest fractions, respectively. It varies between 0 and 1. The entropy increment:

$$\Delta S = -\frac{1}{\ln 2} \sum_{x_i \neq 0} x_i \ln x_i \quad (5)$$

is the usual entropy due to the mixing of the various fractions. The normalized entropy increment *B:*

$$B = \frac{\Delta S}{\ln N} \quad (6)$$

2.1.3 *Entropy diagrams, optimal grading curves, inverse image*

Any grading curve can be represented as a single point in terms of the entropy coordinates. Four maps can be defined between the *N*-1 dimensional, open simplex (fixed *N*) and the two dimensional real Euclidean space of the entropy coordinates.

The map for fixed *N* is continuous on the open simplex and can continuously be extended to the closed simplex. The entropy diagram (Figs 1,2) is compact, like the simplex. In the normalized diagram, the maximum normalized entropy increment line *B* nearly concide for all *N,* differing in scaling from other diagram lines. The inverse image of this line is the optimal line for any N (Fig 3).

The entropy increment *B* is strictly concave function, with a unique conditional maximum point for each *A* = const value [19]. This single optimal point or unique optimal grading curve is defined as follows:

$$x_1 = \frac{1}{\sum_{j=1}^{N} a^{j-1}} = \frac{1-a}{1-a^N}, \; x_j = x_1 \, a^{j-1} \quad (7)$$

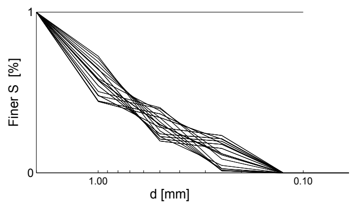
where parameter *a* is the root of the following equation：

$$y = \sum_{j=1}^{N} a^{j-1} \left[j - 1 - A(N-1)\right] = 0 \quad (8)$$

The single positive root *a* varies continuously between 0 and ∞ as *A* varies between 0 and 1, *a*=1 at the symmetry point (*A*=0.5) and *a*>1 on the *A*>0.5 side of the diagram (Imre and Talata 2017).

The internal stability criterion (Figure 8) for elongated grading curves includes a transitional zone, its boundary connects the maximum entropy points with fraction numbers less than *N*. Considering the fractal gradings, the soil is stable if *n* < 2, transitional between *n* at *A*=2/3 (*n* is varying in the function of *N*).

The relative base entropy *A* indicates the relative distance of the mean diameter from the maximum-minimum $\log_2$ diameter values. If A > 2/3 then enough large grains are present in a mixture to form gradually a skeleton and a stable soil matrix The coarse particles "float" in the matrix of the fines if A < 2/3.



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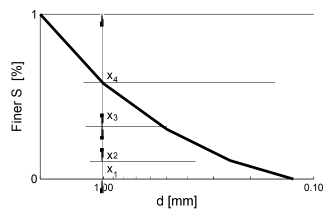


(b)

Figure 3. Mean or fractal gradings. (a), (b) Grading curves with N=4, A=2/3, B=1,2 and the mean or optimal curve.

2.2 *Grading curve characterization by central moments*

The expected value μ of the random variable of a *discrete* distribution is the weighted arithmetic mean of the possible values ($x_1,..., x_k$) of ξ

$$\mu = M(\xi) = \sum_k x_k p_k \tag{10}$$

The variance of a *discrete* variable may be determined by the following expression

$$D^2(\xi) = \sum_i [x_i - M(\xi)]^2 pi = \sum_i x_i^2 pi - M^2(\xi) \tag{11}$$

The coefficient of skewness (C_s) is the quotient of the third central moment of the standard deviation raised to the third power.

$$C_s = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^3 p_i}{D^3} \tag{12}$$

The kurtosis (C_k) is the quotient of the fourth central moment by the fourth power of the standard deviation minus 3.

$$C_k = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^4 p_i}{D^4} - 3 \tag{13}$$

3 METHODS AND MATERIALS

3.1 *Fractal curves & their role in the analyses*

The grading entropy parameters can be used to define a single o ptimal or mean grading curve for each fixed entropy parameter A value (which generally mean infinite many grading curves). . The optimal grading curves are with finite fractal distribution a s follows (Imre and Talata 2017):

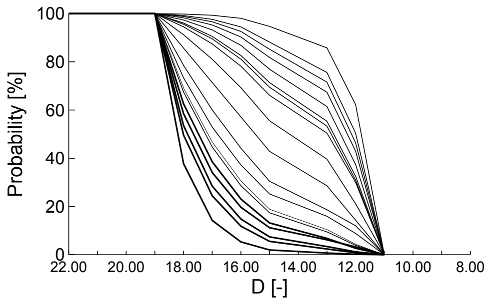
$$a = 2^{(3-n)} \tag{14}$$

$$n = 3 - \frac{\log a}{\log 2} \tag{15}$$

The n depends on N for a given A except at the symmetry point (A = 0.5, B = 1/ln2, n = 3, a =1). The dimension n varies between 3 and -∞ on the A>0.5 side of the normalised diagram as a varies between 1 and ∞; and n varies between 3 and -∞ on the A<0.5 side of the normalized diagram, as a varies between 1 and 0. The grading curves with the same A are a kind of mean grading curve (Figure3).

3.2 *Soils and parameters*

The Central Organisation For Flood Protection And Inland Waterway, Hungary collected 10 different materials from pits along the dikes varying from silt to coarse material. The mixture series were prepared from these at 4 different, fixed, intended d_{10} values, and increasing amount of coarse materials. The artificial mixtures of natural soil grains were prepared from fluvial soils for the permeability testing (Nagy 2011), in this paper the GDS of the artificial mixtures of natural soil grains are analysed. In the reanalysis of the original data, the gradation parameters (d_{10}, d_{30}, d_{50} and d_{60} and Cu= d_{60}/ d_{10}) were determined. The four entropy parameters and central moments were computed. Computing the central moments, both the usual (sieve) fractions system and the theoretical (D~log d) fraction system were used. Using 74 artificial soil mixtures of natural fluvial granular soils, prepared originally for permeability research, in 4 series (differing in fixed d_{10} (Nagy 2011)), the fractal relations and the empirical relations are compared.



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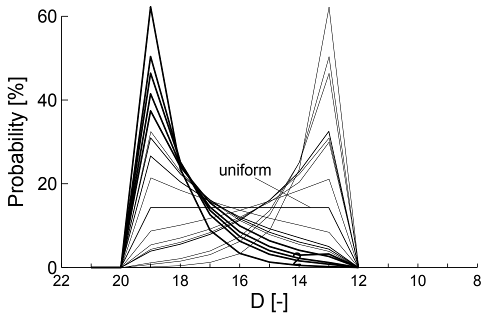

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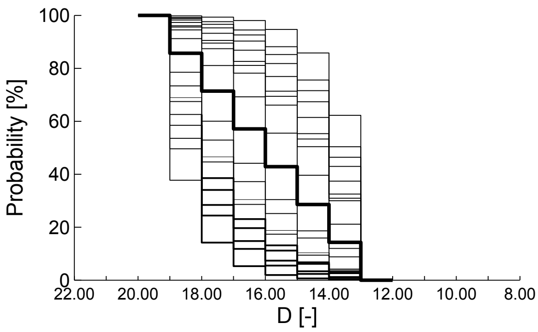
Figure. 4. Fractal grading curves. Continuous representation. (a) measured cumulative and (b) density PSD.



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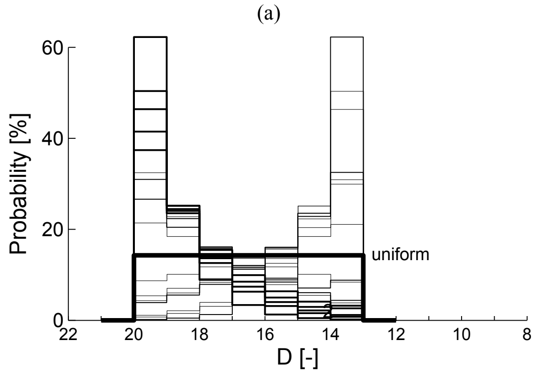


(b)

Figure 5. Fractal grading curves. Discrete representation of (a) measured cumulative and (b) density PSD.

4 RESULTS

Figures 4, 5 indicate the numerically simulated fractal gradings. Results concerning the fractal gradings between the entropy parameters and the usual grading curve parameters are shown in Figures 6 and 7. The central moments and the entropy parameters for the fractal gradings are shown in Figure 8 and in Table 2. Concerning the 4 series, some data are are shown in non-normalized entropy diagram in Figure 9 and represented together with the fractal gradings in Figure 10.

The expected value is the mass center of a distribution, the variance measures how far a set of (random) numbers are spread out from their expected or average value. To assess the general rule for central moments and grading entropy coordinates, about 5 to 18 fractal grading curves with various A values and with fraction number N=5,7..40 were simulated and tested (Figs. 4 to 11). The statistical central moments and Pearson data were computed for d and D. The four grading entropy coordinates were computed, the usual grading curve parameters were determined. Once the relationships were assessed, the data bank grading curve series were also started to be investigated. The first results can be summarized as follows.

Simulations with fractal grading curves by numerical ex periment with N=7 statistical cells are as follows. The rela tionships of the first four central moments in terms of the abstract diameter D [-] and the four entropy coordinates are basically independent of N (except the case of varianc e and mean/non-normalised coordinates).

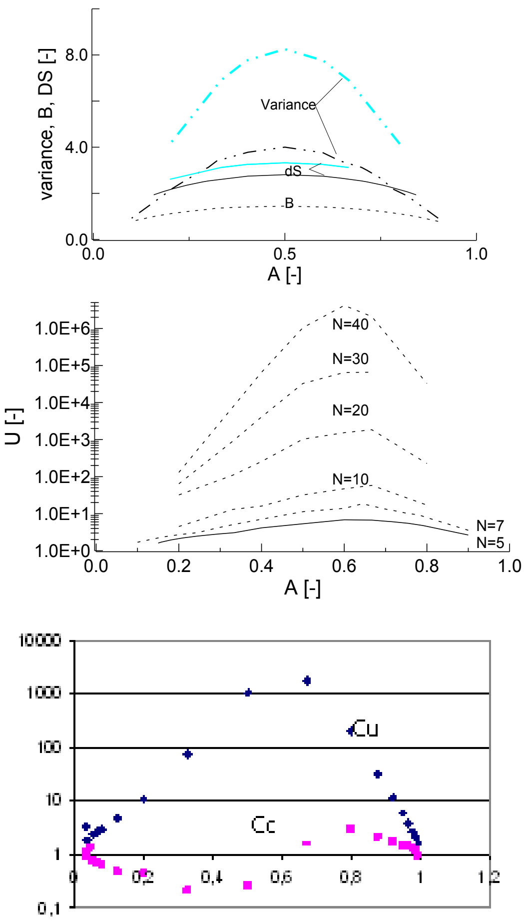


Figure 6. Fractal grading curves. Upper: Variance, Cu and DS depend on N. Lower: N=19 data for Cu and Cc.

Table 2. The skewness and kurtosis+3 in terms of D, fractal gradings

A	Ck+3	CS	B
0,899987	7,74020107	-2,0135569	0,7840322
0,75548	3,32494232	-1,069248	1,2095222
0,5	1,74999894	06	1,442695
0,322138	2,36695412	0,6936551	1,3332234
0,18111	4,63166511	1,5819316	1,062141
0,100003	7,73572768	2,0129315	0,7839906

Figs. 6, 11 demonstrate that C_U has a maximum around A = 2/3 in terms of A while DS and the variance of D at A=1/2. The maximum C_U is dependent on N similarly to DS. Its value for N=5 is around A=0.6, for N=40, the C_U maximum value at around A=2/3 is C_U~160000. Note that A=2/3 marks the boundary between stable and unstable zones of the instability criterion, moreover, the maximum density point as discussed before. The non-fractal soils have generally smaller Cu than the fractal soils.

The analyses discussed with reference to Figs. 6 , 11 indicate that the entropy increments ΔS and their normalised version (*B*), reflect the actual effective number of fractions within the mixture – as a more precise coefficient of uniformity C_U. The Cu and Cz have inverse relations in terms of A and, therefore, it is enough to test C_U. The simulated variance of D and the simulated entropy increment ΔS depend similarly on A, the relative base entropy, the entropy is less accumulated around the mean.

Figure 8 indicates that the characteristic diameteres are increasing in terms of A for fractal gradings, the simulated parameters d50 and d_{10} increase with A, the relative base entropy. Figure 8 shows that d_{50} and d_{10} also depend on the value of N and (as it depends on particle size) it reaches a maximum when A = 1.

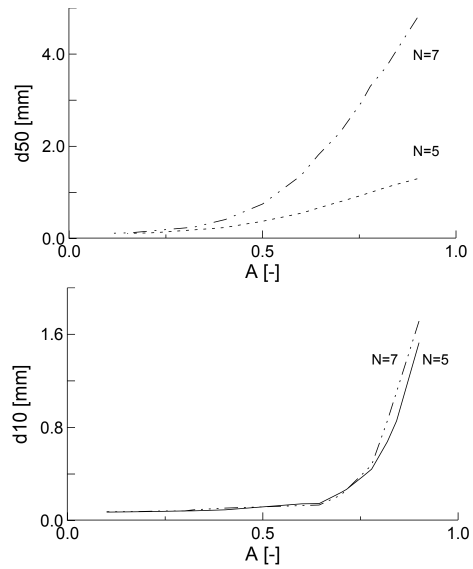


Figure 7. Characteristic diameteres in terms of A for fractal gradings.

The first two central moments are the expected value (mass center of a distribution), the variance (measures how the numbers are spread out from their expected or average value). The first central moment of D is identical to S_0, the second central moment - the variance of D – is similar to DS and C_U as shown in Figure 7.

The relationships for 3^{rd} and 4^{th} central moments of the particle diameter (D) and the entropy represented in terms of kurtosis and skewness are illustrated in Figs 7, 10. Parameters are symmetric for D but not for d. The 3^{rd} and 4^{th} central moments for fractal gradings for given A and N values are basically independent of the fraction number N. The normalized grading entropy parameters A and B have unique, monotonic relationship with the skewness and kurtosis D. The skewness (the measure of the asymmetry of the probability distribution), the kurtosis (the shape of a probability distribution). The non-fractal soils have generally smaller Ck than the fractal soils. The normalized relative base entropy *A* indicates the relative distance of the mean diameter from the maximum-minimum log 2 diameter values and is related to skewness D, the relation is symmetric for abstract diameter D.

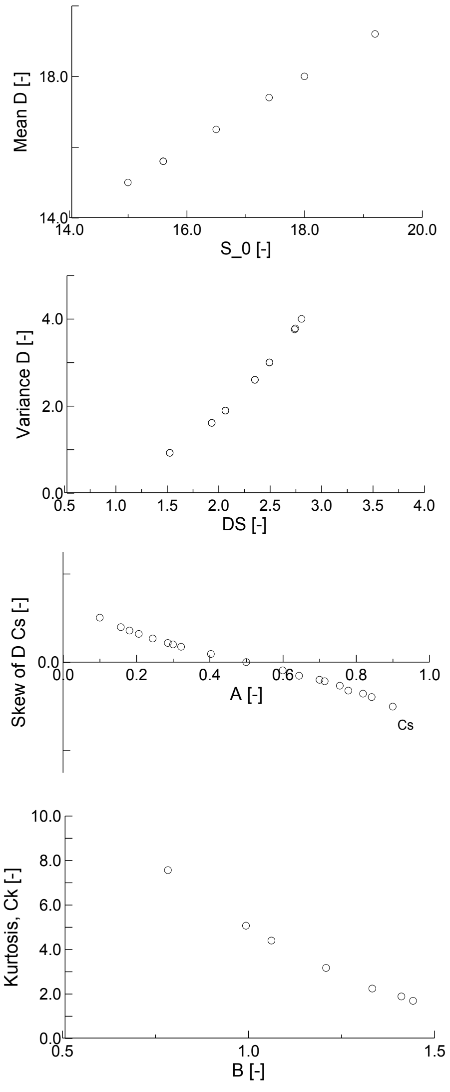


Figure 8. The central moments in terms of D and the grading entropy coordinates for fractal gradings.

5 DISCUSSION & CONCLUSION

5.1 *Grading curve parameters*

The base entropy S_0 is a kind of mean abstract log diameter which varies between S_{0max} and S_{0min}.The relative base entropy A (normalised mean abstract log diameter) varies between 0 and The base entropy S_0 is a kind of dimensionless mean log diameter, a more precise d_m. it is the 1st central moment of D

The relative base entropy parameter *A* indicates the relative distance of the mean diameter from the maximum-minimum $\log_2$ diameter values.

It describes the internal structure. If A > 2/3 then enough large grains are present in a mixture to form gradually a skeleton and a stable soil matrix The coarse particles “float” in the matrix of the fines if A < 2/3. The internal stability for elongated grading curves includes a transitional zone.

The entropy increments ΔS and its normalised version *B,* reflect the actual effective number of fraction within the mixture – similar to the coefficient of uniformity c_u. Its maximum is ln*N*/ln2 and 1/ln2, respectively.

5.2 *Comparing grading curve parameters*

All gradings can equally be represented in terms of both the abstract diameter *D* (see Table 1) and the diameter *d* [mm]. A single fractal grading curve is a mean grading curve for the grading curves with a fixed normalized grading entropy parameter value *A*, fixed fraction number *N* and fixed minimum fraction diameter d_{min}.

These mean grading curves were used to illustrate the relationship between the grading entropy parameters and the central moments of the grading curves; moreover, the relationship between the grading entropy parameters and the traditional parameters of the grading curve.

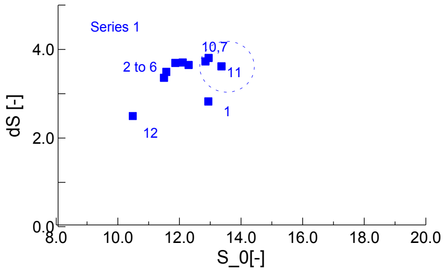
Concerning the relation of grading entropy parameters of grading entropy and the central moments of the grading curves in terms of the abstract diameter *D,* the following is found for fractal gradings.

1. The mean and variance of D are increasing with fraction number *N,* similarly to the base entropy S_0 and the entropy increment ΔS. The kurtosis and skewness of *D* for fractal gradings are about independent of the fraction number *N*, similarly to the normalized parameters *A* and *B*.
2. The base entropy S_0 is the same as the mean abstract diameter *D*. The entropy increment ΔS for fractal gradings is very similar to the variance of *D,* both have the maximum at *A*=0,5. The fractal gradings have about the same kurtosis and skewness of *D* irrespective of *N*.
3. For fractal gradings, the normalized grading entropy parameters *A* and *B* and the skewness and kurtosis, have symmetric relationships, which were given here.
4. For ‘general’ grading curves, these symmetric functions are mean relations.

Concerning the relation of parameters of grading entropy and the traditional parameters of the grading curves*,* the following is found for fractal gradings.

5. The entropy increment ΔS is very similar to the coefficients of uniformity, Cu*,* both are dependent on *N* but they have maximum in terms of different *A*. The entropy increment ΔS has a maximum at *A*=0,5. The fractal gradings had the largest the coefficients of uniformity, Cu at around *A*=2/3.
6. This maximum of Cu at around *A*=2/3, and the Cu-A relation of fractal gradings seems to be a maximum for the Cu-A relation of general gradings at fixed *N*.
7. It is found that the base entropy S0 or the relative base entropy parameter *A* show (a theoretically based) strong, unique relationship with the parameters d_{50} or d_{10}. The slope of the d_{10}- *A* relation is significantly changing in terms of *A*, being different if *A*<2/3 or *A*>2/3. This may significantly influence the empirical correlations for the permeability if they are formulated in terms of the grading entropy parameters instead of d_{10}.

8. The coefficients of uniformity, Cu and Cc of fractal gradings have a kind of inverse relationship in terms of the relative base entropy parameter *A*.



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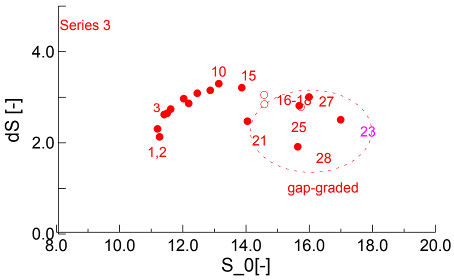

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Figure 9. (a) to (b) Series 1 and 3 data in the non-normalized entropy diagram.

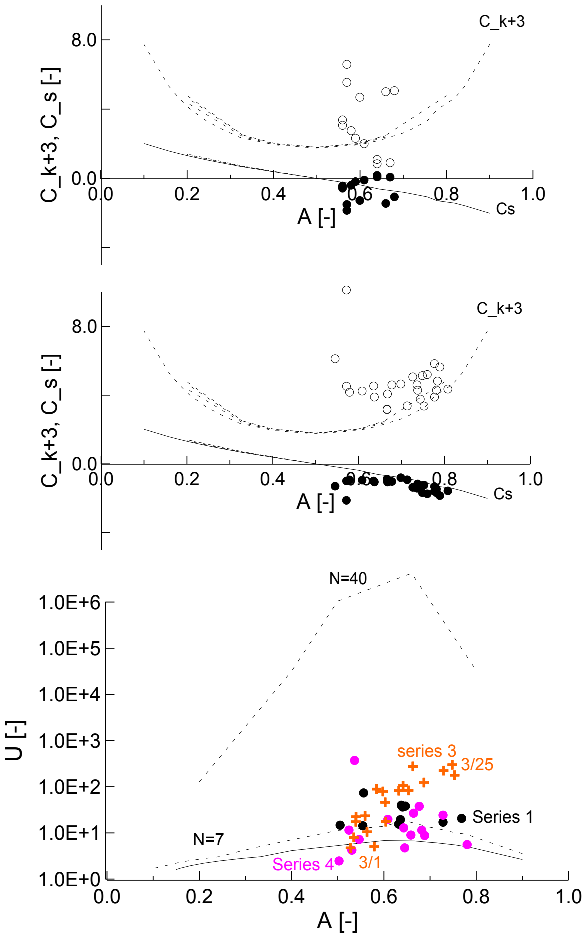

Figure 10. The skewness and kurtosis+3 in terms of D, series 1 and 3. The fractal grading curves reflect mean behaviour. The Cu for 4 series.

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