Proceedings of the 20th International Conference on Soil Mechanics and Geotechnical Engineering– Rahman and Jaksa (Eds)

Effect of soil layering on the serviceability of a large circular slab under uniformly distributed load

Effet de la stratification du sol sur l'aptitude au service d'une grande dalle circulaire sous une charge uniformément répartie

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ABSTRACT: Circular slabs are commonly used as foundations of large-diameter storage tanks. The flexural behavior of a circular slab depends on not only the diameter and thickness of the slab itself but also the stiffness of the supporting soil layering system. This paper investigates the effect of soil layering on the serviceability of a large circular slab under uniformly distributed load. The authors propose a semi-analytical analysis method based on variational principles and energy approach, and the results from the proposed analysis method are validated against those from finite element analyses. The authors then perform parametric studies for various soil layering conditions using the semi-analytical analysis method. Results from the parametric study for circular slabs resting on two-layered soils provide important insights into their serviceability and flexural behavior, capturing a complex nature of soil-slab interactions.

RÉSUMÉ: Les dalles circulaires sont couramment utilisées comme fondations de réservoirs de stockage de grand diamètre. Le comportement en flexion d'une dalle circulaire dépend non seulement du diamètre et de l'épaisseur de la dalle elle-même, mais aussi de la rigidité du système de stratification du sol support. Cet article étudie l'effet de la stratification du sol sur l'aptitude au service d'une grande dalle circulaire sous une charge uniformément répartie. Les auteurs proposent une méthode d'analyse semi-analytique basée sur des principes variationnels et une énergie approche , et les résultats de la méthode d'analyse proposée sont validés par rapport à ceux de l'analyse par éléments finis. Les auteurs effectuons ensuite des études paramétriques pour diverses conditions de stratification du sol en utilisant la méthode d'analyse semi-analytique. Les résultats de l'étude paramétrique pour les dalles circulaires reposant sur des sols à deux strate fournissent des informations importantes sur leur aptitude au service et leur comportement en flexion, capturant la nature complexe des interactions sol-dalle.

KEYWORDS: Circular slabs; Serviceability; Semi-analytical analysis; Settlements

1 INTRODUCTION

Bottom slabs of large circular tanks such as liquid storage tanks and silos are often directly founded on soils. Excessive differential settlement of the bottom slab is very damaging to the structure and often leads to ultimate limit states (Marr et al. 1982, D'orazio and Duncan 1987). Therefore, investigation of the flexural behavior of circular slabs on elastic soils under uniformly distributed load is of significant interest and has been extensively studied using various methods (Cheung and Zienkiewicz 1965, Vlasov and Leont'ev 1966, Selvadurai 1979, Issa and Zaman 1986, Vallabhan and Das 1991). However, most previous studies considered a circular slab resting on a single, homogeneous elastic soil, and the effects of soil layering on differential settlements of large-storage tanks — which is one of the key factors in serviceability limit state (SLS) design of tank foundations — are still largely unknown.

This paper presents a continuum-based, semi-analytical model for analysis of a circular slab resting on a layered soil under uniformly distributed load. The layered soil deposit forms a semi-infinite half space and each soil layer is assumed to behave as a linear-elastic material. The circular slab is modeled as a linear-elastic thin plate based on small-deflection theory of bending. The authors first present the mathematical formulation and the derivation of the governing differential equations. Solutions are obtained by solving the differential equations analytically for soil displacements and numerically, using the finite difference method, for slab deflection. The results from the semi-analytical model are then compared with those from finite element analysis (FEA). Finally, parametric studies are performed for circular slabs resting on two-layered soils. The proposed analysis method effectively captures the complex nature of the interactions between the slab and layered soil and important insights are highlighted.

2 ANALYSIS METHOD

2.1 *Problem Definition*

The analysis considers a circular slab of diameter B (= $2r_p$, where r_p is the radius of the slab) and thickness t_p resting on a layered soil deposit (Fig. 1). There are altogether N discrete soil layers. H_i denotes the depth from the ground surface to the bottom of any layer i (subscript i denotes the i^{th} layer). All soil layers extend to infinity in the radial direction, and the bottom (N^{th}) layer extends downward to infinity in the vertical direction. The soil medium is assumed to behave as linear elastic, isotropic, and homogeneous in each soil layer, with elastic properties described by Young's modulus E_{si} and Poisson's ratio ν_{si}. The circular slab is also described as a linear elastic material with Young's modulus E_p, thickness t_p, and a Poisson's ratio ν_p and assumed to follow Kirchhoff's small-deflection theory of bending. A uniformly distributed load q acts on the slab. It is assumed that there is no separation or slippage between the slab and soil or between the soil layers. Since this problem is axisymmetric, a cylindrical (r-θ-z) coordinate system is used to define the slab-layered soil system.

The vertical displacement of soil u_z at any point in the soil medium is assumed to be a product of separable variables (Vallabhan and Das 1991; Seo et al. 2009; Salgado et al. 2013) and is given by

$$u_{zi}(r,z) = w(r)\phi_i(z) \tag{1}$$

where $u_{zi}(r, z)$ = vertical soil displacement at any point (r, z) in the soil mass within the i^{th} layer; $w(r)$ = function that represents the vertical displacement of the soil profile at ground surface; $\phi_i(z)$ = dimensionless shape function that describes the decay of the vertical soil displacement with increasing depth in the i^{th} soil layer. The soil displacement decay function in the 1st layer, $\phi_1(z)$, is assumed to be equal to one at the slab-soil interface [*i.e.*, $\phi_1(0) = 1$], and this condition ensures proper slab-soil contact [*i.e.*, $u_{z1}(r, 0) = w(r)$] at the ground surface. The vertical displacement in soil must vanish at infinite depth from the slab; therefore, the soil displacement decay function of the bottommost layer, $\phi_N(z)$, is assumed to be zero at $H_N = \infty$ [*i.e.*, $\phi_N(\infty) = 0$]. The radial soil displacement u_r is neglected in the analysis since it is much smaller compared to the vertical soil displacement u_z under the symmetric vertical load. With the absence of u_r, tangential soil displacement u_θ becomes zero owing to rotational symmetry.

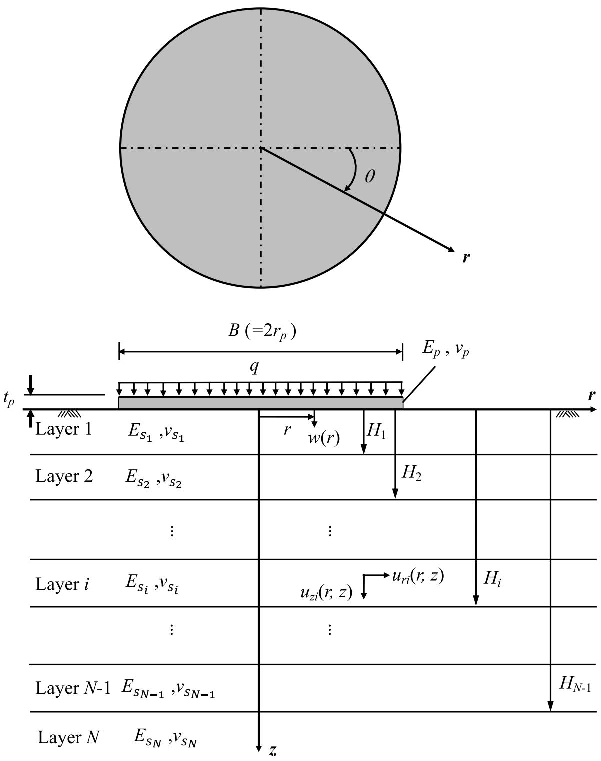

Figure 1. Circular slab resting on a layered soil under uniformly distributed load.

2.2 *Governing Differential Equation*

In this study, the principle of minimum total potential energy is used to obtain the equilibrium of the system. The total potential energy of the slab-soil system, including both internal and external potential energies, is given by

$$\begin{aligned}\Pi = &\frac{D}{2}\int_0^{2\pi}\int_0^{r_p}\left[\left(\nabla^2 w\right)^2 - \frac{2\left(1-\nu_p\right)}{r}\frac{dw}{dr}\frac{d^2w}{dr^2}\right] r\,dr\,d\theta + \\ &+\frac{1}{2}\int_0^{\infty}\int_0^{2\pi}\int_0^{\infty}\left(\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \sigma_{zz}\varepsilon_{zz} + \tau_{rz}\gamma_{rz} + \tau_{r\theta}\gamma_{r\theta} + \tau_{\theta z}\gamma_{\theta z}\right) r\,dr\,d\theta\,dz \\ &-\int_0^{2\pi}\int_0^{r_p} q w r\,dr\,d\theta\end{aligned} \tag{2}$$

where $D = E_p t_p^3/[12(1-\nu_p^2)]$ = flexural rigidity of the slab; ∇^2 = Laplace operator; and σ_{ij} and ε_{ij} are the stress and strain tensors in the soil (i and j represent indicial notation). The first integral term, which contains D, w, and ν_p, of Eq. (2), represents elastic strain energy stored in the slab due to bending. The second integral term, which contains stress and strain tensors σ_{ij} and ε_{ij} represents the elastic strain energy of soil. Finally, the third integral term that contains q represents the work done by the external load on the slab-soil system.

Applying the principle of minimum potential energy (i.e., $\delta\Pi = 0$) to Eq. (2) and using variational calculus, we obtain the governing differential equations for the functions $w(r)$ and $\phi_i(z)$. Detailed derivation process of governing differential equations can be found in Gunerathne et al. (2019).

For the function $w(r)$, two sub-domains are considered: within the slab diameter ($0 \le r \le r_p$) and outside the slab ($r_p \le r < \infty$). From variational principles, the governing differential equation for the function w within the slab ($0 \le r \le r_p$), representing the vertical deflection of the slab itself, is given as

$$D\nabla^4 w - 2\sum_{i=1}^{N} t_i \nabla^2 w + \sum_{i=1}^{N} k_i w = q \tag{3}$$

where the parameters k_i and t_i appearing in Eq. (3) are defined as follows:

$$k_i = \frac{E_{si}\left(1-\nu_{si}\right)}{\left(1+\nu_{si}\right)\left(1-2\nu_{si}\right)}\int_{H_{i-1}}^{H_i}\left(\frac{d\phi_i}{dz}\right)^2 dz \tag{4}$$

$$2t_i = \frac{E_{si}}{2\left(1+\nu_{si}\right)}\int_{H_{i-1}}^{H_i}\phi_i^2\,dz \tag{5}$$

When the slab deforms, the soil mass beneath the slab provides resistances to the slab deformation. The soil beneath the slab can be considered as a series of concentric hollow soil cylinders with infinitesimal thicknesses (see Fig. 2). The parameter k_i in Eq. (4) has a unit of force/length^3 and represents the compressive resistance of the hollow soil cylinder, whereas the parameter t_i in Eq. (5) has a unit of force/length and accounts for the shearing resistance between adjacent hollow soil cylinders (Basu 2006, Salgado et al. 2007). Because an explicit analytical solution to Eq. (3) for the slab deflection is not available, finite difference method is used to solve Eq. (3).

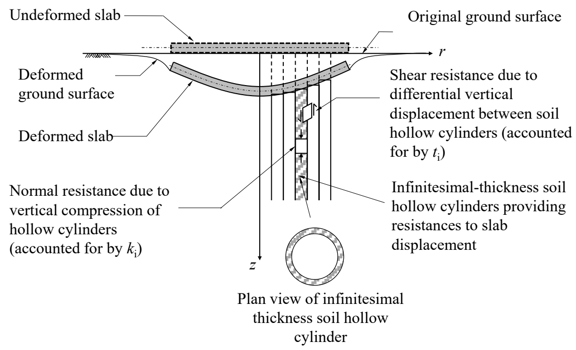

Figure 2. Slab-soil interactions and physical meanings of parameters k and t.

The differential equation for the function w outside the slab ($r_p \le r < \infty$), representing the vertical displacement of the soil at ground surface, is given as

$$-2\sum_{i=1}^{N} t_i \nabla^2 w + \sum_{i=1}^{N} k_i w = 0 \tag{6}$$

with boundary conditions of $w = w(r_p)$ at $r = r_p$ [$w(r_p)$ is a deflection of the slab at $r = r_p$, and this boundary condition represents displacement continuity at $r = r_p$] and $w = 0$ at $r = \infty$. Eq. (6) is a form of the modified Bessel differential equation, and its solution using the boundary conditions is given by

$$w(r) = w(r_p)\frac{K_0(\beta r)}{K_0(\beta r_p)} \tag{7}$$

where

$$\beta = \sqrt{\sum_{i=1}^{N} k_i / 2\sum_{i=1}^{N} t_i} \tag{8}$$

and K_0 = modified Bessel functions of the second kind of zero order.

The governing differential equation for soil displacement decay function $\phi_i(z)$ in the vertical direction is given as

$$\frac{d^2\phi_i}{dz^2} - \left(\frac{\gamma_i}{H_i - H_{i-1}}\right)^2 \phi_i = 0 \tag{9}$$

where

$$\left(\frac{\gamma_i}{H_i - H_{i-1}}\right)^2 = \frac{(1-2\nu_{si})}{2(1-\nu_{si})}\frac{\int_0^\infty \left(\frac{dw}{dr}\right)^2 rdr}{\int_0^\infty w^2 rdr} \tag{10}$$

The parameter γ_i has a unit of per length and determines the rate at which the vertical soil displacement diminishes in the vertical direction (Vlasov and Leont'ev 1966). General solution to Eq. (9) is a form of exponential function, and constants of integration are determined with boundary conditions.

2.3 *Iterative Solution Scheme*

As stated previously, Eqs. (6) and (9) have explicit analytical solutions while Eq. (3) requires a numerical method to solve. Therefore, semi-analytical, iterative solution scheme is employed in this study. To obtain the vertical displacement w at ground surface from Eqs. (3) and (6), the soil resistance parameters k_i and t_i need to be known *a priori*. The k_i and t_i parameters are obtained by integrating the decay function ϕ_i and its first derivative [refer to Eqs. (4) and (5)], which in turn depends on w through γ_i [refer to Eqs. (9) and (10)]. Due to this interdependence of the soil and slab differential equations, an iterative solution scheme is required. The solution scheme first assumes initial values of γ_i (= $\gamma_{i,\text{old}}$) and then obtain ϕ_i by solving Eq. (9) analytically. With the solution of ϕ_i, the k_i and t_i parameters are obtained from Eqs. (4) and (5). Using the known values of k_i and t_i, Eqs. (3) and (6) are solved numerically and analytically, respectively, and the displacement function w is obtained. Then new values of γ_i (= $\gamma_{i,\text{new}}$) are calculated using the computed displacement function w and its derivative dw/dr [see Eq. (10)] and compare $\gamma_{i,\text{new}}$ against $\gamma_{i,\text{old}}$. If the differences are greater than the prescribed tolerance (a value of 10^{-5} was used in this study), iterations are continued with the calculated value of γ_i taken as the new guess ($\gamma_{i,\text{old}}$ = $\gamma_{i,\text{new}}$) until the differences between $\gamma_{i,\text{old}}$ and $\gamma_{i,\text{new}}$ from two successive iterations fall below the prescribed tolerance for all layers.

3 ANALYSIS RESULTS

3.1 *Comparison with FEA*

The results from the semi-analytical method were compared against those from FEA performed using PLAXIS 2D under axisymmetric condition. Fifteen-noded triangular elements were used to represent both soil and slab in the FEA. It was confirmed that the mesh and domain size used in FEA were fine and large enough, respectively, and had no effect on load-displacement response of the slab. For comparison purposes, a concrete tank with a bottom slab of B = 20 m, t_p = 150 mm, E_p = 30 GPa, and ν_p = 0.15 was considered. The bottom slab was subjected to a uniformly distributed load q of 60 kPa representing both fluid weight and self-weight of the tank. The slab rested on a three-layer soil deposit with H_1 = 10 m (= 0.5B), H_2 = 20 m (= 1B), and H_3 = 500 m (= 25B); E_{s1}= 20 MPa, E_{s2} = 80 MPa and E_{s3} = 300 MPa, and ν_{s1} = ν_{s2} = ν_{s3} = 0.2.

Fig. 3(a) shows the vertical displacement profiles at the ground surface obtained from the semi-analytical method and from FEA. The center settlement obtained from Schmertmann's method (Schmertmann et al. 1978), which is often used to estimate the center settlement of a tank foundation in practice for preliminary design purposes (Ramasamy and Kalaiselvan 1998), is also presented in the figure. The vertical displacement profile at the ground surface from the semi-analytical method closely matches that of the FEA, although the semi-analytical method produces a slightly stiffer response (the semi-analytical method produced 12% less displacement than the FEA at the slab center). One of the reasons for such a slightly stiffer response (*i.e.*, slightly smaller slab deflections) from the semi-analytical method is because the radial displacement of soil was assumed to be negligible, thus imposing additional stiffness. Fig. 3(a) further suggests that computing the center settlement using Schmertmann's method, originally developed for isolated footings, may underestimate the actual center settlement of the large-diameter tanks.

Now distributions of bending moment M_r across the concrete slab obtained from both analyses are compared [Fig. 3(b)]. The bending moment of the slab from the semi-analytical method is computed using the following relationship:

$$M_r(r) = -D\left[\frac{d^2w}{dr^2} + \frac{\nu_p}{r}\frac{dw}{dr}\right] \tag{11}$$

Bending moments from both analyses show good agreement from the center of the slab to a radial distance of 5 m. They then deviate from each other toward edge of the slab. However, the same trend is observed for both cases, showing that bending moments are larger near the edge of the slab than at the center.

Overall, results from the semi-analytical model show favorable agreements with those from FEA in terms of vertical displacement and bending moment of the slab with much less computational efforts.

3.2 *Effect of Soil Layering*

To investigate the effect of soil layering on flexural behavior of a slab, a circular concrete slab with B = 10 m, t_p = 150 mm, E_p = 30 GPa, and ν_p = 0.15 is considered in this section. The concrete slab rests on two-layer soil profiles, and a uniformly distributed load q of 24 kPa is applied. The depth H_1 to the bottom of the first layer varies from 0 to 4B; the depth H_2 to the bottom of the second layer is kept constant at 25B. For each of these cases, seven different soil stiffness ratios are considered (E_{s1}/E_{s2} = 0.1, 0.2, 0.4, 1, 2, 3, and 4) while keeping Young's modulus E_{s2} of the 2nd layer constant at 30 MPa (*i.e.*, E_p/E_{s2} = 1000). E_{s1}/E_{s2} < 1 represents a soft-over-stiff soil, E_{s1}/E_{s2} > 1 represents a stiff-over-soft soil, and E_{s1}/E_{s2} = 1 represent a homogeneous soil case.

Figs. 4(a) and 4(b) show profiles of the vertical displacement of ground surface for the concrete slab with H_1/B = 0.25 and 2, as examples, for all seven soil stiffness ratios. For the given thickness of the top layer, the vertical displacement of the slab decreases as the E_{s1}/E_{s2} ratio increases (*i.e.*, the top layer is getting stiffer). When a soft soil overlies a stiff soil (E_{s1}/E_{s2} = 0.1, 0.2, and 0.4), the vertical displacement of the slab increases with increasing H_1/B ratio (*i.e.*, the top soft layer becomes thicker). On the other hand, when a stiff soil overlies a soft soil (E_{s1}/E_{s2} = 2, 3 and 4), the vertical displacement decreases with increasing H_1/B ratio (*i.e.*, the top stiff layer becomes thicker).

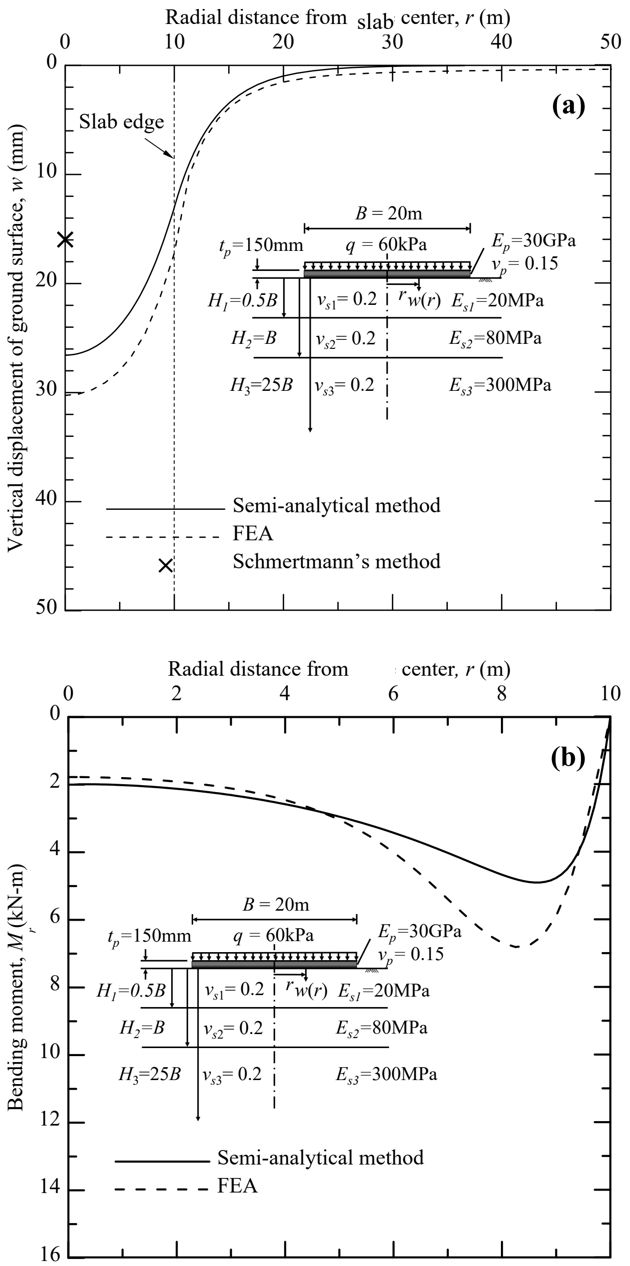

Figure 3. Comparison of the results from the semi-analytical method with those from FEA: (a) vertical displacement of ground surface versus radial distance from the slab center and (b) bending moment of the slab versus radial distance from the slab center.

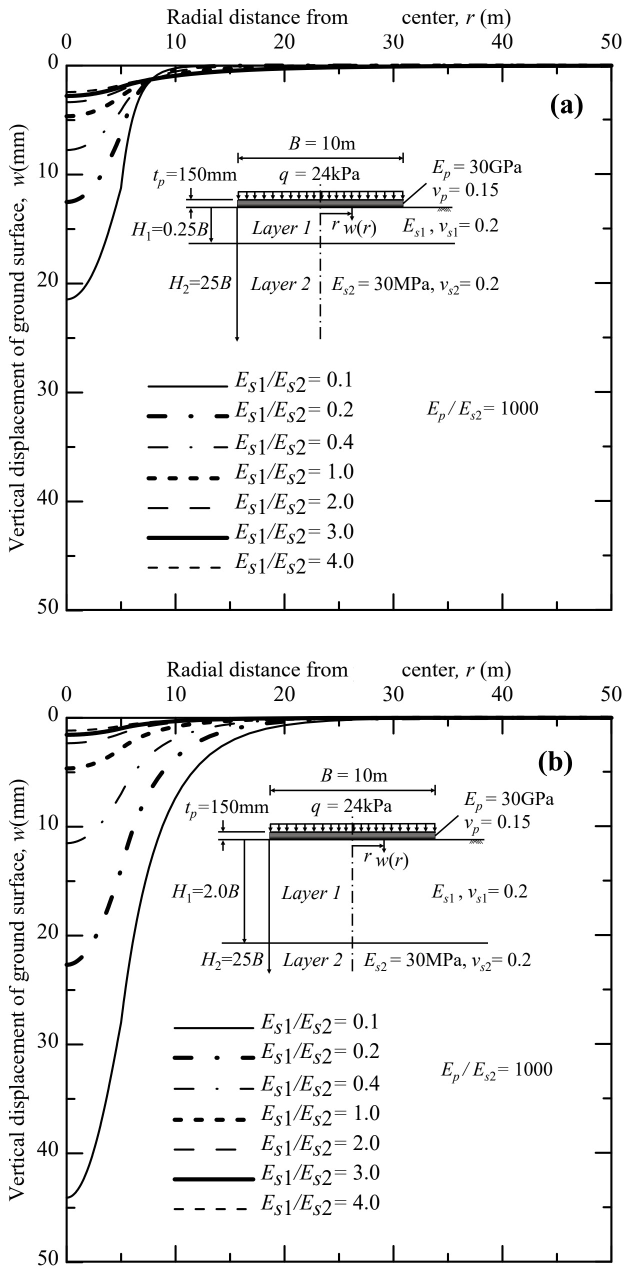

Figure 4. Vertical displacement of ground surface for uniformly loaded circular slabs on a two-layered soil versus radial distance from slab center with (a) H_1/B = 0.25 and (b) H_1/B = 2.

The edge-to-center settlement ratio $w(r_p)/w(0)$ versus relative top layer thickness H_1/B obtained from the analysis is presented in Fig. 5(a). When a layer of soft soil overlies a stiff soil (E_{s1}/E_{s2} < 1), $w(r_p)/w(0)$ initially decreases as H_1/B increases but, then, starts increasing and approaches a plateau. This is because the flexural behavior of the slab is dominated by the bottom stiff soil when the thicknesses of the top soft layer is very small, but, as the top soft layer becomes thicker, flexural behavior of the slab is less affected by the bottom stiff layer and more controlled by the top soft layer. On the other hand, when a stiff soil overlies a soft soil (E_{s1}/E_{s2} > 1), the edge-to-center settlement ratio $w(r_p)/w(0)$ initially increases as H_1/B increases and then decreases, approaching a plateau at H_1/B is about 3.

Although the edge-to-center settlement ratio $w(r_p)/w(0)$ is useful to describe the flexibility of the slab, it is not the best indicator for differential settlement because $w(r_p)/w(0)$ can be large but the differential settlement $w(0) - w(r_p)$ is negligible when the magnitude of the settlement itself is very small. To better represent the effect of soil layering on differential settlement, the differential settlement is normalized with respect to the slab radius r_p [*i.e.*, $\{w(0) - w(r_p)\}/r_p$]. The normalized differential settlement is also known as an angular distortion and is commonly used as a tolerable deflection criterion for SLS design of large storage tank foundations. Tolerable angular distortion values for concrete tank foundations range from 0.002 to 0.0033 (Brannan 2012, USACE 1990).

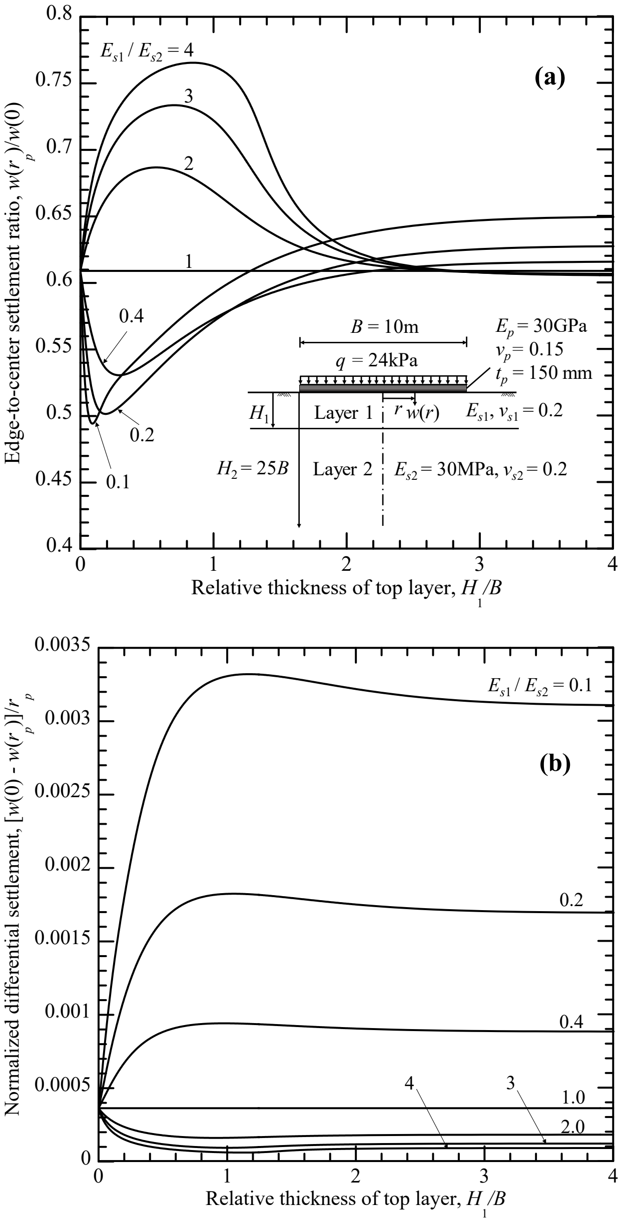

Figure 5. (a) Edge-to-center settlement ratio of the slab vs. relative top layer thickness and (b) normalized differential settlement vs. relative top layer thickness for a two-layered soil.

Fig. 5(b) shows the normalized differential settlement versus relative top layer thickness H_1/B. As expected, for the given thickness of the top layer, the normalized differential settlement decreases as the E_{s1}/E_{s2} ratio increases (*i.e.*, the top layer is getting stiffer). For a stiff-over-soft soil case ($E_{s1}/E_{s2} > 1$), the normalized differential settlement is well below the tolerable limit of 0.002 regardless of H_1/B. On the other hand, for a soft-over-stiff soil case ($E_{s1}/E_{s2} > 1$), the normalized differential settlement increases as H_1/B increases (*i.e.*, the thickness of the top soft layer increases) and exceeds 0.002 when $E_{s1}/E_{s2} = 0.1$ with $E_p/E_{s2} = 1000$. Interestingly, however, the normalized differential settlement does not keep increasing as H_1/B increases and shows the maximum value when about 1B-thick soft soil overlies a stiff soil. In other words, having 1B-thick-soft soil overlying a stiff soil is more detrimental to the slab in terms of an angular distortion than having a 4B-thick-soft soil, although 4B-thick-soft soil would yield more total settlement than 1B-thick-soft soil.

4 SUMMARY AND CONCLUSIONS

A continuum-based, semi-analytical model was developed for the elastic analysis of a uniformly loaded large circular slab resting on a layered elastic soil. The layered elastic soil forms a semi-infinite half space and each layer is assumed to behave as a linear-elastic material. The circular slab follows Kirchhoff's small-deflection theory of bending. The governing differential equations of the slab-soil system are derived based on variational principles and energy approach, and solved semi-analytically. One of the advantages of the semi-analytical model is that it is general enough to consider any number of layers and any flexibility of the plate and requires much less computational efforts than FEA.

The results from the semi-analytical model show favorable agreements with those from FEA in terms of vertical displacement and bending moment of the slab. Furthermore, results from the parametric study show that the semi-analytical model successfully captures the complexity of slab-soil interactions for two-layered profiles with various layer stiffness and thickness ratios.

When a layer of soft soil overlies a stiff soil, the edge-to-center settlement ratio initially decreases as the relative top layer thickness increases but, then, starts increasing and eventually approaches a plateau. On the other hand, when a stiff soil overlies a soft soil, the edge-to-center settlement ratio initially increases as the relative top layer thickness increases and then decreases, approaching a plateau when the thickness of top soil layer H_1 becomes about three times slab diameter B. The normalized differential settlements (i.e., angular distortions) show maximum values when about 1B-thick soft soil overlies a stiff soil, suggesting that having 1B-thick-soft soil overlying a stiff soil might cause more distress in the slab than having a 4B-thick-soft soil.

Finally, the semi-analytical model can be extended to consider the nonlinear behavior of founding soil by employing a piecewise-linear approach.

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