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Is the scale of fluctuation the only important parameter in geotechnical spatial variability?

L'échelle de fluctuation est-elle le seul paramètre important de la variabilité spatiale géotechnique?

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ABSTRACT: This paper summarizes our recent findings in characterizing geotechnical spatial variability. In the literature, the scale of fluctuation (SOF) has been considered as the main parameter that affects the auto-correlation behavior in geotechnical spatial variability. Other parameters have been largely ignored. In this paper, we show that SOF is the only key parameter for spatial variability only when the performance function of a geotechnical structure is completely governed by arithmetic spatial averaging (e.g., friction pile). For a performance function that is not completely governed by arithmetic spatial averaging, we show that two other factors may be also important: sample path smoothness and pseudo-periodicity (hole effect). The sample path smoothness is characterized by the smoothness parameter in the Whittle-Matérn (WM) auto-correlation model, and the hole effect can be characterized by multiplying the auto-correlation model by a cosine function.

RÉSUMÉ : Cet article résume nos récentes découvertes dans la caractérisation de la variabilité spatiale géotechnique. Dans la littérature, l'échelle de fluctuation (SOF) a été considérée comme le paramètre principal qui affecte le comportement d'auto-corrélation dans la variabilité spatiale géotechnique. D'autres paramètres ont été largement ignorés. Dans cet article, nous montrons que SOF est le seul paramètre clé pour la variabilité spatiale uniquement lorsque la fonction de performance d'une structure géotechnique est complètement régie par une moyenne spatiale (par exemple, un pieu de friction). Pour une fonction de performance qui n'est pas complètement régie par la moyenne spatiale, nous montrons que deux autres facteurs sont également importants : la régularité du chemin de l'échantillon et la pseudo-périodicité (effet trou). Le lissage du chemin de l'échantillon est caractérisé par le paramètre de lissage dans le modèle d'auto-corrélation de Whittle-Matérn (WM), et l'effet de trou peut être caractérisé en multipliant le modèle d'auto-corrélation par une fonction cosinus. L'article continue d'illustrer comment identifier systématiquement le SOF, la régularité et la pseudo-périodicité dans un ensemble de données de test de pénétration du cône en utilisant la méthode du maximum de vraisemblance avec le modèle d'auto-corrélation cosinus-WM.

KEYWORDS: Spatial variability, auto-correlation mode, scale of fluctuation, sample path smoothness, hole effect.

1 INTRODUCTION

For decades, the scale of fluctuation (SOF) (Vanmarcker 1977, 1983) has been considered as the only important parameter for the auto-correlation (or spatial correlation) of soil properties. The effect of other parameters on geotechnical problems is not well investigated in the literature until recently. Ching and Phoon (2019) showed that the performance of a geotechnical structure in spatially variable soil can be significantly affected by the sample path smoothness (or simply, smoothness), which is related to the mean-square differentiability of a random field. More recently, Chang et al. (2021) further showed that the performance of a geotechnical structure can also be significantly affected by the pseudo-periodicity (hole effect) in spatial variability. This suggests that SOF alone is not enough to capture the effect of soil spatial variability on geotechnical performance. One may need to consider three parameters, SOF, smoothness, and hole effect, simultaneously.

The purpose of the current paper is two folds. First, the paper will briefly introduce the notions of smoothness and pseudo-periodicity (hole effect) in spatial variability. Second, the paper will demonstrate by numerical examples that smoothness and hole effect have significantly impact on the performance of a geotechnical structure in spatially variable soil.

2 SMOOTHNESS AND HOLE EFFECT

2.1 *Smoothness*

In geotechnical engineering, the property of a spatially variable soil mass is often modeled as the summation of trend function (t) and spatial variability (ε). The spatial variability ε(z) (z is depth) is typically modeled as a zero-mean stationary random field with standard deviation σ and certain auto-correlation function (Vanmarcke 1977). The auto-correlation function (ACF), denoted by ρ(Δz), defines the correlation between two locations with Δz apart:

$$\rho(\Delta z)=\rho\left[\varepsilon(z),\varepsilon(z+\Delta z)\right]=\frac{CV\left[\varepsilon(z),\varepsilon(z+\Delta z)\right]}{\sqrt{Var\left[\varepsilon(z)\right]}\cdot\sqrt{Var\left[\varepsilon(z+\Delta z)\right]}} \quad (1)$$

where Var(.) denotes variance; CV(.,.) denotes covariance. Several ACF models have been adopted in the literature (e.g., Vanmarcke 1983; Jaksa et al. 1999; Uzielli et al. 2005):

$$\rho(\Delta z)=\exp\left(-2\times|\Delta z|/\delta\right) \quad \text{Single exponential (SExp)} \quad (2)$$

$$\rho(\Delta z)=\left(1+4\frac{|\Delta z|}{\delta}\right)\times\exp\left(-4\frac{|\Delta z|}{\delta}\right) \quad \text{Second order Markov (SMK)} \quad (3)$$

$$\rho(\Delta z)=\exp\left(-\pi\times\Delta z^2/\delta^2\right) \quad \text{Squared exponential (QExp)} \quad (4)$$

where δ (in m) is the scale of fluctuation (SOF). Figure 1 shows these ACF models with δ = 1. These ACF models are finite-scale models in the sense that the area under ρ(Δz) is finite. Figure 1a show the general view, whereas Figure 1b shows the zoom-in near Δz = 0.

Seldom mentioned in the literature is that the random field realizations obtained from different ACF models may have very different appearances. Figure 2 shows realizations of a zero-mean random field with different ACF models. All random fields have a unit SOF, i.e., $\delta = 1$. It is clear that for SExp, the random field realizations have significant local jitters, whereas the local jitters are not present for QExp and SMK. The QExp model produces a very smooth realization. Whether or not the ACF is differentiable at zero lag has direct impact on the local jitters (Rasmussen and Williams 2006): the ACFs for QExp and SMK are differentiable at zero lag, whereas that for SExp is not (see Figure 1b). Therefore, significant local jitters are present in the random field realizations for SExp, but not in those for QExp.

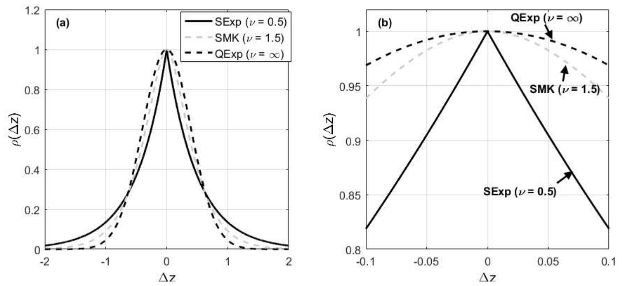

Figure 1. Three different ACF models with $\delta = 1$: (a) general view; (b) zoom-in near $\Delta z = 0$.

The Whittle-Matérn (WM) auto-correlation model (Stein 1999; Guttorp and Gneiting 2006) is a suitable ACF model that can produce random field realizations with various degrees of local jitters. Its ACF model has the following format:

$$\rho(\Delta z)=\frac{2}{\Gamma(\nu)}\cdot\left(\frac{\sqrt{\pi}\cdot\Gamma(\nu+0.5)\cdot|\Delta z|}{\Gamma(\nu)\cdot\delta}\right)^{\nu}K_{\nu}\left(\frac{2\sqrt{\pi}\cdot\Gamma(\nu+0.5)\cdot|\Delta z|}{\Gamma(\nu)\cdot\delta}\right) \quad (5)$$

where ν is the smoothness parameter; Γ is the Gamma function (Abramowitz and Stegun 1970); K_ν is the modified Bessel function of the second kind with order ν (Abramowitz and Stegun 1970). For ν = 0.5, 1.5, and ∞, the W-M auto-correlation model reduces to the SExp, SMK, and QExp model, respectively.

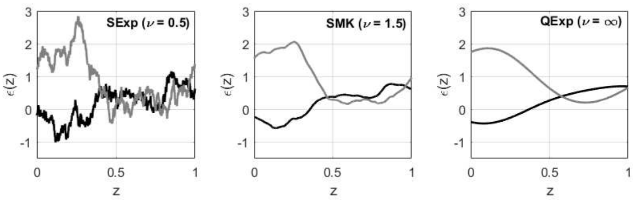

Figure 2. Realizations of a zero-mean random field with different ACF models

2.2 *Hole effect*

The hole effect refers to the phenomenon of a non-monotonic ACF (Journel and Froidevaux 1982; Ma and Jones 2001). An ACF model without a hole effect is usually characterized by monotonic decreasing trend with lag distance (e.g., Figure 3a), whereas an ACF model with the hole effect is usually characterized by certain degree of periodicity as a function of lag distance (e.g., Figure 3c). The hole effect is usually the reflection of pseudo-periodicity in the spatial variability (Journel and Froidevaux 1982). Figures 3d-f show the random field realizations for the ACF models in Figures 3a-c, respectively. The model in Figure 3a has no hole effect, so there is no pseudo-periodicity in the realization in Figure 3d. In contrast, the model in Figure 3c has a strong hole effect, so the pseudo-periodicity in Figure 3f is clearly visible. Note that pseudo-periodicity in the random field realization is not always clearly visible, e.g., the pseudo-periodicity in Figure 3e is not very clear, although its underlying model (Figure 3b) has certain hole effect.

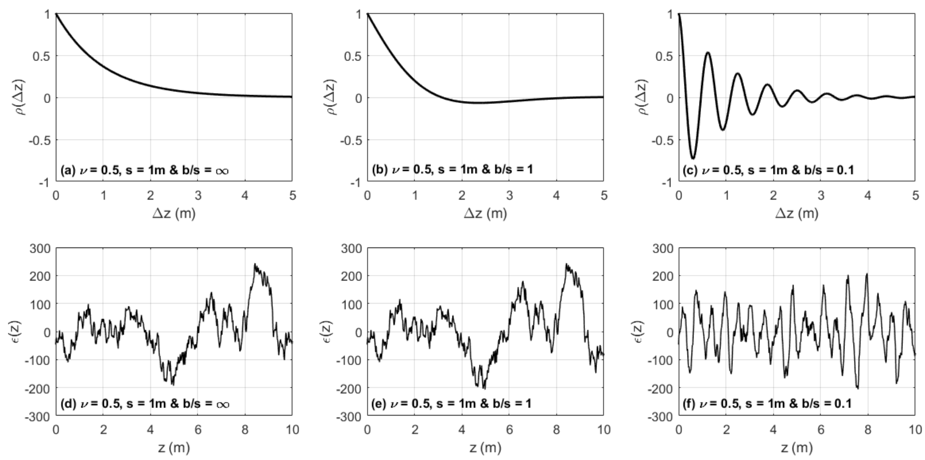

Figure 3. (a-c) ACF models with various degrees of hole effect; (d-f) random field realizations.

The following cosine Whittle-Matérn (CosWM) model (Chang et al. 2021) is a suitable ACF model that can produce random field realizations with various degrees of smoothness and hole effect:

$$\rho(\Delta z)=\frac{2^{1-\nu}}{\Gamma(\nu)}\cdot\left(\frac{\sqrt{2\nu}\cdot|\Delta z|}{s}\right)^{\nu}K_{\nu}\left(\frac{\sqrt{2\nu}\cdot|\Delta z|}{s}\right)\cdot\cos\left(\frac{\Delta z}{b}\right) \quad (6)$$

where s (in m) is the scale parameter; the parameter b (in m) controls the period of the consine function. The dashed lines in Figure 4 illustrate the CosWM model with ν = 0.5 (CosSExp), 1.5 (CosSMK), and ∞ (CosQExp). All illustrations are with s = 1 m and b/s = 0.3. The hole effect is quantified by b/s. When b/s approaches ∞, the hole effect disappears, and CosWM reduces to WM. The smaller b/s, the stronger the hole effect. b/s = 1 can be regarded as the threshold for a significant hole effect. For b/s < 1 (e.g., Figure 4 with b/s = 0.3), the hole effect is significant. For b/s $\geq$ 1, the hole effect is insignificant. Although the CosWM model does not adopt SOF as its basic parameter, the ratio δ/s can be expressed as a function of ν and b/s, but unfortunately the analytical expression for this relationship is not available. Figure 5 shows how δ/s varies with ν and b/s. When b/s approaches ∞, the hole effect disappears, and δ/s approaches a constant.

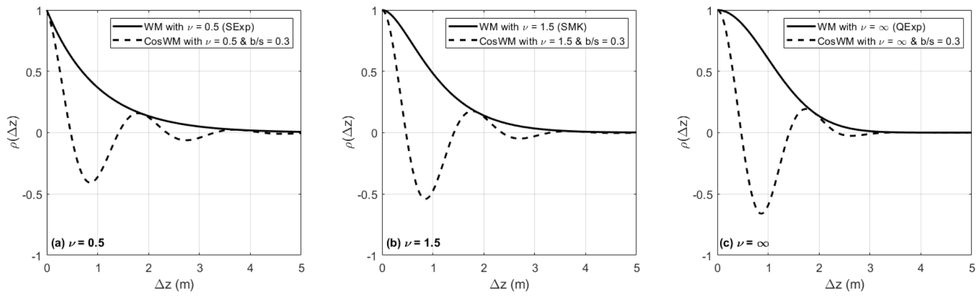

Figure 4. ACF models with various smoothness: (a) $\nu = 0.5$; (b) $\nu = 1.5$; (c) $\nu = \infty$. Solid lines are for monotonic models, whereas dashed lines are for hole-effect models.

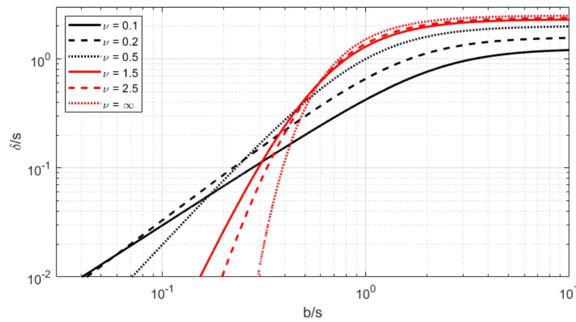

Figure 5. Relationship between δ/s and (ν, b/s) for the CosWM model.

3 NUMERICAL EXAMPLES

The current paper investigates the impact of the hole effect on the failure probability (p_f) of a geotechnical structure. The WM and CosWM models are adopted to simulate random fields. Four geotechnical examples are investigated: (a) friction pile; (b) one-dimensional seepage; (c) infinite slope; and (d) differential settlement between two footings.

3.1 *Friction pile*

Consider a friction pile embedded in clay with a total length L subjected to a vertical dead load DL. The pile has a diameter B = 1 m. The spatially variable undrained shear strength (s_u) of the clay is modeled as a stationary normal random with mean μ = 100 kN/m^2 and standard deviation σ = 30 kN/m^2:

$$s_u(z) = \mu + \sigma \cdot \varepsilon(z) \tag{7}$$

where $\varepsilon(z)$ is the zero-mean spatial variability with standard deviation σ. Suppose that the horizontal SOF is larger than the diameter B, so the horizontal variability can be ignored. The unit side resistance $f_s(z)$ is expressed as

$$f_s(z) = \alpha \cdot s_u(z) \tag{8}$$

where α = 0.5 is adopted. For a friction pile, the end bearing is negligible, so the total resistance Q is equal to the total shaft resistance:

$$Q_u = \pi B \int_0^L f_s(z)\,dz = \alpha\pi BL\left(\mu + \sigma \cdot \bar{\varepsilon}_L\right) \tag{9}$$

where $\bar{\varepsilon}_L$ is the spatial averaged $\varepsilon(z)$ over the depth range L. The performance function G can be defined as

$$G = Q_u - DL = \alpha\pi BL\left(\mu + \sigma \cdot \bar{\varepsilon}_L\right) - DL \tag{10}$$

The pile fails if G < 0. The spatial average $\bar{\varepsilon}_L$ can be computed as the arithmetic average of the $\varepsilon(z)$ values simulated over the dense grid points (z_1, z_2, …, z_n).

3.2 *One-dimensional seepage*

Consider a soil mass with spatially variable hydraulic conductivity (k). The total depth of the soil mass is D, and it is underlain by an impermeable layer. Suppose the vertical variability in k is much more significant than the horizontal one, so k is modeled as a function of depth z only. The vertically variable k(z) is modeled as a stationary lognormal random with mean $\mu = 1\times10^{-4}$ cm/s and COV = 100%:

$$k(z) = \exp\left[\lambda + \xi \cdot \varepsilon(z)\right] \tag{11}$$

where $\lambda = \ln[\mu/(1+COV^2)^{0.5}]$ and $\xi = [\ln(1+COV^2)]^{0.5}$ are the mean value and standard deviation of ln[k(z)], respectively; $\varepsilon(z)$ is modeled as a zero-mean stationary normal random field with standard deviation = 1. The main focus of the one-dimensional seepage example is on the equivalent vertical k of the soil mass, denoted by k_V:

$$\frac{1}{k_V} = \frac{1}{D}\sum_{j=1}^{n}\frac{dz}{k(z_j)} = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{k(z_j)} \tag{12}$$

which involves the harmonic average. Suppose that the failure is defined as the exceedance of k_V over a prescribed critical value k_{cr}:

$$G = k_{cr} - k_V \tag{13}$$

where k_{cr} is taken to be 1×10^{-4} cm/s. Each sample path of $\varepsilon(z)$ produces a realization of G.

3.3 *Infinite slope*

Consider an infinite slope with an inclination angle α and a vertical depth D (see Figure 6). Suppose the friction angle ϕ' of the cohensionless soil is spatially variable in a way that the variability is only in the vertical direction. It is homogeneous in the direction parallel to the slope. The ground water is deep, so it has no effect on the stability. The spatially variable $\phi'(z)$ is modeled as a stationary normal random with mean $\mu = 30^o$ and standard deviation $\sigma = 3^o$:

$$\phi'(z) = \mu + \varepsilon(z) \tag{14}$$

where $\varepsilon(z)$ is the zero-mean spatial variability with standard deviation σ. A potential slip plane with depth z fails if $\phi'(z) < \alpha$, where $\phi'(z)$ denotes the friction angle at depth z. The infinite slope fails if any one potential slip plane fails. Therefore, the performance function G for the infinite slope can be written as

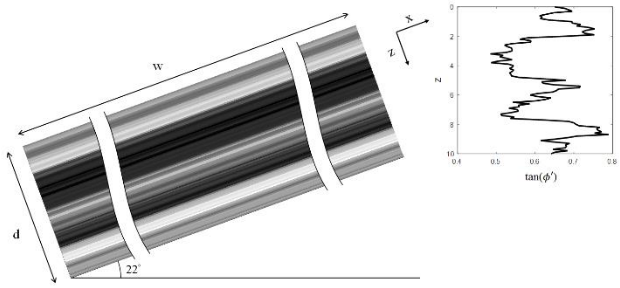

Figure 6. Random field sample paths of tan[ϕ′(z)] for the infinite slope.

$$G = \min_{z \in [0,10]} \phi'(z) - \alpha = \mu + \varepsilon_{min} - \alpha \tag{15}$$

where ε_{min} denotes the minimum value of the ε(z) realization. The infinite slope fails if $G < 0$.

3.4 *Differential settlement between two footings*

Let us consider two square rigid footings on undrained clay, with width B = 1 m and separation distance = L shown in Figure 7. The depth of a hard stratum is assumed to be very deep. Each footing is subjected to a vertical load of Q, producing a bearing pressure of $q = Q/B^2$. The horizontal direction is denoted by x. The Young's modulus E for the undrained clay underlying the footing is denoted by E(x). It is assumed that E(x) has taken into account the (weighted) average of E over 4B to 5B beneath the footing. The spatially variable E(x) is modeled as a stationary lognormal random field with mean μ = 20 MN/m² and COV = 50%. This suggests that ln[E(x)] is a stationary normal random field with variance $\xi^2 = \ln(1+COV^2)$ mean $\lambda = \ln(\mu) - 0.5\times\xi^2$:

$$\ln\left[E(x)\right] = \lambda + \xi \cdot \varepsilon(x) \tag{16}$$

where ε(x) is the zero-mean spatial variability with standard deviation = 1. The settlement S_e at the center of each rigid footing can be estimated as (Janbu et al. 1956; Christian and Carrier 1978)

$$S_e(x) = 0.93 \times \frac{q \cdot B}{E(x)} \times A_1 A_2 \tag{17}$$

where (x_1, x_2) are the x coordinates for the two footings. The performance function G can be written as

$$G = \beta - \frac{1}{500} \tag{18}$$

where 1/500 is the maximum acceptable angular distortion (European Committee for Standardization 1994).

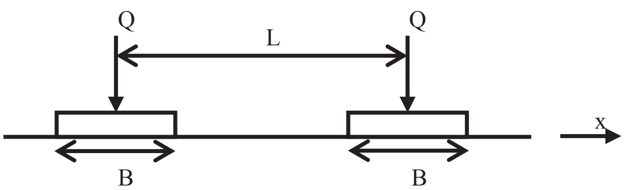

Figure 7. Two footings.

4 ANALYSIS RESULTS

For all four examples, Monte Carlo simulation (MCS) is adopted to estimate the failure probability $p_f = P(G < 0)$. Ten thousands (N = 10,000) realizations of ε(z) are simulated. Each ε(z) sample produces a realization of G. Hence, N = 10,000 realizations of G are simulated, and the failure probability p_f can be estimated:

$$p_f \approx \frac{1}{N}\sum_{i=1}^{N} I\left[G^i < 0\right] \tag{19}$$

where I[.] is the indicator function; G^i is the i-th realization of G. For cases with small p_f, N increases to 100,000.

4.1 *Impact of smoothness on performance*

To investigate the impact of smoothness on p_f, the WM model with SOF = δ and smoothness parameter = ν is adopted as the ACF model for ε(z).

4.1.1 *Friction pile*

For the friction pile example, Figure 8 shows how p_f changes with ν for several chosen δ/L values. It is clear that p_f does not depend much on ν, but it depends on δ/L. The effect of the smoothness of ε(z) realizations is minimal for the friction pile example because it is completely governed by arithmetic spatial averaging.

4.1.2 *One-dimensional seepage*

Figure 9 shows how p_f changes with ν for several chosen δ/D values. It is interesting to see that ν does not significantly affect p_f, except when ν is small. Note that both the friction pile and one-dimensional seepage examples are completely governed by spatial averaging. The only difference is that the friction pile example is governed by the arithmetic average, whereas the one-dimensional seepage example is governed by the harmonic average.

4.1.3 *Infinite slope*

Figure 10 shows how p_f changes with ν for several chosen δ/D values. It is clear that ν now has a significant effect on p_f. In particular, p_f produced by SExp (ν = 0.5) is significantly larger than those produced by SMK (ν = 1.5) and QExp (ν = ∞) even if they share the same δ/D. This observation stands in strong contrast to that obtained in the friction pile example. For the friction pile example, arithmetic spatial averaging completely governs, hence δ is the *only* parameter that matters. On the contrary, for the infinite slope example, there is no spatial averaging, and weakest-path seeking mechanism completely governs. The local jitters produced by a small ν lead to lots of apparent weak layers that affect the stability of the infinite slope. As a result, ν has a significant effect on p_f for the infinite slope problem.

4.1.4 *Differential settlement between two footings*

Figure 11 shows how p_f changes with ν for several chosen δ/L values. It is clear that ν has a significant effect on p_f when δ/L is relatively large. When δ/L is large, the separation distance between the footings is only a fraction of δ. In this case, the differential settlement between the two footings is governed by the short range auto-correlation, which is affected significantly by ν. This explains why ν has a significant effect on p_f when δ/L is relatively large.

4.2 *Impact of hole effect on performance*

To investigate the impact of hole effect on p_f, the CosWM model with various b/s is adopted as the ACF model for ε(z). The smoothness ν in the CosWM model is fixed at 0.5.

4.2.1 *Friction pile*

For the friction pile example, Figure 12 shows how p_f changes with b/s for several chosen δ/L values. It is clear that p_f does not depend much on b/s. The impact of the hole effect of ε(z) realizations is minimal for the friction pile example.

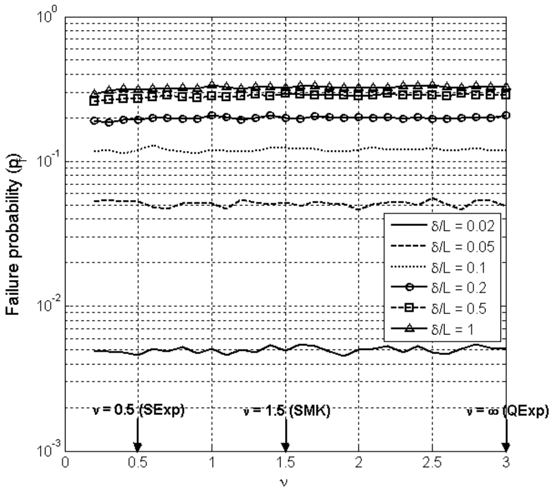

Figure 8. Variation of p_f with respect to ν for several chosen δ/L values (friction pile with L = 10 m & DL = 1400 kN).

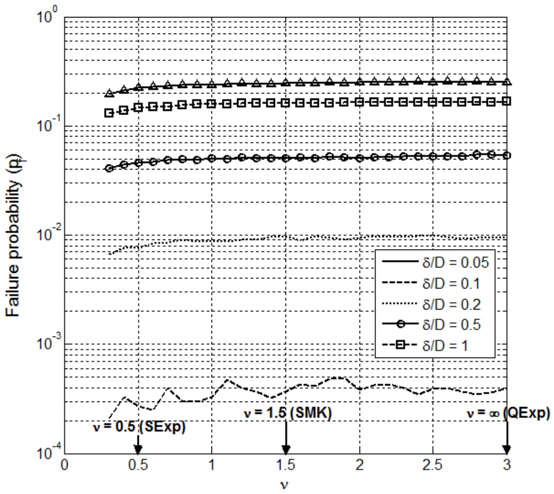

Figure 9. Variation of p_f with respect to ν for several chosen δ/D values (one-dimensional seepage with D = 10 m).

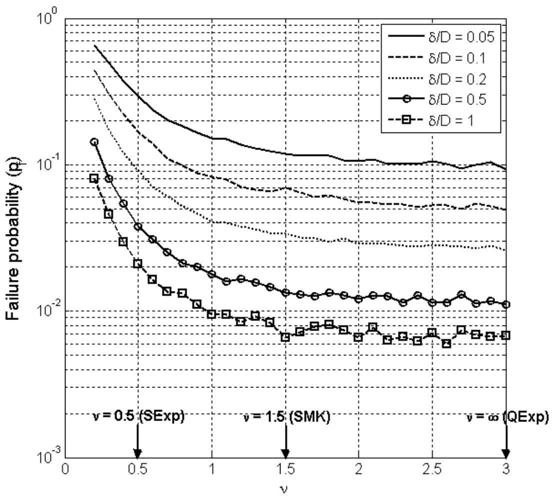

Figure 10. Variation of p_f with respect to ν for several chosen δ/D values (infinite slope with α = 18° & D = 10 m).

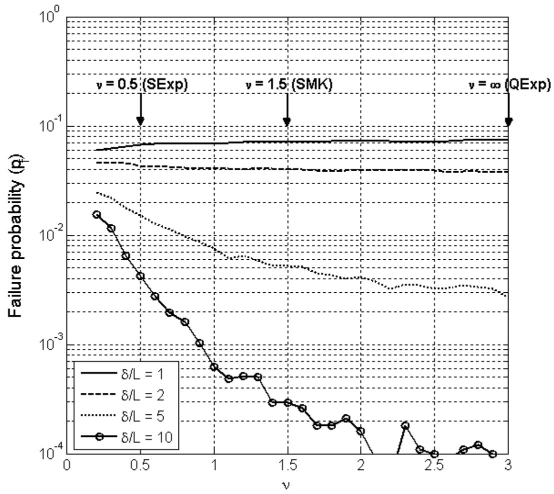

Figure 11. Variation of p_f with respect to ν for several chosen δ/L values (differential settlement with L = 5 m & Q = 200 kN).

4.2.2 *One-dimensional seepage*

Figure 13 shows how p_f changes with b/s for several chosen δ/D values: b/s now significantly affects p_f. Note that the one-dimensional seepage example is governed by harmonic spatial averaging, not by arithmetic spatial average.

4.2.3 *Infinite slop*

Figure 14 shows how p_f changes with b/s for several chosen δ/D values: b/s significantly affects p_f.

4.2.4 *Differential settlement between two footings*

Figure 15 shows how p_f changes with b/s for several chosen δ/L values: b/s significantly affects p_f.

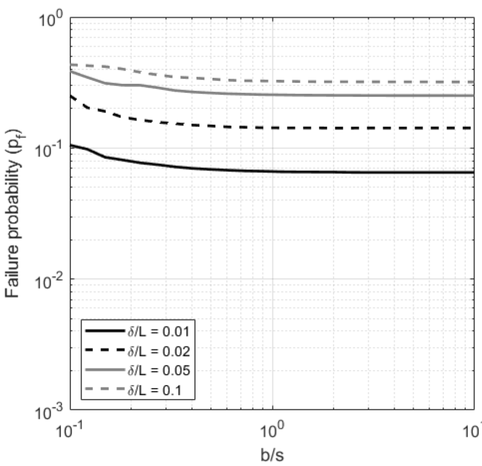

Figure 12. Variation of p_f with respect to b/s for several chosen δ/L values (friction pile with L = 10 m & DL = 1500 kN).

5 CONCLUSIONS AND DISCUSSIONS

The scale of fluctuation (SOF) has been regarded as the only essential parameter for soil spatial variability. Numerous 1-parameter auto-correlation function (ACF) models are available in the literature. Recently, Ching and Phoon (2019) showed that the sample-path smoothness (governed by ν) has significant impact on the performance of a geotechnical structure and that SOF and smoothness can be simultaneously modeled by the Whittle-Matérn (WM) model. Chang et al. (2021) further showed that the pseudo-periodicity (hole effect) (governed by b/s) also has significant impact on the performance of a geotechnical structure and that SOF, smoothness, and hole effect can be simultaneously modeled by the cosine Whittle-Matérn (CosWM) model. The only exception is the friction pile example whose performance is completely governed by arithmetic spatial average: both ν and b/s has insignificant impact on the performance of a geotechnical structure.

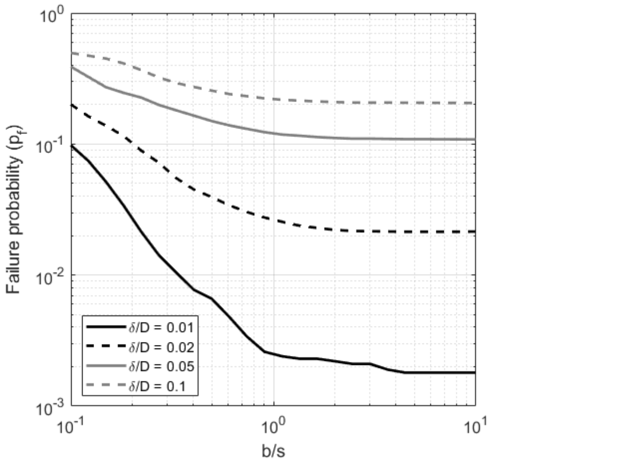

Figure 13. Variation of p_f with respect to b/s for several chosen δ/D values (one-dimensional seepage with D = 10 m).

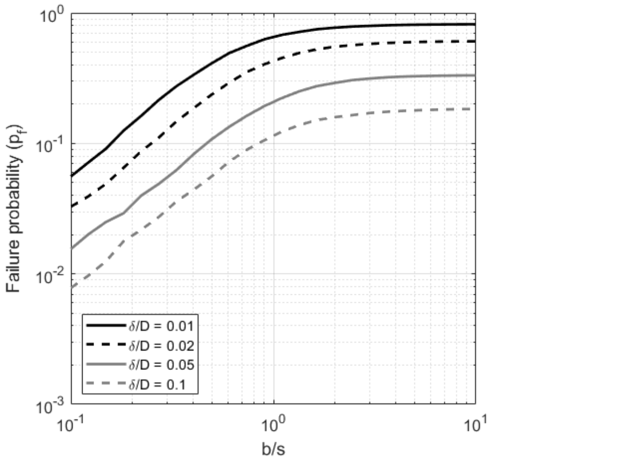

Figure 14. Variation of p_f with respect to b/s for several chosen δ/D values (infinite slope with α = 22º & D = 10 m).

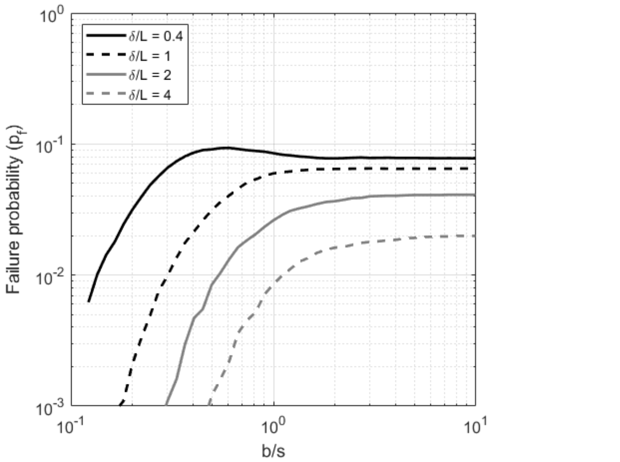

Figure 15. Variation of p_f with respect to b/s for several chosen δ/L values (differential settlement with L = 5 m & Q = 200 kN).

Vanmarcke (1977) stated that the probability distribution of "point" soil properties may be less important, whereas the probability distribution of the "spatial averaged" soil properties is more important. This statement has a profound influence on research works conducted later in the area of geotechnical reliability. Because the (arithmetic) spatial averaging effect (or the variance reduction effect) can be well captured by a single parameter, the scale of fluctuation δ, this parameter δ has been widely accepted to be a key parameter for spatial correlation. In fact, δ probably has become the *only* parameter that people concern about the spatial correlation. Indeed, δ should be the *only* concern if the problem is *completely* governed by arithmetic spatial averaging, e.g., the friction pile.

However, the observation obtained in the current paper rejects the claim that δ should be the *only* concern. There are other important mechanisms, e.g., seeking of the weakest path, that are apart from spatial averaging. It is evident that δ *alone* may not capture these important mechanisms. We need more parameters such as the smooth parameter ν in the WM model and the hole-effect parameter b/s in the CosWM model to capture these mechanisms. However, in the literature, extra parameters have never been addressed. When characterizing spatial correlation, nearly all past works focus on the identification of δ. There is no past work discussing how to characterize the "smoothness" or "hole effect" of the spatial variability. Nevertheless, the observation obtained in the current paper indicates that these characteristics (smoothness and hole effect) may be important for problems not completely governed by arithmetic spatial averaging.

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