

Formulation of a constitutive soil model based on threshold strain and energy approach

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ABSTRACT: The report examines the fundamental features of dispersed media, and, in particular, soils, which do not allow the theory of elasticity and plasticity of a continuous medium to be used to describe their behavior. It is shown that elasticity as the ability to restore deformations in dispersed soils has a more complex nature than in solid bodies - elastic behavior is due to structural mechanism of soil particles. Using the example of identifying and investigating points of strain instability, it is shown that the flow conditions for dispersed media should be formulated in strains, not in stresses. These threshold strains are characteristic for each soil type. The analysis of various strength factors influence on the failure surface is given. It is shown, that inner friction, cohesion and particles engagement should be considered separately. Based on these prerequisites, a constitutive soil model is formulated, in which each process occurring in the soil skeleton under loading is described by a separate physical equation using non-vector energy parameters, and the specified threshold strain parameters allow each of the processes to be activated. The report presents a comparison of the simulation results using the described model with real experimental data. It is shown that the model has a satisfactory predictive ability and can be easily supplemented with extra options, such as creep or dynamic behavior.

KEYWORDS: Constitutive model, energy approach, soil mechanics, threshold strain, disperse soil modelling.

1 INTRODUCTION

When discussing the significance of various branches of solid mechanics, N.I. Bezukhov identifies rheology as a science that establishes general laws of any substance strain formation and development over time (Bezukhov, 1961). In this formulation, a substance is understood to be anything: solid or liquid, elastic or plastic, viscous, etc. The author emphasizes that the task of rheology is to answer the question: what are the strains and stresses at a given point in a given body at a certain point in time with known parameters of external influence and its known history in the past.

In this formulation, the answer to this question is a very difficult task that has not been solved comprehensively for decades. In this regard, certain areas of deformable body mechanics apply various assumptions and hypotheses that simplify the solution of applied problems for certain types of bodies: solid, liquid, gaseous, discrete, composite. At the same time, each such simplification has application limitations due to the structural features of a particular body.

Dispersed soils are multicomponent materials and cannot be classified as continuous solids, primarily because the soil components interact with each other, and the patterns of this effect were studied by specialists in the field of engineering geology in the 20th century, after the formation of the classical theory of soil mechanics. And if the earliest solutions of soil mechanics obtained by Coulomb (1776), Rankine (1857), and a group of specialists in the field of elasticity theory (Boussinesq, 1885; Love, 1927) fully satisfied the needs of solving applied problems, but subsequently the degree of discrepancy with the accumulated experimental data and the results of observations of the soil masses behavior required the use of an increasing number of refinements. Thus, the most important hypotheses of soil mechanics arose: the concept of elementary volume (Yasinsky, 1898), the principle of effective stresses (Terzaghi, 1925), the theory of linear deformation (Gersevanov, 1933), the unassociated law of plastic flow (Radenkovic, 1961), the theory of critical state (Roscoe, Schofield and Wroth, 1958). Each of them tries to eliminate the discrepancy between the theoretical solutions of soil mechanics (mainly based on solid and liquid mechanics) and the observed behavior of dispersed soil.

This is the paradox of modern soil mechanics: its classical solutions are based on theories, the initial hypotheses of which were built for fundamentally different substances that cannot be considered as continuous, but at the same time do not require consideration of discreteness, since their elementary particles interact with each other quite unambiguously.

If we consider each of the initial assumptions and hypotheses used in the mechanics of deformable solids, taking into account modern ideas about the internal structure of dispersed soils and the interaction of their components with each other, we can identify a number of contradictions.

But if we take a step back and begin to consider the simplest processes occurring in dispersed soils, it turns out that they have been sufficiently studied and described by experts in the field of soil science, and it remains only to find a way to move from the micro-level task to the description of macro-deformations. Similar examples are known, for example, the Sekiguchi-Ohta creep model is based on this principle (Ohta and Hata, 1976; Sekiguchi, 1977).

The principle of energy conservation can be a convenient tool. Indeed, it is impossible to determine the number of contacts per unit volume of dispersed soil, as well as the energy spent on deforming or destroying such a contact. But at the macro level, energy per unit volume is easily measurable, and at the same time, the resulting parameter retains its physical meaning.

The pioneers of energy method application were specialists in the field of metal plasticity theory. The hypothesis of the constancy of internal energy during plastic flow made it possible to formulate the von Mises strength theory. Most theories of plastic flow (first of all, Prandtl–Reissner, Saint-Venant–Mises, Hill) take into account the work of plastic strain or its power. For example, an analysis of the strain work makes it possible to describe hardening during plastic flow based on the Prandtl diagram.

As applied to soil mechanics, this approach was first adapted by Drucker & Prager (1952), who proposed the law of unassociated flow and a specialized flow condition. There are also well-known works by a group of researchers in the field of critical state theory (Roscoe, Schofield and Wroth, 1958), the

main provisions of which were derived from energy assumptions. A group of researchers considered a method for determining the plastic flow point based on the work expended (Crooks and Graham, 1976). Currently, the energy approach is used in the study of soil properties under dynamic loading, which makes it possible to assess the degree of energy absorption by comparing the work of elastic strain and the energy dissipated during one loading cycle (the hysteresis loop area) (Ishihara, 2006). Based on the work of strain, it is proposed to evaluate the so-called "deformation instability" - the levels of strain at which structural changes occur in dispersed soils (Usov and Voznesensky, 2016; Voznesensky, 2018). Energy methods are also used to determine the loading history (recompaction parameters), among them the most widely used methods are Becker (Becker et al., 1987) and Wang & Frost (Wang and Frost, 2004).

2 GEOMSU SOIL MODEL

2.1 Model description

The main difficulty in describing the mechanical behavior of dispersed soils is the parallel flow of various processes, reacting to stress or strain in their own way. In addition, these processes can be activated and attenuated when stress or strain thresholds are reached. As a result, the experimental dependence is a complex function, the phenomenological description of which does not allow us to establish the actual nature of the processes under mechanical action on the soil.

On the other hand, a separate examination of these processes shows that each of them is based on fairly simple and well-studied patterns. For example, volumetric strain under isotropic compression is described by a logarithmic function; shear resistance due to friction linearly depends on the level of acting stresses in a fairly wide range, etc.

As a result, a combination of simple functions can be used to describe the mechanical behavior under complex loading, similar to how a complex periodic oscillation is described by a Fourier series.

In addition to determining the type and parameters of each function describing a particular process, it is also necessary to determine the stress/strain thresholds at which this process will be activated or deactivated. Constitutive models based on the solid body plasticity theory use the concept of yield functions that separate the elastic strain region from the plastic one. But for dispersed soils, this approach is not justified from a physical point of view, since the dispersed medium constantly changes the internal structure, unlike a solid body. As a result, it becomes necessary to introduce complex patterns of nonlinear elasticity, additional yield functions with kinematic hardening/softening.

In the proposed model, the threshold values of stress and strain are used instead, determined based on the analysis of the soil microstructure.

It is most convenient to implement the model using the energy approach. All processes are divided into two large groups – volumetric strain and shearing. Each of the processes is described by an energy function that sets the specific strain work required to realize the strain at a given stress level (or vice versa). As the straining occurs at each step of the calculation, it is possible to determine the required specific strain work by summing up the parameters of each of the processes. Further, based on the obtained energy parameters, it is possible to determine the values of the volumetric compression and shear moduli.

These values can be used to calculate the strain increments, after which new values of the energy parameters will be determined in the next calculation step. It should be noted that

the pseudoelastic moduli in this case are only intermediate parameters that simplify the transition from the physical equations of the model to the implementation in the FEM.

Thus, the model is realized as non-linearly elastic, but the type of function is constantly changing depending on the level of strain and stress.

It is noteworthy that the model makes it possible to realize hardening and softening only due to the accumulation of strain, without using the mathematical apparatus of the theory of plasticity. In addition, the model does not use the laws of plastic flow, which greatly simplifies the mathematical implementation and eliminates possible errors in determining the direction of the plastic strain increment vector. Similarly, the model lacks the concept of a failure function, since plastic flow is still an energy-consuming process, even if the resistance is low. If it is necessary to model brittle fracture, it is enough to introduce another threshold strain, after which the energy parameters are reset to zero. In this case, the model will use the Marriot strength theory approach.

The energy parameters of the model, unlike the more familiar constitutive models based on the theory of elasticity and plasticity of a continuous medium, have a clear physical meaning and can be easily related to the composition and structure of soils.

An important feature of the model is also its versatility and the possibility of fairly simple modification: as the concepts of each of the selected processes are clarified, new functions can be integrated into the model without significantly changing its architecture.

2.2 Volumetric strain

In case of volumetric (isotropic) compression, the work of an external force is spent on volumetric compression of the skeleton, deformation of solid particles at the contacts, and compression of the pore fluid (in the case of a two-phase medium). If necessary, the hardness of the gas contained in the pores can also be taken into account.

The laws of volumetric compression are set based on stresses, which are the determining factor in terms of strain magnitude and prevailing processes. For example, elastic strain of solid particles at the contacts is possible only until the contact stresses exceed the strength limit of the particle material, at which crushing begins. This threshold stress will be determined by both the mineral composition of the particles and their relative size: in more finely dispersed soils, the stress concentration ratio is significantly lower. To a first approximation, the dependence of strain on stress can be given by a linear function, that is, the specific work of elastic forces is a constant - W_K^e . Similarly, the elastic strain of pore fluid and gas can be described by its parameters W_K^{ew} and W_K^{eg} , and to a first approximation, these will be constants, since both gas and fluid are perfectly elastic.

Skeletal compaction is traditionally described by a logarithmic function, which has been repeatedly confirmed by experiments with various types of soils (Terzaghi, 1925; Casagrande, 1936). The specific work of strain in that case increases linearly with the stress increase. For non-cohesive soils, this will be a continuous smooth function, but for clay soils, two threshold stresses should be introduced: the cohesive pressure p_c , which characterizes the strength of brittle contacts, or the historical pressure p_p , which characterizes the level of previously operating sealing loads (1).

$$W_K^p = k_K^p(p - p_c) \quad (1)$$

At each calculation step, the bulk modulus K is used to determine the increment of volumetric strain in a particular

element, determined based on the current total energy parameter in expression (2).

$$K = \frac{\Delta p^2}{2(W_K^e + W_K^{ew} + W_K^{eg} + W_K^p)} \quad (2)$$

2.3 Shear strain

During shear strain, the work of an external force is spent on shear strain of the skeleton. The shape changes of the particles themselves can be considered insignificant. If necessary, the resistance to viscous flow of the pore fluid can additionally be taken into account.

The laws of shear stiffness are set based on strain, and strain also act as thresholds, since the lifetime of contacts of various nature is determined by mutual displacement.

The resistance to elastic shearing is determined by the shear stiffness of the contacts, but it depends on the level of average stresses, which is described by a power law (Puebla and Byrne, 1997).

As a result, the energy parameter can be written by the function (3).

$$W_G^e = k_G^e p^{ref} \left(\frac{p}{p^{ref}} \right)^{ne} \quad (3)$$

Elastic shear strain is realized until the first threshold strain γ_I is reached, after which the rigid contacts collapse and the skeleton stores elastic strain energy very limited, or does not store it at all, since viscous contacts are unable to do so.

The resistance to plastic shearing is caused by losses due to viscous and dry friction during mutual displacement of particles. The energy parameter of the resistance to plastic shearing is described by a linear function, the parameters of which are, respectively, constant and variable (stress-dependent) components of the resistance forces (4).

$$W_G^p = k_G^p p + b_G^p \quad (4)$$

Plastic shear is activated when the first threshold strain is reached.

The shear modulus is expressed similarly to elastic (5), but is based on the strain level.

$$G = \frac{2(W_G^e + W_G^p)}{\Delta \gamma^2} \quad (5)$$

2.4 Dilatancy

When the second threshold strain γ_{II} is reached, plastic volume strain associated with displacement due to dilatancy/contraction begins to develop. The magnitude of this strain is determined using an energy parameter expressed as a fraction of the work of plastic shear strain (6).

$$W_K^\psi = \sin \psi_{mob} W_G^p \quad (6)$$

The value of the mobilized dilatation angle, which determines the rate of volume growth during plastic shear, can be assumed according to any known theories (Rowe, 1962; Bolton, 1986). If we use a theory that takes into account the gradual stabilization of volume when reaching critical state, then the model will reproduce the softening.

The resulting energy parameter is used to calculate an additional increment of volumetric strain $-\Delta \varepsilon_v^\psi$, which may be positive or negative.

In general, many other effects can be realized in a similar way, for example, a viscous formulation of the solution – if we use functions of energy parameters with a time factor.

2.5 Further calculation

After determining the values of the shear and bulk moduli for a specific calculation step, the current stiffness matrix is formed based on them. Mean stress tensor (average stress increment Δp) determines the increment of volumetric strain, the stress deviator determines the increment of shear strain. In the simplest formulation, the invariant q can be used as a measure of deviatoric load, however, the dependence of the model behavior on the type of stress state is lost. As a further development, it is necessary to switch to the use of invariant values of threshold strain, this issue is the topic of further research.

Considering that for volumetric strain the energy coefficients are determined based on the stress state, and for shear strain, the level of strain achieved, it is most convenient to use the total work of external and internal forces as a convergence criterion.

This method of implementing the model has a disadvantage: the use of elastic modulus does not allow us to simulate softening – for example, with triaxial compression, the tangential shear modulus may become negative. To eliminate this disadvantage, an additional parameter can be entered into expression (5), which will take a negative value when passing through the peak shear resistance.

3 GEOMSU MODEL PARAMETERS

A special feature of the model is the dependence of its parameters on the physical properties, composition and structure of the soil. This allows one to use reference values instead of defining a number of parameters.

The determination of model parameters can be based on a combination of any tests. Preference should be given to the simplest methods, which combine the least number of parallel processes. In the future, the obtained patterns can be used in the interpretation of more complex methods. For example, the law of volumetric stiffness change is determined from isotropic compression tests. Its comparison with the results of oedometric compression (OED) makes it possible to determine the lateral pressure coefficient. Next, we can proceed to the interpretation of the results of the consolidated-drained triaxial compression (TX-CD), from which we can exclude the loss of energy on the volumetric strain component on the basis of the already known law. Tests of consolidated-undrained triaxial compression (TX-CU) and direct simple shear (DSS) make it possible to obtain shear stiffness in its pure form. Triaxial compression tests with constant average stress (TX with TC trajectory) allow direct measurement of volume change due to dilatation. An example of the interpretation of the results of these tests was considered by the authors earlier (Mirnyy, 2024).

Methods for obtaining the model parameters are presented in Table 1.

Table 1. Methods for obtaining the model parameters.

Process	Method
Elastic volumetric strain of the skeleton	Isotropic compression OED, DSS (CD), TX (CU, CD)
Plastic volumetric strain of the skeleton	Isotropic compression OED, DS (CD), TX (CD)
Plastic volumetric strain – dilatancy/contraction	OED, DSS (CD), TX (CD – CTC, TC)
Elastic shearing	OED, DSS (CD, CU), TX (CU, CD)
Plastic shearing	OED, DSS (CD, CU), TX (CU, CD)
Pore fluid pressure change	TX (CU)

The list of model parameters may seem quite significant, but some of them are determined by reference values based on data on the granulometric and mineral composition of the soil. The model parameters are given in Table 2.

Table 2. GeoMSU model parameters.

Parameter	Physical meaning
$W_K^e, W_K^{ew}, W_K^{eg}$	Specific work of elastic volumetric strain for soil particles, pore fluid and gas, respectfully
k_K^p	Plastic volumetric strain of soil skeleton parameter
p_c, p_p	Threshold mean pressures – structural strength and historical pressure, respectfully
k_G^e, n_e	Elastic shear power law parameters
p^{ref}	Reference mean pressure for elastic shear power law
k_G^p, b_G^p	Plastic shear linear law parameters (“strength” parameters)
γ_I	Threshold strain of rigid contacts destruction
γ_{II}	Threshold strain of dilatancy activation
γ_{III}	Threshold strain of critical state
ψ_{max}	Maximum value of dilatancy angle

4 IMPLEMENTATION AND VALIDATION

The proposed model is intended for implementation in numerical software systems implementing the finite element method. The applied energy approach makes it possible to determine the increments of the stress-strain state components at each step of the calculation, after which new values of the energy parameters are determined. At the same time, since the physical equations of the model ultimately boil down to shear and volumetric compression moduli, the model can be implemented as an add-on to the Linear Elastic model.

In the current version, the model uses increments of stress invariants p and q to calculate, respectively, increments of volume and shear strain, as a result of which it is isotropic, however, with some modifications it may take into account the type of stress state.

Currently, the model is in process of implementation in the SiO 2D software package.

5 CONCLUSIONS

The paper describes a fundamentally new constitutive soil model GeoMSU based on an energetic description of elementary mechanical processes. Due to this, a hypo-elastic model has been obtained that does not use the assumptions of the theories of elasticity, plasticity and strength of a solid body, as a result of which it better corresponds to the physical meaning of the processes occurring in dispersed soils.

Each of the elementary mechanical processes is described by its own law, which relates specific work to the stress-strain state. Next, the obtained energy parameters related to volumetric and shear strain are summarized, on the basis of which the shear and volumetric compression moduli corresponding to a specific calculation step are determined. The obtained moduli are used to determine the increments of the volumetric and shear component of strain.

The advantage of the model is its mathematical simplicity and the possibility of easy development without changing the architecture: the model does not use methods of plasticity theory, and each of the laws used can be modified independently.

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