Critical State for Unsaturated Soils and Steady State of Thermodynamic Process

Etat critique de s sols non saturés et état stable thermodynamique

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ABSTRACT: Critical state is an important concept in modern soil mechanics. It was developed by Roscoe, Schofield and Wroth (1958) and Schofield and Wroth (1968) for saturated soils. Currently, the critical state for unsaturated soils has not been clearly defined and the necessary conditions and constraints to attain to critical state for unsaturated soils have not been definitely developed. More studies on critical state of unsaturated soils are required. Based on theory of thermodynamics, the conditions and constraints to attain to steady state of thermodynamic process for unsaturated soils are proposed in this paper. This paper points out that the steady state of a deformation process of unsaturated soils is the ultimate state of the deformation process and it is, by concept, the critical state of unsaturated soils. The conditions and constraints for critical state of unsaturated soils presented in the paper are more general and complete, and are based on more rigorous thermodynamic theory instead of only being based on the laboratory test results of particular unsaturated soil samples.

RÉSUMÉ :

Le concept d'état critique est un concept important en mécanique des sols. Il a été développé pour les sols saturés par Roscoe et al ; (1958), et Schofield et Worth (1968). En revanche, ce concept n'est pas encore clairement décrit pour les sols non saturés. Il est donc nécessaire de mettre en place des études plus approfondies dans ce sens. Dans cet article, en se basant sur la théorie de la thermodynamique, les conditions et les limites pour atteindre l'état critique dans les sols non saturés sont exposées. Ici on considère que l'état permanent de la déformation dans les sols non saturés est l'état ultime du processus de déformation, il peut donc être considéré comme étant l'état critique. Les conditions et les limites pour l'état critique des sols non saturés présenté dans cet article sont plus générales et complète , car elles sont basées sur la théorie de la thermodynamique au lieu d'être issues des essais de laboratoire sur des échantillons de sols non saturés.

KEYWORDS: unsaturated soils; critical state; steady state of thermodynamic process

1 INTRODUCTION

Critical state is an important concept in modern soil mechanics. It acts as the cornerstone of critical state soil mechanics since it gives the final point of a soil deformation process, which is necessary for establishing soil constitutive models. Without the concept of critical state, it would not be able to know which direction a soil deformation process evolves, and where and when the deformation process ends. If the end point of a deformation process is not defined, significant deviations might occur in constructing soil constitutive models. Once the initial state and the end point (the critical state) of a deformation process are defined, relatively more accurate models can be established by taking advantage of interpolations between the initial state and the critical state, rather than only using one-sided extrapolation from the initial state (without critical state). Therefore critical state is a fundamental concept that most constitutive models for soils are based on.

The critical-state concept was proposed by Roscoe, Schofield and Wroth (1958), and Schofield and Wroth (1968), and is defined as the end or ultimate state of a deformation process for saturated soils, in which soil keeps shear deforming with constant stress and volume in large strains. In three-dimensional axisymmetric space (i.e. in triaxial stress state), when a soil deformation process evolves to a critical state, the stress and strain should satisfy the following necessary conditions and constraints:

$$\dot{p}' = 0; \quad \dot{q} = 0; \quad \dot{\nu} = 0; \quad \dot{\varepsilon}_s \neq 0 \tag{1}$$

The critical state can be expressed in $q - p'$ space and $e - \ln p'$ space by the following two equations, respectively:

$$(q / p')_c = M; \qquad \nu = \nu_c = \Gamma + \lambda \ln(p') \tag{2}$$

where p' is the mean effective stress of saturated soils, q is the deviator stress, ν is the specific volume, ε_s is the deviator strain, M is the slope of the critical state line in $q - p'$ space, Γ and λ are the intercept (at $p' = 1\text{kPa}$) and the slope of the critical state line, respectively, in $e - \ln p'$ space.

Previous and current studies of critical state have been conducted mainly on saturated soils. Although the constitutive models for unsaturated soils within the framework of critical state soil mechanics have been developed by Alonso, Gens and Josa (1990), Gens and Alonso (1992), Wheeler and Sivakumar (1995), Wheeler, Sharma and Buisson (2003), Sheng, Sloan and Gens (2004), Li (2007), Sun, Sheng, Cui and Sloan (2007) to mention only a few from the existing literature, the definition and the necessary conditions and constraints for critical state of unsaturated soils have not been provided explicitly and clearly. So it is affected to determine the value of critical state for unsaturated soils correctly and accurately.

The engineering properties and behaviors of unsaturated soils are much more complicated than those of saturated soils because of the existence of air in unsaturated soils. Comparing with saturated soils, the stress states and the phase volume changes of unsaturated soils are more complicated. The effective stress for unsaturated soils is not only related to the pore water pressure, but also related to the pore air pressure and even the degree of saturation (Zhao, Liu and Gao 2010). Therefore the critical state of unsaturated soils could not be accurately defined by Eq. (1). For this reason it is important to determine which conditions and constraints should be satisfied when the unsaturated soils attain to critical state, and which additional conditions and constraints should be added to Eq. (1) to completely and accurately describe critical state for unsaturated soils. It is impossible for the theory of critical state unsaturated soil mechanics and the corresponding constitutive models to be further developed if these problems are not fully solved.

Until now most researches on critical state of unsaturated soils have been based on laboratory triaxial tests, such as Toll (1990), Wheeler and Sivakumar (1995), Maatouk et al.(1995), Adams and Wulfsohn (1997), Rampino et al. (1998), Wang et al. (2002), Kayadelen et al. (2007), etc. By means of laboratory triaxial tests, these researches have been focused on whether the critical state for unsaturated soils exists, and if the critical state for unsaturated soils exists, then what necessary conditions and constraints should be satisfied when unsaturated soils attain to critical state. Many researchers, such as Wheeler and Sivakumar (1995), Maatouk et al. (1995), Adams and Wulfsohn (1997), etc., selected the mean net stress, the deviator stress, the suction and the specific volume as the state variables to describe critical state of unsaturated soils. In addition to the above-mentioned state variables, other researchers such as Wang et al. (2002), Kayadelen et al. (2007), etc., suggested an additional variable, specific water volume v_w or degree of saturation S_r, to define critical state of unsaturated soils. Some laboratory test results, such as from Wheeler and Sivakumar (1995), showed that v_w did not attain to the steady value ultimately, possibly due to the limitations of equipment and experimental conditions. Whereas Rampino et al. (1998) reported that v_w could keep constant at the end of deformation test for unsaturated soils. Wang et al. (2002) pointed out that it may not be reliable to use v_w as an indicator of critical state for unsaturated soils, and more data from tests and more researches are required before a conclusion can be made on this matter.

Thermodynamics is a universally applicable theory. The deformation process for unsaturated soils must follow thermodynamic laws. By using thermodynamic theory, the deformation process and behavior of unsaturated soils can be investigated with more common situations and on more general perspectives in order to reveal the complex soil behaviors and properties. The objective of this paper is to establish the necessary conditions and constraints for unsaturated soils to reach critical state based on thermodynamic theory.

With more general perspectives based on the theory of thermodynamics, critical state for unsaturated soil is studied in this paper. Steady state in thermodynamic process is more common and general than critical state in soils. Steady state describes the final state of a thermodynamic process, and critical state of soils is a special case of it. This paper demonstrates that steady state in a deformation process of unsaturated soils is the final state of the process just as the critical state in soil mechanics, and it includes the more state variables and restrictions than those in critical state of saturated soils.

2 THERMODYNAMIC CONDITIONS FOR STEADY STATE

It is assumed that unsaturated soils satisfy the assumption of local equilibrium thermodynamics, and the theory of local equilibrium thermodynamics can be used to approximately describe the irreversible deformation process of unsaturated soils. The theory of local equilibrium thermodynamics (Kuiken 1994, Wark and Richards 1999) demonstrates that under the environmental force disturbance, a closed system that keeps constant temperature but cannot exchange both mass and heat energy with its surroundings can evolve from a non-equilibrium state to a steady equilibrium state with system entropy reaching the maximum. In accordance with the assumptions widely used in unsaturated soil mechanics, it is assumed that a representative volume element (RVE) of unsaturated soils is a closed system that cannot exchange mass with its surroundings (actually the real system can exchange mass with its surroundings. If the system were not supposed to be a closed system, Gibbs's thermodynamic theory would not be applied to it. As usual applied thermodynamics, here a representative volume element (RVE) of unsaturated soils is supposed to be a closed system). Therefore, a RVE evolves from a non-equilibrium state to a steady equilibrium state with the maximum entropy. Under the isothermal and isometric conditions, the Helmholtz free energy Ψ of the RVE reaches the minimum value and remains constant at steady state according to the theory of local equilibrium thermodynamics (Kuiken 1994, Wark and Richards 1999).

The first law and the second law of thermodynamics (Wark and Richards 1999) are given as following:

$$\begin{cases} \dot{W} + \dot{Q} = \dot{U} & \text{First law} \\ S_i = \dot{S} - \dot{Q}/T \geq 0 & \text{Second law} \end{cases} \tag{3}$$

where W, Q, U, T, S_i and S are work, heat supplied, internal energy, temperature, internal entropy and total entropy, respectively. From Eq. (3), the following equation can be developed:

$$\dot{W} - TS_i = \dot{U} - T\dot{S} = \dot{\Psi} \tag{4}$$

In light of Gibbs's theory of local equilibrium thermodynamics (Kuiken 1994, Wark and Richards 1999) when the RVE attains to steady equilibrium, Helmholtz free energy, Ψ, reaches the minimum value, i.e. $\dot{\Psi} = 0$, and all state variables must keep constant, which can be expressed as:

$$\dot{W} - T_s S_i = \dot{U}_s - T_s \dot{S}_s = \dot{\Psi}_s = 0 \tag{5}$$

where the subscript s stands for steady state.

3 CONDITIONS AND CONSTRAINTS TO ATTAIN TO CRITICAL STATE FOR UNSATURATED SOILS

To analyze the problems of soil mechanics, it is important that some independent state variables should be selected at first. In continuum mechanics, total stress σ and corresponding strain ε are used to describe material mechanical properties. As there are solid, liquid and gas phases in soils, and each phase has its dual variables, stress and the volume fraction, the properties of soils cannot be analyzed strictly by continuum mechanics. However, theory of porous media can be used to describe behaviors and properties of soils. According to theory of porous media, such as de Boer (2000), the volume fraction and the stress of each phase are selected as the dual state variables. In theory of porous media, the velocity is defined as a mass weighted average quantity, but in soil mechanics or hydrology, the velocity of soil skeleton v_s is generally used as a reference configuration to construct seepage or other equations. The control equations based on the theory of porous media may have different forms from the ones developed in some engineering fields, such as soil mechanics and hydrology, due to the selection of different basic kinematical variables. In this paper, the variables and parameters widely used in soil mechanics are selected as the state variables and parameters, the same as those used in Zhao, Liu and Gao (2010), e.g., the total stress σ and the dual variables in Eq. (6), effective stress $\tilde{\sigma}$ and strain ε^s of soil skeleton that determines soil deformation, suction s and degree of saturation S_r, and air pressure P_a and air volume strain $\varepsilon_v{}^a$.

Based on porous media theory, it is assumed that the solids and the pore water are incompressible, neither heat nor mass is transferred among the three phases, and the velocities of seepage and airflow are sufficiently small such that the diffusion effects on internal energy, stress, heat and entropy are all negligible, then, as in Zhao, Liu and Gao (2010), the work for unsaturated soils can be expressed as:

$$\dot{W} = [\mathrm{tr}(\tilde{\sigma} \cdot \dot{\varepsilon}^s) + sn\dot{S}_r + P_a n^a \dot{\varepsilon}_v^a] \tag{6}$$

Houlsby (1997) gave a similar form as Eq. (6). Substituting Eq. (6) into Eq. (4) results in the following:

$$[\mathrm{tr}(\tilde{\sigma} \cdot \dot{\varepsilon}^s) + sn\dot{S}_r + P_a n^a \dot{\varepsilon}_v^a] - TS_i = \dot{U} - T\dot{S} = \dot{\Psi} \tag{7}$$

where ε^s is soil skeleton strain, $s = P_a - P_w$ is suction, P_a is the pore air pressure, P_w is the pore water pressure, n is the porosity of soil, S_r is degree of saturation, $n^a = n(1-S_r)$ is volume fraction of air phase, ε_v^a is volume strain of air phase and $\tilde{\sigma}$ is the effective stress of unsaturated soils, as given by Zhao, Liu and Gao (2010) as:

$$\tilde{\sigma} = \sigma - [S_r P_w + (1-S_r)P_a]\delta \tag{8}$$

where δ is unit tensor.

In soil mechanics, some fundamental concepts and constitutive models are usually developed based on the results of triaxial tests, and the conditions and constraints for critical state of saturated soils given by Eq. (1) and Eq. (2) are in three-dimensional axisymmetric space. Following this convention, Eq. (8) is rewritten in the three-dimensional axisymmetric stress and strain space as:

$$[\tilde{p}\dot{\varepsilon}_v^s + q\dot{\varepsilon}_s^s + sn\dot{S}_r + P_a n(1-S_r)\dot{\varepsilon}_v^a] - TS_i = \dot{U} - T\dot{S} = \dot{\Psi} \tag{9}$$

where ε_v^s and ε_s^s are volume and deviator strains, respectively, of solid phase in triaxial strain space, $\tilde{p} = 1/3(\tilde{\sigma}_{11} + \tilde{\sigma}_{22} + \tilde{\sigma}_{33})$ $= p - [S_r P_w + (1-S_r)P_a]$ is the mean effective pressure dual to ε_v^s, and $p = 1/3(\sigma_{11} + \sigma_{22} + \sigma_{33})$ is the mean total pressure.

The objective of this paper is to discuss the stress and strain conditions and constraints of each phase to attain to the ultimate state or steady state of deformation process for unsaturated soils from thermodynamic perspective. According to the theory of thermodynamics as mentioned in the above section, when the unsaturated soil continually being sheared to reach the ultimate point of the deformation process, a local steady equilibrium state should be attained, and then the local state variables should not change with time, and Helmholtz free energy, Ψ, reaches the lowest value, i.e. $\dot{\Psi} = 0$. In other words, all state variables except shear strain should be constant. When soil deformation attains to steady state, according to Eq. (5), Eq. (9) equals to zero. Following the principle proposed by Roscoe, Schofield and Wroth (1958), and Schofield and Wroth (1968), the shear strain energy is entirely dissipated at steady state, and then Eq. (9) can be decomposed into:

$$\begin{cases} [q\dot{\varepsilon}_s^s]_s = T_s S_i \\ [\tilde{p}\dot{\varepsilon}_v^s + sn\dot{S}_r + P_a n(1-S_r)\dot{\varepsilon}_v^a]_s = \dot{U}_s - T_s\dot{S}_s = \dot{\Psi}_s = 0; \dot{U}_s = 0; \dot{S}_s = 0; \dot{\Psi}_s = 0 \end{cases} \tag{10}$$

The first equation in Eq. (10) demonstrates that when the deformation process of unsaturated soils attains to steady state, its state variables should not include shear strain. The shear strain energy is supposed to be the only dissipated energy, and it should be dissipated (assume that the elastic shear strain is sufficiently small and its corresponding energy is ignored) to make the system entropy increase. The system entropy reaches the maximum value, or the Helmholtz free energy reaches the lowest value at steady state. From the second equation in Eq. (10), it can be learnt that because the pressure of each phase is not null when the deformation process attains to steady state, the volume increment of each phase in the left side of the equation must be zero. This result is very important since currently there is not a clear definition about whether the volume or volume fraction of each phase keeps constant at the steady state or critical state. Some researchers might believe intuitively the above point, but there has not been any rigorous theoretical proof. Based on the theory of thermodynamic, it has been proven theoretically that the volume or volume fraction of each phase must keep constant at steady state. So according to the second equation in Eq. (10), the necessary conditions and constraints for steady state of unsaturated soils can be expressed as:

$$\dot{\varepsilon}_v^s = 0; \quad \dot{S}_r = 0; \quad \dot{\varepsilon}_v^a = 0; \quad \dot{\varepsilon}_s^s \neq 0 \tag{11}$$

where n is the porosity of the soil. It should be constant in order to keep the volume of each phase constant, otherwise the change of n would make ε_v^s, S_r and ε_v^a change, which is contradictory to Eq. (11).

On the other hand, when the deformation process of unsaturated soil attains to steady state, in addition to ε_v^s, S_r and ε_v^a keeping constant, the stress variables $\tilde{p}$, q, s and P_a in the second equation of Eq. (10) should also keep constant, otherwise the deformation process would not attain to steady state in light of thermodynamics. If s and $\tilde{p}$ are constant at steady state, it is obvious that P_w, p and net pressure $\hat{p} = p - P_a$ must be constant.

From above discussion, a thermodynamic process evolves continuously and finally attains to steady state. For the deformation process of unsaturated soils, steady state or steady balance means that deformation process of unsaturated soils attains to the ultimate state, namely the critical state in soil mechanics or the steady state in thermodynamics, and at this state all the state variables do not change. The necessary conditions and constraints for critical state of unsaturated soils based on thermodynamics are being stated as: 1) the volume change of each phase should satisfy the requirement of Eq. (11) and porosity n should keep constant; 2) the stress variables: $\tilde{p}$, q, s and P_a (also including P_w, p and $\hat{p}$) should also be constant. Comparing with the conditions and constraints for critical state of saturated soils, i.e. Eq. (1), more conditions and constraints are needed for critical state of unsaturated soils.

Toll (1990), Wheeler and Sivakumar(1995), Maatouk et al. (1995), Adams and Wulfsohn (1997), Rampino et al. (1998), Wang et al. (2002), and Kayadelen et al. (2007) have conducted some pioneering work on critical state of unsaturated soils by laboratory triaxial tests, and suggested some conditions required for critical state of unsaturated soils. Comparing the conditions from these researchers, the conditions proposed in this paper are more complete and generalized with rigorous theoretical basis. In some laboratory tests on samples of unsaturated soils, at the end of the deformation processes, the conditions and constraints given in this paper are not all satisfied. This does not mean that the conditions and constraints provided in this paper are incorrect, since they are established based on the universally applicable laws of thermodynamics. The explanation may be that these tests might be limited by laboratory equipment and experimental conditions, and the deformation of unsaturated soil samples might not be able to attain to critical state, just as those with saturated soils.

Based on above the necessary conditions and constraints for critical state of unsaturated soils, two special cases need further discuss: 1) When air pressure, P_a, is not considered as an independent state variable, identical to ignoring the item of P_a in Eq. (10), the conditions and constraints to attain to critical state can be rewritten as: the volume increment of each phase should satisfy $\dot{\varepsilon}_v^s = 0$ and $\dot{S}_r = 0$, the porosity n keeps constant, and the stress variables p , $\hat{p}$, $\tilde{p}$, q , s and P_w remain constant. 2) When both air phase pressure, P_a , and degree of saturation, S_r , are not considered as independent state variables, the conditions and constraints to attain to critical state can be rewritten as: the volume increment should satisfy $\dot{\varepsilon}_v^s = 0$, the porosity n keeps constant, and the stress variables p , $\hat{p}$, $\tilde{p}$, q , s and P_w remain unchangeable. Alonso, Gens and Josa (1990) developed the Barcelona model, in which they selected mean net stress $\hat{p} = p - P_a$, matric suction s , deviator stress q , specific volume v , and shear strain ε_s^s as state variables, the conditions to attain to critical state given by them belong to the second special case described above. In the models of coupling of hydraulic hysteresis and stress-strain behaviour developed by Wheeler, Sharma and Buisson (2003), Sheng, Sloan and Gens(2004), Li (2007), Sun, Sheng, Cui and Sloan (2007), etc., the conditions to attain to critical state belong to the first special case described above. Besides when the degree of saturation, S_r , equals to 1, the soil is saturated, and matrix suction and pore air presser are zero, then the necessary conditions and constraints for critical state of unsaturated soils are degraded into those for saturated soils, e.g. equal to what are given in Eq. (1). Of course, more study is needed for some special circumstances whether the conditions and constraints for critical state of unsaturated soils are all necessary.

It should be noted that when the deformation process of soils attains to critical state, the soil structure might be anisotropic. Li and Dafalias (2010) discussed the uniqueness of critical state line with anisotropic structure and the enhanced critical state conditions for saturated soils. But more research on critical state with anisotropic structure and its uniqueness for unsaturated soils is needed.

4 CONCLUSIONS

This paper develops the necessary conditions for the deformation of unsaturated soils to reach critical state based on the theory of local equilibrium thermodynamics. The necessary conditions are developed using state variables and parameters widely used in soil mechanics and the expression of work $\dot{W}$ for unsaturated soils proposed by Zhao, Liu and Gao (2010). Comparing with the conditions given by other pioneering researchers based on the laboratory triaxial tests, the conditions given in this paper are more complete and generalized with rigorous theoretical basis, and these conditions are not based on the laboratory test results on some special samples. In addition, it is shown that when some variables are not treated as independent variables, the conditions given by other researchers are a special case of the conditions presented in this paper.

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6 REFERENCES

Adams B.A. and Wulfsohn D. 1997. Variation of the critical-state boundaries of an agriculture soil. *European Journal of Soil Science*, 48(4), 739-758.

Alonso E. E., Gens A. and Josa A. 1990. A constitutive model for partially saturated soils. *Géotechnique*. 40(3), 405-430.

de Boer R. 2000. *Theory of porous media*. Springer, Berlin.

Gens A. and Alonso E. E. 1992. A framework for the behavior of unsaturated expansive clays. *Canadian Geotechnical Journal*, 29(6), 1013-1032.

Houlsby G.T. 1997. The work input to an unsaturated granular material. *Géotechnique*, 47(1), 193-196.

Kayadelen C., Sivrikaya O., Taskiran T. and Guneyli H. 2007. Critical-state parameters of an unsaturated residual clayey soil from Turkey. *Engineering Geology*, 94(1-2), 1-9.

Kuiken G.D.C. 1994. *Thermodynamics of irreversible processes*. John Wiley & Sons, Chichester.

Li X.S. 2007. Thermodynamics-based constitutive framework for unsaturated soils. 1: theory. *Géotechnique*. 57(5), 411-422.

Li X.S. and Dafalias Y.F. 2010. Anisotropy at critical state: the role of fabric. *Proceedings of the 9th HSTAM International Congress on Mechanics: Vardoulakis mini-symposia*. International Congress on Mechanics: Vardoulakis Mini-Symposia (9th, HSTAM), Limassol, Cyprus, 57-60.

Maatouk A., Leroueil S. and La Rochelle P. 1995. Yielding and critical state of a collapsible unsaturated silty soil. *Géotechnique*, 45(3), 465-477.

Rampino C., Mancuso C. and Vinale F. 1998. Behavior of a compacted silty sand during suction controlled tests. *Proceedings of the 2nd International Conference on Unsaturated Soils*, Beijing, China, 1, 108-113.

Roscoe K.H., Schofield A.N. and Worth C.P. 1958. On the yielding of soils. *Géotechnique*, 8(1), 22-53.

Schofield A.N. and Wroth C.P. 1968. *Critical state soil mechanics*. McGraw-Hill, New York.

Sheng D.C., Sloan S.W. and Gens A. 2004. A constitutive model for unsaturated soils: thermomechanical and algorithmic aspects. *Computational Mechanics*, 33, 453-465.

Sun D.A., Sheng D.C., Cui H.B. and Sloan S.W. 2007. A density-dependent elastoplastic hydro-mechanical model for unsaturated compacted soils. *International Journal for Numerical and Analytical Methods in Geomechanics*. 31(11), 1257-1279.

Toll D.G. 1990. A framework for unsaturated soil behavior. *Géotechnique*, 40(1), 31-44.

Wang Q., Pufahl D.E. and Fredlund D.G. 2002. A study of critical state on an unsaturated silty soil. *Canadian Geotechnical Journal*, 39(1), 213-218.

Wark Jr., K. and Richards D.E. 1999. *Thermodynamics, 6th Edition*. McGraw-Hill, New York.

Wheeler S.J. and Sivakumar V. 1995. An elasto-plastic critical state framework for unsaturated soil. *Géotechnique*, 45(1), 35-53.

Wheeler S.J., Sharma R.S. and Buisson M.S.R. 2003. Coupling of hydraulic hysteresis and stress-strain behaviour in unsaturated soils. *Géotechnique*, 53(1), 41-54.

Zhao C.G., Liu Y. and Gao F.P. 2010. Work and energy equations and the principle of generalized effective stress for unsaturated soils. *International Journal for Numerical and Analytical Methods in Geomechanics*, 34(6), 920-936.