Application of the method of initial parameters to the laterally loaded pile problem

Application de la méthode de paramètres initiaux de la pieu latéralement chargée problème

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ABSTRACT
The method of initial parameters (MIP) has been used to analyze beams on elastic foundations with discontinuities along the span due to applied concentrated forces and moments. MIP is modified in this paper to analyze laterally loaded piles with discontinuities, caused by soil layering, along the length of the piles. Using MIP, the pile deflection, slope, bending moment and shear force can be calculated analytically within each soil layer. The use of MIP to solve the problem of a laterally loaded pile is illustrated for a pile embedded in a three-layer soil profile, although the method is, in principle, applicable to problems with any number of soil layers. One advantage of MIP is that complete symbolic solutions can be obtained for problems with up to four soil layers. The method is equally applicable to problems in which forces act not only at the pile head but also along the pile shaft.

RÉSUMÉ
La méthode de paramètres initiaux (MIP) a été utilisé pour analyser les *faisceau* sur les fondations élastique avec discontinuités le long de la portée en raison de forces et les moments appliquées. MIP est modifiée dans le présent document pour analyser des pieux chargées latéralement avec de discontinuitiés causées par les couches de sol sur la longueur des pieux. L'utilisation de MIP, le tas de déviation, la pente, le moment de flexion et de cisaillement de la force ne peut être calculé analytiquement à l'intérieur de chaque couche de sol. L'utilisation de MIP pour résoudre le problème de la pieu chargée latéralement est illustré pour une pile inséré dans un profil de sol avex trois couches, bien que la méthode est, en principe, applicable à des problèmes avec n'importe lequel nombre de couches de sol. Un avantage de MIP est des solutions symbolique complètes qui peuvent être obtenus en cas de problème avec jusqu'à quatre couches de sol. La méthode est également applicable à des problèmes dans lesquels des forces agissent non seulement sur le tas la tête mais aussi le long de la pile arbre.

Keywords: pile, analytical solution, lateral load, initial parameter

1 INTRODUTION

The method of initial parameters (MIP) was originally developed to analyze beams on elastic foundations with discontinuities along the span arising due to the application of concentrated forces (Vlasov and Leont'ev 1966, Harr et al. 1969, Rao et al. 1971). Laterally loaded piles are often treated as Euler-Bernoulli beams embedded in a bed of mechanical springs that resist lateral pile movement (Matlock and Reese 1960, Reese et al. 1974). Thus, laterally loaded piles can be looked upon as vertical beams supported laterally by soil. Consequently, MIP can be used to analyze the problem. However, in the case of piles in layered soil, the discontinuities along the span of the pile are created by soil layering, for which the existing MIP needs modification. In this paper, we present the modified MIP applicable to laterally loaded piles in layered soil. We illustrate the method with the help of an example of a pile in a three-layer soil medium.

2 EQUATIONS DESCRIBING PILE DEFLECTION

The governing differential equation for pile deflection within any layer i for a pile of length L_p embedded in a multi-layered elastic medium with n layers is (Basu and Salgado 2007a):

$$d^4\tilde{w}_i / d\tilde{z}^4 - 2\tilde{t}_i d^2\tilde{w}_i / d\tilde{z}^2 + \tilde{k}_i\tilde{w}_i = 0 \qquad (1)$$

where $\tilde{w}_i(\tilde{z})$ is the normalized pile deflection in the i^{th} layer ($\tilde{w}_i = w_i/L_p$); $\tilde{z}$ is the normalized depth ($\tilde{z} = z/L_p$); and $\tilde{k}_i$ and $\tilde{t}_i$ are the normalized compressive and shear resistances of soil, both of which are related to soil modulus and Poisson's ratio. The boundary conditions required to solve equation (1) are: (1) at the pile head (i.e., at $z = \tilde{z} = 0$), $\tilde{w}_1 = 0$ or $d^3\tilde{w}_1/d\tilde{z}^3 - 2\tilde{t}_1\, d\tilde{w}_1/d\tilde{z} = \tilde{F}_a$ and $d\tilde{w}_1/d\tilde{z} = 0$ or $d^2\tilde{w}_1/d\tilde{z}^2 = \tilde{M}_a$; (2) at the interface between any two layers (i.e., at $z = H_i$ or $\tilde{z} = \tilde{H}_i$), $\tilde{w}_i = \tilde{w}_{i+1}$, $d\tilde{w}_i/d\tilde{z} = d\tilde{w}_{i+1}/d\tilde{z}$, $d^3\tilde{w}_i/d\tilde{z}^3 - 2\tilde{t}_i\, d\tilde{w}_i/d\tilde{z} = d^3\tilde{w}_{i+1}/d\tilde{z}^3 - 2\tilde{t}_{i+1}\, d\tilde{w}_{i+1}/d\tilde{z}$ and $d^2\tilde{w}_i/d\tilde{z}^2 = d^2\tilde{w}_{i+1}/d\tilde{z}^2$; and (3) at the pile base (i.e., at $z = L_p$ or $\tilde{z} = 1$), $\tilde{w}_n = 0$ or $d^3\tilde{w}_n/d\tilde{z}^3 - 2\tilde{t}_n\, d\tilde{w}_n/d\tilde{z} = \sqrt{(2\tilde{k}_n\tilde{t}_{n+1})}\,\tilde{w}_n$ and $d\tilde{w}_n/d\tilde{z} = 0$ or $d^2\tilde{w}_n/d\tilde{z}^2 = 0$. $\tilde{F}_a$ ($= F_a L_p^2/E_p I_p$) and $\tilde{M}_a$ ($= M_a L_p/E_p I_p$) are the normalized applied force and moment ($E_p I_p$ is the flexural rigidity of the pile).

The above boundary conditions can be interpreted physically by drawing analogy with the Euler-Bernoulli beam theory. Given that w is the pile deflection, the slope θ, bending moment M and shear force S at any cross section of the pile (or any Euler-Bernoulli beam resting on an elastic foundation) can be expressed, in their normalized form, as: $\tilde{\theta} = d\tilde{w}/d\tilde{z}$, $\tilde{M} = ML_p/E_pI_p = d\tilde{w}^2/d\tilde{z}^2$ and $\tilde{S} = SL_p^2/E_pI_p = d\tilde{w}^3/d\tilde{z}^3 - 2\tilde{t}\, d\tilde{w}/d\tilde{z}$. The shear force S at any cross section is the sum of the shear forces arising due to pile (beam) flexure and soil deformation.

According to the above definitions, the boundary conditions satisfy the continuity of deflection, slope, bending moment and shear force across the interfaces of adjacent layers. At the pile head, the shear force must be equal to the applied horizontal force and either the slope must equal zero (if a pile cap is present that completely restrains pile head rotation) or the bending moment must equal the applied moment. At the pile

Proceedings of the 17[th] International Conference on Soil Mechanics and Geotechnical Engineering
M. Hamza et al. (Eds.)

doi:10.3233/978-1-60750-031-5-1297

base, either the deflection is equal to zero or the shear force just above the base of the pile is equal to the shear force just below the base (the normalized shear force just below the pile base is equal to $\sqrt{(2\tilde{k}_n\tilde{t}_{n+1})}\ \tilde{w}_n|_{\tilde{z}=1}$. The other boundary condition active at the pile base is that either the slope is zero (valid for a rigidly socketed base) or the bending moment is zero (valid for a floating base).

The general solution for equation (1) is given by:

$$\tilde{w}_i(\tilde{z}) = C_1^{(i)}\Phi_1 + C_2^{(i)}\Phi_2 + C_3^{(i)}\Phi_3 + C_4^{(i)}\Phi_4 \quad (2)$$

where $C_1^{(i)}$, $C_2^{(i)}$, $C_3^{(i)}$ and $C_4^{(i)}$ are integration constants for the i^{th} layer; and Φ_1, Φ_2, Φ_3 and Φ_4 are functions of $\tilde{z}$ that are individual solutions of the differential equation (1) (see Basu and Salgado 2007b for a detailed discussion). If $\tilde{k}_i > \tilde{t}_i^2$, then $\Phi_1 = \sinh a\tilde{z}\cos b\tilde{z}$, $\Phi_2 = \cosh a\tilde{z}\cos b\tilde{z}$, $\Phi_3 = \cosh a\tilde{z}\sin b\tilde{z}$, $\Phi_4 = \sinh a\tilde{z}\sin b\tilde{z}$; with $a = \sqrt{\{½(\sqrt{\tilde{k}_i} + \tilde{t}_i)\}}$ and $b = \sqrt{\{½(\sqrt{\tilde{k}_i} - \tilde{t}_i)\}}$. If on the other hand $\tilde{k}_i < \tilde{t}_i^2$, then $\Phi_1 = \sinh a\tilde{z}$, $\Phi_2 = \cosh a\tilde{z}$, $\Phi_3 = \sinh b\tilde{z}$, $\Phi_4 = \cosh b\tilde{z}$; with $a = \sqrt{\{\tilde{t} + \sqrt{(\tilde{t}_i^2 - \tilde{k}_i)}\}}$ and $b = \sqrt{\{\tilde{t} - \sqrt{(\tilde{t}_i^{\ 2} - \tilde{k}_i)}\}}$. The constants $C_1^{(i)}$, $C_2^{(i)}$, $C_3^{(i)}$ and $C_4^{(i)}$ must be determined for each set of boundary conditions. We avoid direct calculation of the constants and obtain solutions using the method of initial parameters.

3 METHOD OF INITIAL PARAMETERS

According to MIP, the normalized deflection $\tilde{w}_i$, the slope $\tilde{\theta}_i$, the bending moment $\tilde{M}_i$ and the shear force $\tilde{S}_i$ within any layer i can be expressed as:

$$\tilde{w}_i(\tilde{z}) = \tilde{w}_0^{(i)}K_{ww}^{(i)} + \tilde{\theta}_0^{(i)}K_{w\theta}^{(i)} + \tilde{M}_0^{(i)}K_{wM}^{(i)} + \tilde{S}_0^{(i)}K_{wS}^{(i)} \quad (3)$$

$$\tilde{\theta}_i(\tilde{z}) = \tilde{w}_0^{(i)}K_{\theta w}^{(i)} + \tilde{\theta}_0^{(i)}K_{\theta\theta}^{(i)} + \tilde{M}_0^{(i)}K_{\theta M}^{(i)} + \tilde{S}_0^{(i)}K_{\theta S}^{(i)} \quad (4)$$

$$\tilde{M}_i(\tilde{z}) = \tilde{w}_0^{(i)}K_{Mw}^{(i)} + \tilde{\theta}_0^{(i)}K_{M\theta}^{(i)} + \tilde{M}_0^{(i)}K_{MM}^{(i)} + \tilde{S}_0^{(i)}K_{MS}^{(i)} \quad (5)$$

$$\tilde{S}_i(\tilde{z}) = \tilde{w}_0^{(i)}K_{Sw}^{(i)} + \tilde{\theta}_0^{(i)}K_{S\theta}^{(i)} + \tilde{M}_0^{(i)}K_{SM}^{(i)} + \tilde{S}_0^{(i)}K_{SS}^{(i)} \quad (6)$$

where $\tilde{w}_0^{(i)}$, $\tilde{\theta}_0^{(i)}$, $\tilde{M}_0^{(i)}$ and $\tilde{S}_0^{(i)}$ are the initial parameters, and the K's are the influence coefficients for the i^{th} layer. The initial parameters are the normalized deflection, slope, shear force and bending moment at a particular chosen pile section called the initial section. The influence coefficients express the influence of one variable on the other; e.g., K_{wM} denotes the influence of bending moment in the pile on pile deflection. Note that, although there are sixteen influence coefficients in equations (3)—(6), only ten of them are actually unknown because, from the theorem of reciprocal deflection by Maxwell and Betti, we get $K_{MM} = K_{\theta\theta}$, $K_{\theta S} = K_{wM}$, $K_{MS} = K_{w\theta}$, $K_{SS} = K_{ww}$, $K_{S\theta} = K_{Mw}$ and $K_{SM} = K_{\theta w}$. If the initial parameters and the influence coefficients are known for a layer, then the deflection, slope, bending moment, and shear force can be obtained as a function of depth.

We now illustrate how MIP can be used in layered soil with the help of an example. We consider, for our example, a pile embedded in a three-layer soil medium with $\tilde{k}_i > \tilde{t}_i^2$ for all the layers (Figure 1). There is no restraint at the pile head (i.e., it is free to translate and rotate). The same is true for the pile base.

The first step is to choose an appropriate initial section for each layer. We choose the pile head and base as the initial sections for the top and the bottom layers, respectively. For the middle layer, we choose the upper interface as the initial section. Thus, the initial section for the top layer (layer 1) is at $\tilde{z} = 0$. The four initial parameters are $\tilde{w}_0^{(1)}$, $\tilde{\theta}_0^{(1)}$, $\tilde{M}_0^{(1)}$ and $\tilde{S}_0^{(1)}$, which are the normalized deflection, slope, bending moment and shear force at $\tilde{z} = 0$. The initial section for the bottom layer (layer 3) is at $\tilde{z} = 1$. The corresponding initial parameters are $\tilde{w}_0^{(3)}$, $\tilde{\theta}_0^{(3)}$, $\tilde{M}_0^{(3)}$ and $\tilde{S}_0^{(3)}$ (these are the normalized deflection, slope, bending moment and shear force at $\tilde{z} = 1$). The initial section for the middle layer (layer 2) is at $\tilde{z} = \tilde{H}_1$ with the initial parameters $\tilde{w}_0^{(2)}$, $\tilde{\theta}_0^{(2)}$, $\tilde{M}_0^{(2)}$ and $\tilde{S}_0^{(2)}$ (the normalized deflection, slope, bending moment and shear force at $\tilde{z} = \tilde{H}_1$).

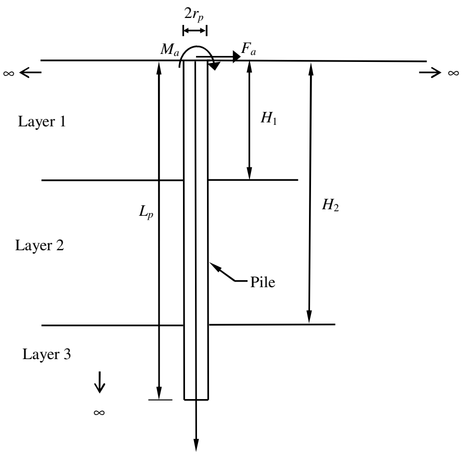

Figure 1. Laterally loaded pile in a three-layer soil medium.

Next, we impose the boundary conditions for the pile head and base on the initial parameters for the two end layers. For a free pile head, the shear force and bending moment boundary conditions are valid, resulting in $\tilde{S}_0^{(1)} = \tilde{F}_a$ and $\tilde{M}_0^{(1)} = \tilde{M}_a$. For a free pile base, the shear force and bending moment boundary conditions are valid, which gives $\tilde{S}_0^{(3)} = \sqrt{(2\tilde{k}_3\tilde{t}_4)}\,\tilde{w}_0^{(3)}$ and $\tilde{M}_0^{(3)} = 0$. Thus, the initial parameters $\tilde{S}_0^{(1)}$, $\tilde{M}_0^{(1)}$, $\tilde{S}_0^{(3)}$ and $\tilde{M}_0^{(3)}$ are now known. The equations for the normalized deflection, slope, bending moment and shear force for layers 1 and 3 can now be rewritten by substituting the known initial parameters into equation (3)—(6) (for i = 1 and 3):

$$\tilde{w}_1 = \tilde{w}_0^{(1)}K_{ww}^{(1)} + \tilde{\theta}_0^{(1)}K_{w\theta}^{(1)} + \tilde{M}_aK_{wM}^{(1)} + \tilde{F}_aK_{wS}^{(1)} \quad (7)$$

$$\tilde{\theta}_1 = \tilde{w}_0^{(1)}K_{\theta w}^{(1)} + \tilde{\theta}_0^{(1)}K_{\theta\theta}^{(1)} + \tilde{M}_aK_{\theta M}^{(1)} + \tilde{F}_aK_{\theta S}^{(1)} \quad (8)$$

$$\tilde{M}_1 = \tilde{w}_0^{(1)}K_{Mw}^{(1)} + \tilde{\theta}_0^{(1)}K_{M\theta}^{(1)} + \tilde{M}_aK_{MM}^{(1)} + \tilde{F}_aK_{MS}^{(1)} \quad (9)$$

$$\tilde{S}_1 = \tilde{w}_0^{(1)}K_{Sw}^{(1)} + \tilde{\theta}_0^{(1)}K_{S\theta}^{(1)} + \tilde{M}_aK_{SM}^{(1)} + \tilde{F}_aK_{SS}^{(1)} \quad (10)$$

$$\tilde{w}_3 = \tilde{w}_0^{(3)}K_{ww}'^{(3)} + \tilde{\theta}_0^{(3)}K_{w\theta}^{(3)} \quad (11)$$

$$\tilde{\theta}_3 = \tilde{w}_0^{(3)}K_{\theta w}'^{(3)} + \tilde{\theta}_0^{(3)}K_{\theta\theta}^{(3)} \quad (12)$$

$$\tilde{M}_3 = \tilde{w}_0^{(3)}K_{Mw}'^{(3)} + \tilde{\theta}_0^{(3)}K_{M\theta}^{(3)} \quad (13)$$

$$\tilde{S}_3 = \tilde{w}_0^{(3)}K_{Sw}'^{(3)} + \tilde{\theta}_0^{(3)}K_{S\theta}^{(3)} \quad (14)$$

where $K_{*w}'^{(3)} = K_{*w}^{(3)} + \sqrt{2\tilde{k}_3\tilde{t}_4}\,K_{*S}^{(3)}$. Equations (7)—(10) are valid for layer 1 (i.e., for $0 \le \tilde{z} \le \tilde{H}_1$) while equations (11)—(14) are valid for layer 3 (i.e., for $\tilde{H}_2 \le \tilde{z} \le 1$).

Now we impose the interface boundary conditions on the initial parameters. This leads to a set of algebraic equations consisting of the unknown influence coefficients and initial parameters. In our example problem, there are two interfaces ($\tilde{z} = \tilde{H}_1$ and $\tilde{z} = \tilde{H}_2$). There are four equations per interface, resulting in eight simultaneous equations, which can be written in matrix form as:

$$\begin{bmatrix} K_{ww}^{(1)} & K_{w\theta}^{(1)} & -1 & 0 & 0 & 0 & 0 & 0 \\ K_{\theta w}^{(1)} & K_{\theta\theta}^{(1)} & 0 & -1 & 0 & 0 & 0 & 0 \\ K_{Mw}^{(1)} & K_{M\theta}^{(1)} & 0 & 0 & -1 & 0 & 0 & 0 \\ K_{Sw}^{(1)} & K_{S\theta}^{(1)} & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & K_{ww}^{(2)} & K_{w\theta}^{(2)} & K_{wM}^{(2)} & K_{wS}^{(2)} & -1 & 0 \\ 0 & 0 & K_{\theta w}^{(2)} & K_{\theta\theta}^{(2)} & K_{\theta M}^{(2)} & K_{\theta S}^{(2)} & 0 & -1 \\ 0 & 0 & K_{Mw}^{(2)} & K_{M\theta}^{(2)} & K_{MM}^{(2)} & K_{MS}^{(2)} & -K_{Mw}'^{(3)} & -K_{M\theta}^{(3)} \\ 0 & 0 & K_{Sw}^{(2)} & K_{S\theta}^{(2)} & K_{SM}^{(2)} & K_{SS}^{(2)} & -K_{Sw}'^{(3)} & -K_{S\theta}^{(3)} \end{bmatrix} \begin{bmatrix} \tilde{w}_0^{(1)} \\ \tilde{\theta}_0^{(1)} \\ \tilde{w}_0^{(2)} \\ \tilde{\theta}_0^{(2)} \\ \tilde{M}_0^{(2)} \\ \tilde{S}_0^{(2)} \\ \tilde{w}_0^{(3)} \\ \tilde{\theta}_0^{(3)} \end{bmatrix}$$

$$= \begin{bmatrix} -\tilde{M}_a K_{wM}^{(1)} - \tilde{F}_a K_{wS}^{(1)} \\ -\tilde{M}_a K_{\theta M}^{(1)} - \tilde{F}_a K_{\theta S}^{(1)} \\ -\tilde{M}_a K_{MM}^{(1)} - \tilde{F}_a K_{MS}^{(1)} \\ -\tilde{M}_a K_{SM}^{(1)} - \tilde{F}_a K_{SS}^{(1)} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (15)$$

We refer to equation (15) as the "matrix equation" of the problem. In the matrix equation, the first four rows are valid at $\tilde{z} = \tilde{H}_1$ while the other four are valid at $\tilde{z} = \tilde{H}_2$. If, in equation (15), we assume that the influence coefficients K's are known, then all the unknown initial parameters for the different layers can be obtained.

In order to determine the influence coefficients, we refer back to equation (2) of normalized deflection and successively differentiate the equation to obtain the normalized slope, bending moment and shear force. For layer 1, this gives:

$$\tilde{w}_1 = C_1\Phi_1 + C_2\Phi_2 + C_3\Phi_3 + C_4\Phi_4 \quad (16)$$

$$\tilde{\theta}_1 = C_1(a\Phi_2 - b\Phi_4) + C_2(a\Phi_1 - b\Phi_3) + C_3\left(a\Phi_4 + b\Phi_2\right) + C_4\left(a\Phi_3 + b\Phi_1\right) \quad (17)$$

$$\tilde{M}_1 = C_1\{(a^2 - b^2)\,\Phi_1 - 2ab\Phi_3\} + C_2\{(a^2 - b^2)\,\Phi_2 - 2ab\Phi_4\} + C_3\{(a^2 - b^2)\,\Phi_3 + 2ab\Phi_1\} + C_4\{(a^2 - b^2)\Phi_4 + 2ab\Phi_2\} \quad (18)$$

$$\tilde{S}_1 = C_1\{a(a^2 - 3b^2)\,\Phi_2 + b(b^2 - 3a^2)\,\Phi_4\} + C_2\{a(a^2 - 3b^2)\,\Phi_1 + b(b^2 - 3a^2)\,\Phi_3\} + C_3\{a(a^2 - 3b^2)\,\Phi_4 - b(b^2 - 3a^2)\,\Phi_2\} + C_4\{a(a^2 - 3b^2)\,\Phi_3 - b(b^2 - 3a^2)\,\Phi_1\} - 2\tilde{t}_1\{C_1(a\Phi_2 - b\Phi_4) + C_2(a\Phi_1 - b\Phi_3) + C_3(a\Phi_4 + b\Phi_2) + C_4(a\Phi_3 + b\Phi_1)\} \quad (19)$$

where C_1, C_2, C_3 and C_4 are integration constants corresponding to layer 1; and $a = \sqrt{\{½(\sqrt{\tilde{k}_1} + \tilde{t}_1)\}}$ and $b = \sqrt{\{½(\sqrt{\tilde{k}_1} - \tilde{t}_1)}$. At the initial section of layer 1, i.e., at $\tilde{z} = 0$, $\Phi_1 = \Phi_3 = \Phi_4 = 0$ and $\Phi_2 = 1$. Substituting these in the above equation and noting that $\tilde{t}_1 = a^2 - b^2$, we express the initial parameters of layer 1 as:

$$\tilde{w}_0^{(1)} = C_2 \quad (20)$$

$$\tilde{\theta}_0^{(1)} = (d\tilde{w}_1/d\tilde{z})_{\tilde{z}=0} = C_1 a + C_3 b \quad (21)$$

$$\tilde{M}_0^{(1)} = \tilde{M}_a = (d^2\tilde{w}_1/d\tilde{z}^2)_{\tilde{z}=0} = C_2(a^2 - b^2) + C_4(2ab) \quad (22)$$

$$\tilde{S}_0^{(1)} = \tilde{F}_a = (d^3\tilde{w}_1/d\tilde{z}^3 - 2\tilde{t}_1\, d\tilde{w}_1/d\tilde{z})_{\tilde{z}=0} = C_1 a(a^2 - 3b^2) - C_3 b(b^2 - 3a^2) - 2(a^2 - b^2)(C_1 a + C_3 b) \quad (23)$$

Solving the above equations simultaneously for C_1, C_2, C_3 and C_4 and subsequently replacing them in equations (16)—(19) yields:

$$\tilde{w}_1(\tilde{z}) = [\Phi_2 - (a^2 - b^2)\Phi_4/2ab]\tilde{w}_0^{(1)} + [(a\Phi_3 + b\Phi_1)/2ab]\tilde{\theta}_0^{(1)} + [\Phi_4/2ab]\tilde{M}_a + [(\Phi_3 - b\Phi_1)/2ab(a^2 + b^2)]\tilde{F}_a \quad (24)$$

$$\tilde{\theta}_1(\tilde{z}) = [\{(a^2 b + b^3)\Phi_1 - (ab^2 + a^3)\Phi_3\}/2ab]\tilde{w}_0^{(1)} + [\{(a^2 - b^2)\Phi_4 + 2ab\Phi_2\}/2ab]\tilde{\theta}_0^{(1)} + [(a\Phi_3 + b\Phi_1)/2ab]\tilde{M}_a + [\Phi_4/2ab]\tilde{F}_a \quad (25)$$

$$\tilde{M}_1(\tilde{z}) = [-(a^4 + 2a^2b^2 + b^4)\Phi_4/2ab]\tilde{w}_0^{(1)} + [\{(3a^2 b - b^3)\Phi_1 - (3ab^2 - a^3)\Phi_3\}/2ab]\tilde{\theta}_0^{(1)} + [\{(a^2 - b^2)\Phi_4 + 2ab\Phi_2\}/2ab]\tilde{M}_a + [(a\Phi_3 + b\Phi_1)/2ab]\tilde{F}_a \quad (26)$$

$$\tilde{S}_1(\tilde{z}) = [\{(-3a^4 b - 2a^2b^3 + b^5)\Phi_1 + (-3ab^4 - 2a^3b^2 + a^5)\Phi_3\}/2ab]\tilde{w}_0^{(1)} + [\{-(a^4 + 2a^2b^2 + b^4)\Phi_4\}/2ab]\tilde{\theta}_0^{(1)} + [\{(a^2 b + b^3)\Phi_1 - (ab^2 + a^3)\Phi_3\}/2ab]\tilde{M}_a + [\Phi_2 - \{(a^2 - b^2)\Phi_4\}/2ab]\tilde{F}_a \quad (27)$$

We now compare the coefficients of $\tilde{w}_0^{(1)}$, $\tilde{\theta}_0^{(1)}$, $\tilde{M}_a$ and $\tilde{F}_a$ in equations (24)—(27) with those in equations (7)—(10), which gives the influence coefficients for layer 1 as:

$$K_{ww}^{(1)} = K_{SS}^{(1)} = \Phi_2 - (a^2 - b^2)\Phi_4/(2ab) \quad (28)$$

$$K_{w\theta}^{(1)} = K_{MS}^{(1)} = (a\Phi_3 + b\Phi_1)/(2ab) \quad (29)$$

$$K_{wM}^{(1)} = K_{\theta S}^{(1)} = \Phi_4/(2ab) \quad (30)$$

$$K_{wS}^{(1)} = (a\Phi_3 - b\Phi_1)/\{2ab(a^2 + b^2)\} \quad (31)$$

$$K_{\theta w}^{(1)} = K_{SM}^{(1)} = \{(a^2 b + b^3)\Phi_1 - (ab^2 + a^3)\Phi_3)\}/(2ab) \quad (32)$$

$$K_{\theta\theta}^{(1)} = K_{MM}^{(1)} = \{(a^2 - b^2)\Phi_4 + 2ab\Phi_2\}/(2ab) \quad (33)$$

$$K_{\theta M}^{(1)} = (a\Phi_3 + b\Phi_1)/(2ab) \quad (34)$$

$$K_{Mw}^{(1)} = K_{S\theta}^{(1)} = -\{(a^4 + 2a^2b^2 + b^4)\Phi_4\}/(2ab) \quad (35)$$

$$K_{M\theta}^{(1)} = \{(3a^2 b - b^3)\Phi_1 - (3ab^2 - a^3)\Phi_3\}/(2ab) \quad (36)$$

$$K_{Sw}^{(1)} = \{(-3a^4 b - 2a^2b^3 + b^5)\Phi_1 + (-3ab^4 - 2a^3b^2 + a^5)\Phi_3\}/(2ab) \quad (37)$$

For the second (middle) layer, the following equations must be solved simultaneously in terms of C_1, C_2, C_3 and C_4 (note that these constants are different from the constants of layer 1):

$$\tilde{w}_0^{(2)} = C_1\phi_1 + C_2\phi_2 + C_3\phi_3 + C_4\phi_4 \quad (38)$$

$$\tilde{\theta}_0^{(2)} = C_1(a\phi_2 - b\phi_4) + C_2(a\phi_1 - b\phi_3) + C_3(a\phi_4 + b\phi_2) + C_4(a\phi_3 + b\phi_1) \quad (39)$$

$$\tilde{M}_0^{(2)} = C_1\{(a^2 - b^2)\phi_1 - 2ab\phi_3\} + C_2\{(a^2 - b^2)\,\phi_2 - 2ab\phi_4\} + C_3\{(a^2 - b^2)\phi_3 + 2ab\phi_1\} + C_4\{(a^2 - b^2)\phi_4 + 2ab\phi_2\} \quad (40)$$

$$\tilde{S}_0^{(2)} = C_1\{a(a^2 - 3b^2)\phi_2 + b(b^2 - 3a^2)\phi_4\} + C_2\{a(a^2 - 3b^2)\phi_1 + b(b^2 - 3a^2)\phi_3\} + C_3\{a(a^2 - 3b^2)\,\phi_4 - b(b^2 - 3a^2)\phi_2\} + C_4\{a(a^2 - 3b^2)\phi_3 - b(b^2 - 3a^2)\phi_1\} - 2\tilde{t}_2\{C_1(a\phi_2 - b\phi_4) + C_2(a\phi_1 - b\phi_3) + C_3(a\phi_4 + b\phi_2) + C_4(a\phi_3 + b\phi_1)\} \quad (41)$$

where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are the values of Φ_1, Φ_2, Φ_3 and Φ_4, respectively, at the initial section of the second layer (i.e., at $\tilde{z} = \tilde{H}_1$). Note that the parameters $a = \sqrt{\{½(\sqrt{\tilde{k}_2} + \tilde{t}_2)\}}$ and $b = \sqrt{\{½(\sqrt{\tilde{k}_2} - \tilde{t}_2)}$ in the above equations are different from those of layer 1. After obtaining the expressions of C_1, C_2, C_3 and C_4 from equations (38)—(41), the same procedure as for layer 1 (i.e., substitution of the constants C_1, C_2, C_3 and C_4 in the equations of $\tilde{w}_2$, $\tilde{\theta}_2$, $\tilde{M}_2$, and $\tilde{S}_2$ obtained from equation (2) with $i = 2$ and its successive differentiations, and subsequent comparison of the equations with equations (3)—(6) with $i = 2$ needs to be followed to obtain the influence coefficients of layer 2. Similarly, the influence coefficients of layer 3 can be obtained by solving the following equations for C_1, C_2, C_3 and C_4:

$$\tilde{w}_0^{(3)} = C_1\phi_1 + C_2\phi_2 + C_3\phi_3 + C_4\phi_4 \quad (42)$$

$$\tilde{\theta}_0^{(3)} = C_1(a\phi_2 - b\phi_4) + C_2(a\phi_1 - b\phi_3) + C_3(a\phi_4 + b\phi_2) + C_4(a\phi_3 + b\phi_1) \quad (43)$$

$$0 = C_1\{(a^2 - b^2)\,\phi_1 - 2ab\phi_3\} + C_2\{(a^2 - b^2)\,\phi_2 - 2ab\phi_4\} + C_3\{(a^2 - b^2)\phi_3 + 2ab\phi_1\} + C_4\{(a^2 - b^2)\phi_4 + 2ab\phi_2\} \quad (44)$$

$$\sqrt{2\tilde{k}_3\tilde{t}_4}\,\tilde{w}_0^{(3)} = C_1\{a(a^2 - 3b^2)\phi_2 + b(b^2 - 3a^2)\phi_4\} + C_2\{a(a^2 - 3b^2)\phi_1 + b(b^2 - 3a^2)\phi_3\} + C_3\{a(a^2 - 3b^2)\phi_4 - b(b^2 - 3a^2)\phi_2\} + C_4\{a(a^2 - 3b^2)\phi_3 - b(b^2 - 3a^2)\phi_1\} - 2\tilde{t}_3\{C_1(a\phi_2 - b\phi_4) + C_2(a\phi_1 - b\phi_3) + C_3(a\phi_4 + b\phi_2) + C_4(a\phi_3 + b\phi_1)\} \quad (45)$$

where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are the values of Φ_1, Φ_2, Φ_3 and Φ_4, respectively at $\tilde{z} = 1$; and $a = \sqrt{\{½(\sqrt{\tilde{k}_3} + \tilde{t}_3)\}}$ and $b = \sqrt{\{½(\sqrt{\tilde{k}_3} - \tilde{t}_3)}$.

Once the influence coefficients are known, the unknown initial parameters are determined by solving the matrix equation (equation (15)). The initial parameters and influence coefficients are then used to find the normalized pile deflection, slope, bending moment and shear force at any pile section by using equations (7)—(10) for layer 1, (3)—(6) for layer 2 (with $i = 2$), and (11)—(14) for layer 3.

4 SOLUTION ALGORITHM AND NUMERICAL EXAMPLE

In order to obtain pile response, the soil resistances k and t need to be known. Expressions for $\tilde{k}_i$ and $\tilde{t}_i$ have been obtained using the principle of minimum potential energy (Basu and Salgado 2007a):

$$\tilde{k}_i = \frac{9\pi G_{si}(1+0.75\upsilon_{si})L_p^4}{4E_pI_p}\frac{[K_1(\gamma)+\gamma K_0(\gamma)]^2-(\gamma^2+1)[K_1(\gamma)]^2}{[K_0(\gamma)]^2} \tag{46}$$

$$\tilde{t}_i = \begin{cases} \dfrac{3\pi r_p^2 G_{si}(1+0.75\upsilon_{si})L_p^2}{8E_pI_p}\dfrac{[K_1(\gamma)]^2-[K_0(\gamma)]^2}{[K_0(\gamma)]^2}; & i=1,2,\dots n \\ \dfrac{3\pi r_p^2 G_{sn}(1+0.75\upsilon_{sn})L_p^2}{8E_pI_p}\dfrac{[K_1(\gamma)]^2}{[K_0(\gamma)]^2}; & i=n+1 \end{cases} \tag{47}$$

where G_{si} and υ_{si} are the shear modulus and Poisson's ratio of the i^{th} soil layer; r_p is the radius of the pile cross section, $K_0(\cdot)$ and $K_1(\cdot)$ are the zero-order and first-order modified Bessel function of the second kind; and γ is a dimensionless parameter given by:

$$\frac{\gamma^2}{\psi^2} = \frac{\sum_{i=1}^{n}(1+0.75\upsilon_{si})G_{si}\int_{\tilde{H}_{i-1}}^{\tilde{H}_i}\left[\frac{d\tilde{w}_i}{d\tilde{z}}\right]^2 d\tilde{z} + (1+0.75\upsilon_{sn})G_{sn}\sqrt{\frac{\tilde{k}_n}{8\tilde{t}_{n+1}}}[\tilde{w}_n|_{\tilde{z}=1}]^2}{1.5\{\sum_{i=1}^{n}(1+0.75\upsilon_{si})G_{si}\int_{\tilde{H}_{i-1}}^{\tilde{H}_i}\tilde{w}_i^2 d\tilde{z} + (1+0.75\upsilon_{sn})G_{sn}\sqrt{\frac{\tilde{t}_{n+1}}{2\tilde{k}_n}}[\tilde{w}_n|_{\tilde{z}=1}]^2\}} \tag{48}$$

where ψ is the pile slenderness ratio.

An iterative technique (Basu and Salgado 2007a) is used to solve the problem because the parameter γ is not known a priori. Assuming an initial value of γ, the pile deflection, slope, bending moment and shear force are calculated applying MIP as the first iteration. At the end of the iteration, γ is calculated by performing integrations of the squares of calculated pile deflections and slope according to equation (48) and compared with the assumed initial value of γ. If the difference between the two values is more than the tolerable limit, a second iteration is performed. Iterations are continued until the values of γ obtained from two consecutive iterations fall below the prescribed limit, which we prescribe as $|\gamma_{i+1}-\gamma_i|<0.001$. An initial value of $\gamma = 1$ was used in the calculations. The final solutions were obtained in seconds and took not more than six iterations.

We consider an example problem of a four-layer soil profile with the thickness of the top three layers being 2 m, 3 m and 3 m, respectively. The pile, which is assumed to be 10-m long with 0.6 m diameter, is embedded into the fourth layer, which extends to great depth. $H_1 = 2$ m, $H_2 = 5$ m, and $H_3 = 8$ m; $E_{s1} = 20$ MPa, $E_{s2} = 35$ MPa, $E_{s3} = 50$ MPa and $E_{s4} = 80$ MPa (E_{si} is the Young's modulus of the i^{th} layer; $G_{si} = E_{si}/2(1+\upsilon_{si})$); $\upsilon_{s1} = 0.35$, $\upsilon_{s2} = 0.25$, $\upsilon_{s3} = 0.2$ and $\upsilon_{s4} = 0.15$. A force $F_a = 300$ kN is applied at the head and the pile modulus assumed is $E_p = 24 \times 10^6$ kPa. Figures 2(a) and (b) show the pile deflection, bending moment and shear force, respectively. Also plotted in Figure 2(a) is the deflection profile of a three-dimensional finite element (FE) analysis of the problem. The results match reasonably well (the difference between the head deflection values is less than 10%).

5 CONCLUSIONS

An analytical solution was presented for the problem of a laterally loaded pile embedded in a layered soil medium. The method of initial parameters was modified to obtain the pile deflection, slope, bending moment and shear force for a laterally loaded pile embedded in a layered soil medium. The method was illustrated using an example of a laterally loaded pile embedded in a three-layer soil deposit. An example problem has been solved and compared with finite element analysis; it is evident that this method of solution produces satisfactory results and provides important insights into pile response.

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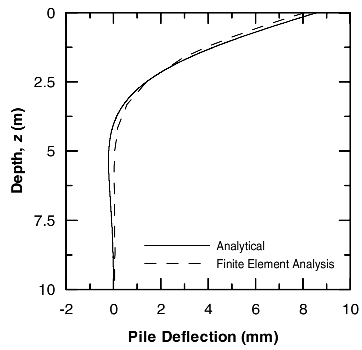
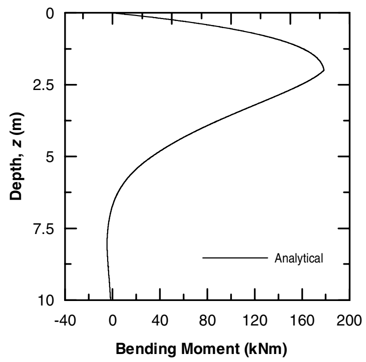


(a)



(b)

Figure 2. (a) Deflection and (b) bending moment of a 10-m long laterally loaded pile.