Rigid plastic shakedown analysis and its application for a bearing capacity problem of a multi-footing system

Analyse de shakedown et cette application au probléme de la capacité portante de la structure avec multi-semelles

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ABSTRACT: Bearing capacity characteristics of a multi-footing system under various repeated loads are investigated. To consider the effects of repeated loads on the bearing capacity characteristics, rigid-plastic shakedown analysis is proposed and investigated in this paper. This method is recognised as an approximation of elasto-plastic shakedown analysis of static approach. Numerical calculation is carried out to check its validity. This method may be a good estimator in practical cases.

RÉSUMÉ: Caractères de capacité portante d'un structure avec multi-semelles sous des forces variétés et répétés sont recherchés. Analyse de shakedown rigide et plastique est proposé pour exprimer un effet des forces répétés. Ce méthode est traité en l'approximation d'analyse de shakedown élastique et plastique. Des calculs numériques sont faits pour montrer ses validités. On espère que ce méthode est utile pour des pratiques en ingénierie.

1 INTRODUCTION

A bearing capacity problem of a multi-footing system, which consists of several footings in its foundation is investigated. A typical example of a multi-footing system is an oil platform structure in ocean areas. This kind of structure is subject to severe loading conditions; external forces due to waves, tides, floating ices etc. are always acting repeatedly in a complicated manner. Therefore, we aim to establish a concise but quantitative estimation method for a bearing capacity problem subject to enormous repeated and combined loads.

Recently, many experimental studies have been carried out to express bearing capacity characteristics of a single shallow foundation which is subject to combined loads. In order to express these characteristics concisely, a yielding locus in a generalised loading space is sometimes introduced (Georgiadis & Butterfield 1988, Tan 1990, Nova & Montrasio 1991, Houlsby & Martin 1993, Butterfield & Gottardi 1994, Sekiguchi & Kobayashi 1994, Dean et al. 1993, Dean et al. 1997a, Bransby & Randolph 1999). This locus separates loading conditions in the following two situations; (i) If a loading condition is within this yielding locus, a foundation behaves safely. However, (ii) if a loading condition is on the locus, a foundation behaves plastically. A general form of this yielding locus in the generalised loading space is as follows,

$$f(V, H, M/B) = 0, \tag{1}$$

where V, H and M are vertical, horizontal and momentum loads acting on a foundation, respectively, and B means a width of a foundation. Based on many experimental evidences and theoretical considerations, a concrete form of this locus has been revealed.

On the contrary, a bearing capacity problem of a multi-footing system has not been investigated enough. By centrifugal experiments (Dean et al. 1997b) or conventional limit analysis calculations (Murff 1994), they presented a quantitative estimation method for its bearing capacity. However, an enormously repeated loading condition, which is typical in ocean areas, was not considered in their studies. This severe loading condition is very important, especially for ocean engineering practices.

To this end, a concise and quantitative estimation method for bearing capacity characteristics under repeated and combined loads is proposed and investigated in this study.

2 PROPOSED ESTIMATION METHOD

2.1 *Basic ideas and assumptions*

There may be two ways to deal with the effects of repeated loading on the bearing capacity characteristics. One way is the elasto-plastic step by step calculation in the time domain. This approach is widely used in many engineering problems. However, as we want to deal with enormous numbers of repeated loads in various manners, it may be difficult to use this method directly. On the contrary, the other way is the shakedown analysis based on shakedown theorems (Martin 1975, Maier 1979). Because the step by step calculation is not necessary for shakedown analysis, shakedown analysis is suitable for the loading condition of our interest. It has also advantages in calculation time and cost. For this reason, we use shakedown analysis in this paper.

Shakedown is defined as follows. Let us consider a structure is subject to repeated loads. Although at the beginning stage of loading, plastic deformation occurs, and the response of a structure becomes elastic after certain numbers of loading cycles. If such behaviour is observed, we call that this structure *(elastically) shakes down*. We can say that a structure is stable if it shakes down, because the response becomes elastic and no more plastic deformation is accumulated. Thus, it is important to estimate a maximum loading level which causes shakedown response, from the point of stability of a structure under repeated loads.

The following assumptions are used in this study.

- A structure has plural footings. Bearing capacity character-

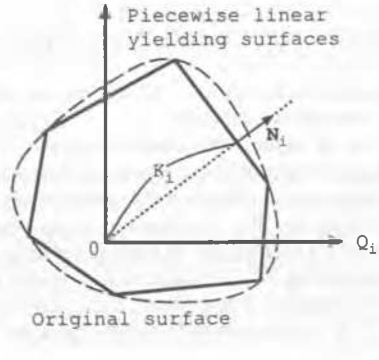

Figure 1. Illustration of piecewise linear yielding locus.

istics of each footing can be expressed as a yielding locus in a generalised loading space. Behaviour of the footings is independent of each other.

- External loads acting on the upper part of the structure are distributed to each footing. The upper structure is assumed to be rigid. Therefore, the deformation of the upper structure can be negligible for the distribution of the external loads to the footings.

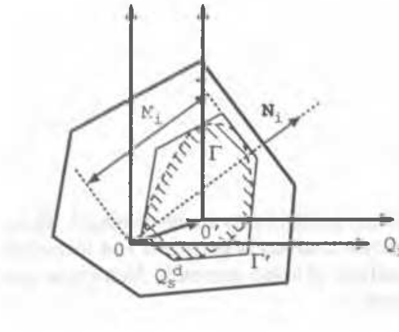

Figure 2. Definition of region Γ'.

- The elastic deformation of a footing is negligible.

Under these assumptions, the distribution of external loads to each footings can be estimated statically by equilibria of forces and moments and yielding conditions of each footing. Our main concern is how to estimate the maximum shakedown limit. Static approach of shakedown analysis based on Melan's theorem is adopted. Concrete formulation is presented in the next section.

2.2 *Formulation*

At first, a general form of static shakedown analysis for elasto-plastic problems is explained. Secondly, rigid plastic shakedown analysis is presented.

Let vector Q be generalised stresses, which expresses forces and moments acting on each footing. Let vector F be external forces acting on the structure. Then, equilibria of forces and moments can be expressed as

$$C^T Q = F. \tag{2}$$

Moreover, external forces can be decomposed into two components; steady term $F_s(s)$ and fluctuated term $F_f(s,t)$,as

$$F(s,t) = F_s(s) + \alpha F_f(s,t), \tag{3}$$

where α is a loading factor, vector s is a position vector in the body D and a scalar t is time.

Let Q^e, Q^e_f and Q^r be elastic stress solutions for steady loads and fluctuated loads, and residual stresses equilibrating to zero external forces, respectively. Then, a following statement can hold from Melan's theorem. If we can find a certain time-independent residual stress field with zero external forces Q^r, which satisfies the following inequality everywhere in the body D for all the time t,

$$f(Q) = f\left(Q^e_s(s,t) + \alpha Q^e_f(s,t) + Q^r(s)\right) \le 0,\ C^T Q^r(s) = \mathbf{0}. \tag{4}$$

Then, a structure shakes down. This is an important result of elasto-plastic shakedown analysis.

However, as we assume that elastic response of each footing is negligible, it is impossible to calculate elastic stress solutions. Thus, we decide to use a minimum norm solution Q^d of equilibria of forces and moments (Eq. 2) instead of elastic stress solutions. The minimum norm solution can be determined uniquely by a generalised inverse matrix $(C^T)^-$ (Hangai & Kawaguchi 1991). As the minimum norm solution is also linear, it can be decomposed into two terms; $Q^d_s(s)$ corresponding to steady loads and $Q^d_f(s,t)$ corresponding to fluctuated loads. Moreover, the residual stress field $Q^r(s)$ equilibrating to zero external forces is possible for statically admissible stress field. These stress fields satisfy the following equations, respectively.

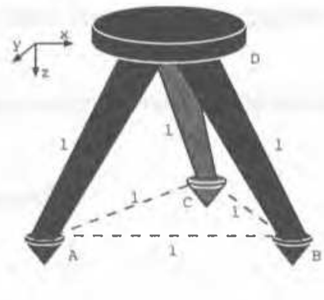

Figure 3. Targeted structure a for numerical example.

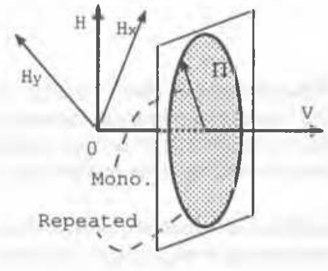

Figure 4. Loading path for shakedown analysis.

$$Q^d_s(s) = (C^T)^- F_s,\ Q^d_f(s,t) = (C^T)^- F_f(t),\ C^T Q^r = \mathbf{0} \tag{5}$$

By analogy to Melan's theorem, a following approximation method to evaluate shakedown limit loads of a rigid-plastic problem is proposed in this study.

Find $\alpha \to \max$ subject to $\forall s \in V$ and $\forall t$

$$f(Q) = f(Q^d_s + \alpha Q^d_f + Q^r) \le 0,\quad C^T Q^r = \mathbf{0} \tag{6}$$

From the theoretical point of view, this approximation method loses uniqueness and existence of exact solutions. Theoretical interpretation of this approximation method will be discussed later in this paper.

For the simplicity of analysis, a nonlinear yielding function of a footing $f(Q) \le 0$ is replaced by piecewisely linearised yielding planes which are inscribed to the original surface. This piecewise linearisation is illustrated in (Fig. 1). Piecewisely linearised yielding loci are expressed as,

$$N^T Q = N^T Q^d_s(s) + \alpha N^T Q^d_f(s,t) + N^T Q^r(s) \le K, \tag{7}$$

where tensor N^T is an assembly of unit normal vectors of each plane and vector K denotes the strength of each plane. Furthermore, a fluctuated term of stresses $Q^d_f(s,t)$ consisting of a region Γ in (Fig. 2) is replaced by a circumscribed region Γ' which is independent of time to avoid optimisation in the time domain. Shift of coordinates due to the steady loads $N^T Q^d_s$ is also illustrated in (Fig. 2). Region Γ' is mathematically defined as

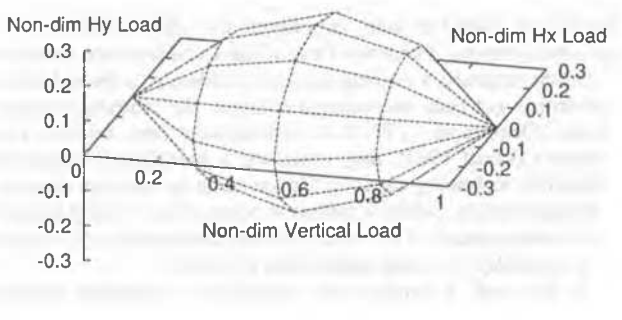

Figure 5. Piecewisely linearised yielding locus.

$$\max\left[N^T Q_f^d(s,t)\right] = M(s),\ \forall t \tag{8}$$

where notation $\max[a(t)]$ takes the maximum value of each component of vector a with time.

Finally, a proposed approximation method to evaluate shakedown limit loads for a rigid-plastic problem is formulated as linear programming in the following form, where unknowns of this method are a loading factor α and a residual stress field Q^r.

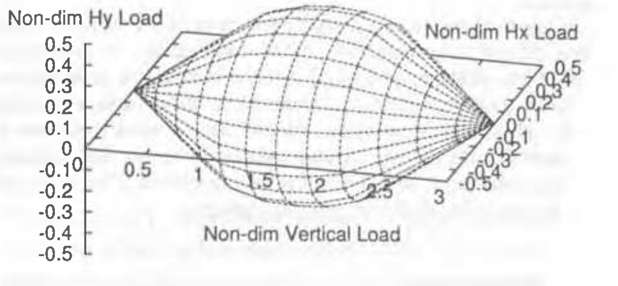

Figure 6. Shakedown limit for repeated loads.

Find $\alpha \rightarrow \max$ subject to $\forall s \in D$

$$\alpha M(s) + N^T Q_s^d(s) + N^T Q^r(s) - K \leq 0,\ C^T Q^r = 0. \tag{9}$$

3 NUMERICAL EXAMPLES

3.1 *Target structure*

Three footing systems shown in (Fig. 3) are considered. Each member with the length l is connected rigidly at node D. Each footing and member are hinge connected to conduct only forces to the footings. Moment loads acting on the footings can be negligible if a dimension of a structure is sufficiently bigger than a size of a footing. For this case, moments due to the vertical and horizontal loads are dominant in the equilibrium of moments of a system.

External loads F are assumed to act only on node D. Load paths of repeated loads considered here are variable horizontal loads within a certain magnitude under constant vertical loads. For a repeated loading case, loads should exist at any point within the hatched region Π with a constant radius in the H_x-H_y plane. On the other hand, monotonic loading which is mainly considered for limit analysis cases are also illustrated in (Fig. 4).

Schematic explanation of shakedown analysis is as follows, we will find the maximum circular area shown in (Fig. 4) as a region Π under a certain constant vertical load V. Contrary to shakedown analysis, we will find the maximum horizontal load in a certain direction to evaluate the ultimate loads in limit analysis calculation. These kinds of analysis will be carried out at various vertical load levels to express the bearing capacity characteristics as closed convex surfaces in a generalised load domain.

Yielding locus of a footing has a following form,

$$(H_x/H_m)^2 + (H_y/H_m)^2 = 16(V/V_m)^2 \cdot (1 - V/V_m)^2, \tag{10}$$

where subscript m implies the maximum loads obtained from monotonic loading tests. This expression of a yielding locus is originally based on the two dimensional experiments (Houlsby & Martin 1993) and developed simply to deal with two horizontal components. As a linearised yielding locus, a polyhedron with 48 planes shown in (Fig. 5) just inscribed to the original locus (Eq. 10) is used in this numerical calculation. Strength ratio of H_m/V_m is assumed to be 0.2, which is typically observed in experiments.

3.2 *Numerical results*

Results are arranged in the non-dimensional form. Non-dimensional vertical load and non-dimensional horizontal load are defined as V/V_m, $H_x/(4.0H_{xm})$ or $H_y/(4.0H_{ym})$, respectively.

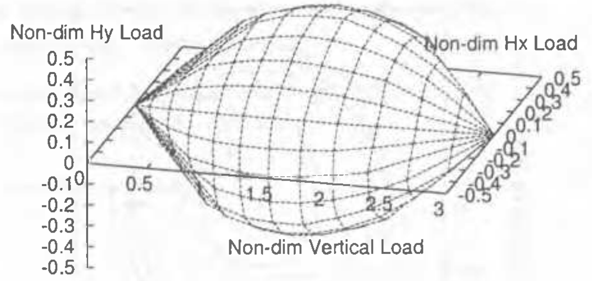

Figure 7. Ultimate limit for monotonic loading.

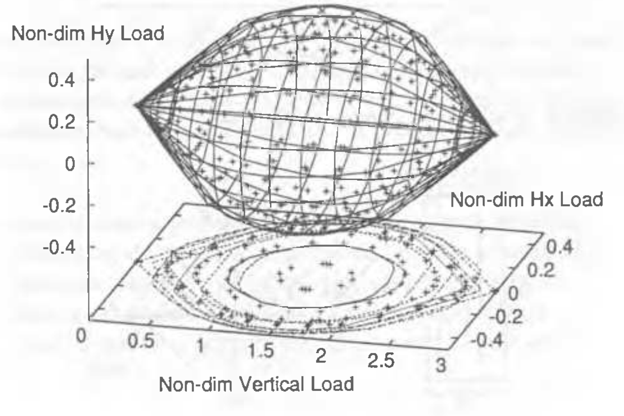

Figure 8. Comparison of shakedown limit and ultimate limit.

Shakedown limit for repeated loads is shown in (Fig. 6). Shape of this shakedown limit is like a cigar, axially symmetric to the vertical load V-axis and symmetry to the $V/V_m = 1.5$ plane. The maximum horizontal capacity $H/(4.0H_m) = 0.416$ appears at the vertical load $V/V_m = 1.5$. On the contrary, ultimate limit for monotonic loading is shown in (Fig. 7).

The maximum horizontal capacity appears at the vertical load $V/V_m = 1.5$, nevertheless its value depends on a direction of a load. If a horizontal load is applied in the H_x-direction, the maximum horizontal capacity is $H/(4.0H_m) = 0.482$. On the other hand, if it is in the H_y-direction, the maximum value is $H/(4.0H_m) = 0.499$.

In order to compare the shakedown limit state and ultimate limit state, these two results are superimposed in (Fig. 8). The ultimate limit state is illustrated as solid frames, whereas the shakedown limit state is plotted as symbols + in (Fig. 8). We can recognise that the shakedown limit state is always located within the ultimate limit surface. Cross sectional view of these two surfaces is shown in (Fig. 9). Solid lines and broken lines imply ultimate limit states and shakedown limit states at constant vertical loads, respectively. There are three pairs of lines which show the limit states at V/V_m = 1.5, 2.0, 2.5 in (Fig. 9), respectively. It can be understood that reduction ratios of horizontal capacities due to repeated loading are depending on the vertical load levels from this figure. These ratios are within the range of 0.6 ~ 0.8 in this example.

4 DISCUSSION

At first, theoretical interpretation of rigid-plastic shakedown analysis for rigid-perfectly plastic media is considered. Let us consider one-dimensional two bar truss example shown in (Fig. 10). If external loads $F(t)$ are acting repeatedly, equilibrium of forces is simply expressed as $N_a + N_b = F(t)$, where N_i means the i-th member force. There are infinite pairs of solutions for this equilibrium, shown as lines $l(t)$ in (Fig. 10). Loci of minimum norm solutions shown as lines **ab** or **a'b'** must be perpendicular to these equilibrium lines $l(t)$. However, loci of elastic solutions are gen-

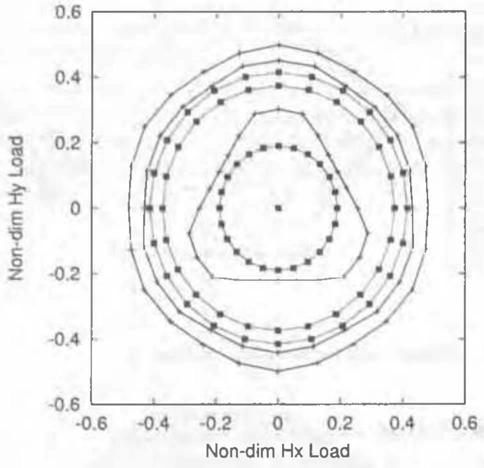

Figure 9. Cross sectional view of two limit states.

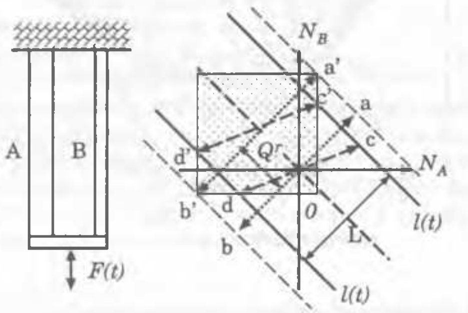

Figure 10. Schematic explanation of shakedown analysis for one-dimensional two-bar-truss example.

erally different from those of minimum norm solutions, say lines **cd** or **c'd'**, because elastic solutions must satisfy the compatibility conditions of deformation at the same time. Let a hatched area be an elastic state of member forces. Then, we estimate a maximum amplitude of repeated loads shown as width L in (Fig. 10) in elasto-plastic shakedown analysis. If we can find a shift stress Q^r from the origin along a broken line (= equilibrium to zero external force) to keep the loci of elastic solutions **c'd'** within the hatched area (= elastic state of member forces) under a certain amplitude L. Then, this truss will shake down.

If we use minimum norm solutions instead of elastic solutions, we will overestimate a shakedown limit. From the schematic explanation in (Fig. 10), it is easily understood that the amplitude of repeated loads L obtained from minimum norm solutions is always greater than the amplitude based on the elastic solutions. For this reason, rigid-plastic shakedown analysis presented in this paper must be recognised as an approximation method.

Secondly, we discuss some merits of this presented method. Let us assume that an elastic response of a footing can be expressed

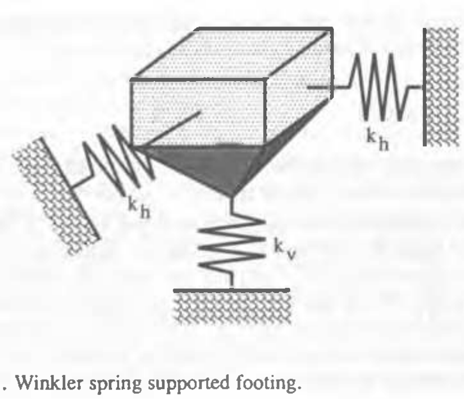

Figure 11. Winkler spring supported footing.

as a combination of linear Winkler springs in vertical and horizontal directions, shown in (Fig. 11). As we tried to calculate the differences between these elastic solutions and minimum norm solutions, we can say fortunately that minimum norm solutions are equal to elastic solutions under only vertical and horizontal loading for this numerical example. We guess that minimum norm solutions are good approximations of elastic solutions if behaviour of footings is similar and distribution of external loads is not so localised.

In practical engineering, elastic response of a footing is sometimes difficult to be measured due to nonlinearity. In such cases, rigid-plastic shakedown analysis proposed here is a good estimator for the bearing capacity characteristics under various repeated loads. This method requires less calculation work and time in comparison with the step by step calculation in the time domain, which is sometimes very difficult to be carried out. Therefore, this concept might be useful for practical designs.

5 CONCLUSIONS

Rigid-plastic shakedown analysis for bearing capacity characteristics of a multi-footing system under various repeated loads is proposed and investigated in this paper. The proposed method should be recognised as an approximation method of elasto-plastic shakedown analysis. However, this method has advantages in calculation time and cost in comparison with step by step calculations in the time domain, which is widely used in many elasto-plastic numerical calculations. Therefore, it might be useful for practical designs.

REFERENCES

Bransby, M. F. & Randolph, M. F. 1999. The effect of embedment depth on the undrained response of skirted foundations to combined loading. *Soils and Foundations* 39(4): 19–33.

Butterfield, R. & Gottardi, G. 1994. A complete three-dimensional failure envelope for shallow footings on sand. *Géotechnique* 44(1): 181–184.

Dean, E. T. R., James, R. G., Schofield, A. N., Tan, F. S. C. & Tsukamoto, Y. 1993. The bearing capacity of conical footings on sand in relation to the behaviour of spudcan footings of jackups. In *Predictive soil mechanics*: 230–253. London: Thomas Telford.

Dean, E. T. R., James, R. G., Schofield, A. N. & Tsukamoto, Y. 1997a. Theoretical modelling of spudcan behaviour under combined load. *Soils and Foundations* 37(2): 1–15.

Dean, E. T. R., James, R. G., Schofield, A. N. & Tsukamoto, Y. 1997b. Numerical modelling of three-leg jackup behaviour subject to horizontal load. *Soils and Foundations* 37(2): 17–26.

Georgiadis, M. & Butterfield, R. 1988. Displacements of footings on sand under eccentric and inclined loads. *Canadian Geotechnical Journal* 25: 199–212.

Hangai, Y. & Kawaguchi, K. 1991. *Shape analysis.* Tokyo: Baifu-Kan. (in Japanese)

Houlsby, G. T. & Martin, C. M. 1993. Modelling of the behaviour of foundations of jack-up units on clay. In *Predictive soil mechanics*: 339–358. London: Thomas Telford.

Maier, G. 1979. Shakedown Analysis. In *Engineering Plasticity by Mathematical Programming*: Chap. 6. New York: Pergamon Press.

Martin, J. B. 1975. *Plasticity.* Cambridge: MIT Press.

Murff, J. D. 1994. Limit analysis of multi-footings foundation systems. In *Proc. of the 8th Int. Conf. on Computer Methods and Advances in Geomechanics*: 223–244.

Nova, R. & Motrasio, L. 1991. Settlements of shallow foundations on sand. *Géotechnique* 41(2): 243–256.

Sekiguchi, H. & Kobayashi, S. 1994. Limit analysis on the bearing capacity characteristics of a near shore structures located on a clay deposit. In *Proc. of 39th Geotechnical Symposium*: 195–202. JCSMFE. (in Japanese)

Tan, F. S. C. 1990. Centrifuge and theoretical modelling of conical footings on sand. *Ph. D. Thesis*, Univ. of Cambridge.