Interaction between two piles inclined in any direction

Interaction entre deux pieux inclinés dans une direction quelconque

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ABSTRACT: The boundary elements method has been used to analyze groups of raked piles in homogeneous soil by Banerjee & Driscoll (1976) and in heterogeneous ground by Banerjee and Davis (1980). In both cases the method is cumbersome and lengthy, and more adequate for research than for applications. The principle of superposition (based upon the interaction between two piles) has been used to reduce the number of equations (v. Pichumani & D'Appolonia, 1967). In this paper the interaction between two piles inclined in any direction in a homogeneous soil has been solved. It is shown that Poulos & Davis (1980) and Randolph's (1980) hypotheses with respect to this interaction are not fulfiled.

RÉSUMÉ: La méthode des élements de frontière a été apliquée par Banerjee et Davis (1976) à l'analyse de groupes de pieux inclinés dans un sol homogène et par Banerjee et Davis (1980) dans un sol sol heterogène. Dans les deux cas la méthode est ennuyante, d'exécution lente, et plus approprié pour la recherche que pour les applications practiques. Pur réduire le nombre des équations, le principe de superposition basé sur l'interaction ente deus pieux, a été employé (v. Pichumani et D'Appolonia, 1967). Dans cette comunication on a résolu, l'interaction entre deux pieux inclinés dans n'importe quelle direction, dans un sol homogène. On a démontré que les hypothèses de Poulos et Davis (1980) et Randolph (1980) sur l'interaction ne sont sont pas accomplis

1 INTRODUCTION

One established procedure to analyze the displacements and stresses in a single pile or a group of piles is based upon the assumption of linear elastic behaviour. The elastic problem may be solved by the boundary element method. The surface of the pile is divided into boundary elements (Figure 1). The stress at each element is assumed constant, and the displacements of the soil and the pile at the centre of the elements are equalized. In a homogeneous soil Mindlin equations are used to find the displacements produced by stresses at the boundaries. This method was initiated by Salas & Belzunce (1965) and Thurman & D'Appolonia (1965) in a single vertical pile, and has been developed by Poulos, Davis and coworkers through several important papers since 1968.

Banerjee and Driscoll (1976) have used this method for the analysis of groups of raked piles in homogeneous soil and Banerjee and Davis (1980) in heterogeneous ground. In both cases the method is cumbersome and lengthy and more adequate for research than for applications.

To apply this method to more than two piles, several simplifying assumptions are required to reduce the number of equations. Pichumani & D'Appolonia (1967) and Poulos & Davis (1968) also solve the problem of interaction between two vertical piles under vertical load using the boundary element method, and Poulos (1971) solves the same problem under lateral load. For a group of piles they apply the principle of superposition, and, in this way, drastically reduce the number of equations. This principle is not rigorously correct, because the addition of a pile involves a change in the stiffness of the overall elastic system. Solutions or symmetrical groups show that the principle of superposition is almost exact (Poulos & Davis, 1980) for displacements, although there are slight alterations in the shear stress distribution, and the proportion of load taken by the base increases as the number of piles in the group increases.

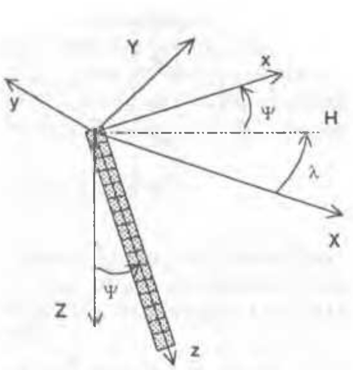

Figure 1. Local and general axes

The applicability of the superposition principle to symmetrical groups suggests that it may be applied to general pile groups.

Poulos & Davis (1980) study the interaction between two piles raked in the same plane, and reduce it to the problem of two equivalent vertical piles by making assumptions about the magnitude and direction of the head displacements. It is assumed that an axial load on pile *j* will cause a deflection of pile *i* that is in the axial direction of pile *i*, and equal to the axial deflection of pile *j* under this axial load multiplied by the interaction factor, supposing that both piles are vertical. Similarly, it is assumed that a normal load on pile *j* will cause a deflection of pile *i*, that is in the normal direction of pile *i*, and equal to the normal deflection of pile *j* under this normal load multiplied by the interaction factor, also supposing that both piles are vertical. To solve the problem of the change of distance between two raked piles with depth, Poulos & Davis (1980) assume that the *equivalent* distance is the centre–to-centre distance at one third the vertical depth of the piles. The interaction between two vertical piles is solved by Randolph & Wroth (1979) for vertical loads, using an approximate closed form solution. Randolph (1981), uses simple algebraic solutions obtained from finite element studies for horizontal loads. The principle of superposition is also applied for the group of piles. Randolph (1980) reduces the problem of interaction between two battered piles to the interaction between vertical piles, adopting the assumption of Poulos & Davis (1980) with respect to the direction of displacement, but taking the distance at the top as equivalent distance.

The true interaction between two piles inclined in any direction, in a homogeneous linear-elastic soil, has been solved by the authors of this paper using the boundary element method. This has allowed comparison with results obtained using Poulos & Davis (1980) and Randolph's (1980) hypotheses.

2 METHOD OF ANALYSIS

The soil is modeled after a Boussinesq half space with an elasticity modulus E_s and a Poisson's ratio ν. There are two piles extending from the surface of the soil of the same diameter, d, length, L, and elasticity modulus, E_p. It is assumed that the presence of the piles does not modify the soil properties, so that Mindlin equations continue to be valid, and there is rough contact between the soil and piles.

It is assumed that the axial displacement under axial load or the normal displacement under a normal load or a moment at the top are the same in a single raked pile as in a vertical pile. Poulos and Madhav (1971) have compared this method with the procedure of decomposing the unknown force at each element into vertical and horizontal forces, finding the displacements at the centre of each element using Mindlin equations and recomposing them. For rakes up to 30° the difference is very small. It is assumed that normal and axial stresses only produce displacements in its direction (v. Poulos & Davis, 1980).

2.1 *Displacement of a single raked pile*

Figure 1 shows the battered pile, and the sets of perpendicular local axes (x,y,z) and general axes (X,Y,Z). The z axis lies in the axial direction of the pile, the Z axis is vertical and H is the projection of z on the horizontal plane. The load acting on the head of the pile is decomposed into its components following the local axes.

As indicated under section 2, the axial displacement under axial load is treated as if the pile were vertical. It is necessary to consider only the compatibility of axial displacements (v. Mattes and Poulos, 1969). The shaft is divided into n equal cilindrical elements (Fig. 1) and the base into m circular crowns of equal area.

In this paper, the calculation scheme of Mattes & Poulos (1969) for a compressible pile is followed, with some variations indicated below.

The displacement of the soil at the centre of an element is:

$$u_{ki} = \sum_{j=1}^{n+m} I_{ij}^{k} p_{kj} \qquad k=z \tag{1}$$

For $j \le n$ $\quad p_{zj}$ = shear stress at j element
For $j > n$ $\quad p_{zj}$ = normal stress at j element
I^z_{ij} = Mindlin function

The equilibrium of the differential pile element subject to an axial load gives:

$$\frac{d^2u_z}{dz^2} = \frac{4p_z}{dE_p} \tag{2}$$

where:
u_z = axial displacement
p_z = shear stress on pile surface

Integrating equation (2) twice, an integral equation is obtained.

Introducing the boundary conditions at the top of the pile, integration constants are found. If the stress at each element is now assumed constant, the displacement at the centre of the i element is obtained (Sánchez Langeber, 2000):

$$u_{zi} = u_{z0} - \frac{4q_z\delta}{\pi d^2E_P}a_i - \frac{4\delta^2}{dE_P}\sum_{j=1}^{n+m}A_{zij}p_{zj} \tag{3}$$

where u_{z0} and q_z are the axial displacement and load at the top of the pile, a_i and A_{zij} coefficients depending upon i and j (v. Sánchez Langeber, 2000), and $\delta = L/n$.

Equalizing the soil and pile displacements (1 and 3), $n+m$ equations are obtained. Using the equilibrium equation they allow the $n+m+1$ unknowns (p_{zj} and u_{z0}) to be found.

Now the displacements of the pile in the x direction will be found, which is subject to a load q_x in this direction and a moment, m_x, of axis y. As indicated in section 2, the x displacement is treated as if the pile were vertical. The calculation scheme of Poulos (1971a) is followed, with some variations indicated below. The x displacement of the soil will be given by equation (1) for $k=x$.

The equilibrium of the differential pile element subject to a normal load at the shaft, p_x, gives:

$$\frac{d^4u_x}{dz^4} = -\frac{p_xd}{E_PI_P} \tag{4}$$

Integrating equation (4), an integral equation is obtained (v. Sánchez Langeber, 2000). Introducing boundary conditions at the top of the pile, integration constants are found. If the stress p_x at each element is assumed constant, the x displacement at the centre of the i element can be obtained:

$$u_{xi} = u_{x0} + u'_{x0}\,(i-0.5) - \frac{m_x\delta^2}{2E_PI_P}(i-0.5)^2 + \frac{q_x\delta^3}{6E_PI_P}(i-0.5)^3 + \frac{d\delta^4}{E_PI_P}\sum_{j=1}^{n}A_{xij}p_{xj} \tag{5}$$

where u_{x0} is the x displacement at the top of the pile, and A_{xij} is a coefficient dependent upon i and j (v. Sánchez Langeber, 2000).

Equalizing the soil and pile displacements in the x-direction (equations 1 with $k=x$ and 5), n equations are obtained. Combined with the two equilibrium equations they allow the $n+2$ unknowns (p_{xj}, u_{x0} and u'_{x0}) to be found.

Exchanging x for y, the stresses and displacements of the pile in the y direction are obtained.

For small rotation angles, the slope may be substituted by the angle in radians. The rotation angle with respect to the axis will be named $\theta_{y0} = -u'_{y0}$ and with respect to the y axis $\theta_{x0} = u'_{x0}$.

A system formed by the 3n+m+5 equations solves the problem of a raked pile subject to any load on the top and moments with respect to axes perpendicular to the pile. The system may be expressed in matrix form (v. Sánchez Langeber, 2000):

$$[\phi]\{p\backslash u_0\}_{3n+m+5} = [S]\{q\}_5 \tag{6}$$

where the subscript indicates the order of the vectors, $[\phi]$ is a square matrix, $[S]$ a $(3n+m+5)x5$ matrix, and:

$$\{p\backslash u_0\} = \begin{Bmatrix} \{p_I\} \\ u_{x0} \\ \theta_{x0} \\ \{p_y\}_n \\ u_{y0} \\ \theta_{y0} \\ \{p_z\}_{n+m} \\ u_{z0} \end{Bmatrix} \tag{7}$$

$$\{q\}_5 = \begin{Bmatrix} q_x \\ m_x \\ q_y \\ m_y \\ q_z \end{Bmatrix} \tag{8}$$

The solution is:

$$\{p\backslash u_0\} = [\phi]^{-1}[S]\{q\}_5 \tag{8}$$

If only the 5 rows corresponding to movements at the top of the pile are taken, the following equation is obtained (Sánchez Langeber, 2000):

$$\{u_o\}_5 = \begin{Bmatrix} u_{x0} \\ \theta_{xo} \\ u_{y0} \\ \theta_{y0} \\ u_{z0} \end{Bmatrix} = [f^1] \begin{Bmatrix} q_x \\ m_x \\ q_y \\ m_y \\ q_z \end{Bmatrix} \quad (10)$$

The rotation at the top produced by a torsional moment, M_z, has been obtained by Poulos (1975):

$$\theta_{z0} = \frac{I_\phi}{G_s d^3} m_z \quad (11)$$

An analytical expression approximating I_θ has been obtained by Sánchez Langeber (2000). In this way equation (9) may be extended to six variables including θ_{z0} and m_z:

$$\{u_o\}_6 = [f'_1]\{q\}_6 \quad (12)$$

An exchange of axes allows appliance of movements to global coordinates:

$$\{U_o\}_6 = [F'_1]\{q\}_6 \quad (13)$$

$$[F_1] = [\Gamma][f'_1][\Gamma]^{-1} \quad (14)$$

$[\Gamma]$ is a 6x6 overall transformational matrix that allows conversion of coordinates from local to global axes:

$$[\Gamma] = \begin{bmatrix} T & \chi \\ \chi & T \end{bmatrix} \quad (15)$$

where $[\chi]$ is a null matrix, and $[T]$ is the transform matrix for forces.

$$[T] = \begin{bmatrix} \cos\lambda\cos\psi & -\sin\lambda & \cos\lambda\sin\psi \\ \sin\lambda\cos\psi & \cos\lambda & \sin\lambda\sin\psi \\ -\sin\psi & 0 & \cos\psi \end{bmatrix} \quad (16)$$

2.2 *Interaction between two raked piles*

It is assumed that the two piles have the same components in local axes. Soil displacement in the k direction, at the centre of the i element in pile 1 will be:

$$u^1_{ki} = u^{11}_{ki} + u^{12}_{ki} \quad (17)$$

where u^{11}_{ki} and u^{12}_{ki} are the displacements produced by the stresses transmitted by pile 1 (equation 1) and pile 2 respectively.

To find u^{12}_{ki} it must be assumed that both piles are reduced to their axes. The global components of stresses acting on the j element of pile 2 are:

$$P^2_{kj} = \sum_{m=x,y,z} t^2_{km}\, c_{mj}\, p^2_{mj} \quad (18)$$

where $k = X, Y, Z$

t_{km} = km component of the transform matrix (16) of pile 2

c_{mj} = area that multiplies the stress p^2_{mj}

p^2_{mj} = stress at element j of pile 2 in the direction of the m-axis

It is assumed that P^2_{kj} produces displacements in pile 1 only in the k direction. In global coordinates:

$$U^{12}_{ki} = \sum_j G^{12}_{kij} P_{kj} \quad (19)$$

where G^{12}_{kij} is a Mindlin function

Now U^{12}_{ki} is transfered to local coordinates using the inverse of $[T]$, $[L]$. Substituting (19), (18) and (1) into (17):

$$u^1_{ki} = \sum_j I^k_{ij}\, p^1_{kj} + \sum_{m=x,y,z} \sum_j B^{12}_{kmij}\, p^2_{mj} \quad (20)$$

where:

$$B^{12}_{kmij} = \sum_{g=X,Y,Z} l^1_{kg} G^{12}_{gij}\, t^2_{gm}\, c_{mj} \quad (21)$$

where:

l_{kg} = kg component of $[L]$ matrix

Equalizing the soil (20) and pile displacements (3 or 5), 3n+m equations are obtained, that must be added to the five equilibrium equations. The same number of equations is obtained from pile 2. This allows the 6n+2m+10 unknowns (p^1_{ij}, p^2_{ij}, $\{u_o\}^1_5$, $\{u_o\}^2_5$) to be found. In matrix form:

$$[\Phi] \begin{Bmatrix} p^1 \\ u^1_0 \\ p^2 \\ u^2_0 \end{Bmatrix} = \begin{bmatrix} S \\ S \end{bmatrix} \{q\}_5 = [\xi]\{q\}_5 \quad (22)$$

The solution is obtained multiplying both members by $[\phi]^{-1}$.

If only the ten rows containing top movements are taken, and it is supposed that rotation, θ_{zo}, with respect to the z axis is produced only by the m_z moment of each pile, equation (12) may be used. In this way the 6x6 flexibility matrices $[f^{12}{}_1]$ and $[f^{21}{}_1]$ that relate movements and actions at the top in each pile in local coordinates may be found:

$$\{u^1_0\} = [f^{12}_1]\{q\}_6 \quad (23)$$

for pile 2, 1 and 2 are switched.

2.3 *Interaction matrix between two piles*

Applying the principle of superposition, the movement at the top of pile 1 subject to loads $\{q^1\}$ in pile 1 and $\{q^2\}$ in pile 2 will be:

$$\{u^1_0\} = [f^1_1]\{q^1\} + [f^2_1]\{q^2\} \quad (24)$$

where $f^2{}_1$ is the interaction matrix of pile 2 on pile 1.

When $\{q^1\} = \{q^2\}$ then equation (23) holds. Then equalizing expressions (23) and (24):

$$[f^2_1] = [f^{12}_1] - [f^1_1] \quad (25)$$

In global coordinates, the corresponding expressions are:

$$\{U^1_0\} = [F^1_1]\{Q^1\} + [F^2_1]\{Q^2\} \quad (26)$$

$$[F^2_1] = [\Gamma_1][f^2_1][\Gamma_2]^{-1} \quad (27)$$

3 RESULTS AND DISCUSSION

So as to test the hypotheses presented by other authors, the case in which two piles are inclined in the same plane will be presented here. The rake angle will be Ψ in both piles but in opposite directions.

Figures 2 and 3 show the interaction on pile 2 of normal and axial forces (respectively) acting on pile 1, where s_z is the settlement of a single pile under a unit axial load, for s/d = 25 and K = 1000, where

$$K = \frac{E_p I_p}{E_s L^4} \quad (28)$$

The full line results are presented for two values of L/d, and rake angles of 0°, 5°, 10°, 15° and 20°, and compared with the hypotheses of Poulos & Davis (1980) and Randolph (1980), represented with broken lines.

The vertical displacement under axial load nearly coincides with that obtained through the hypotheses of Poulos & Davis (1980), and the results for Ψ = 0 evidently coincide. With this exception, the results widely diverge, specially with those of Randolph (1980). Pile length has little influence in the vertical displacements under normal using the hypotheses of Davis &

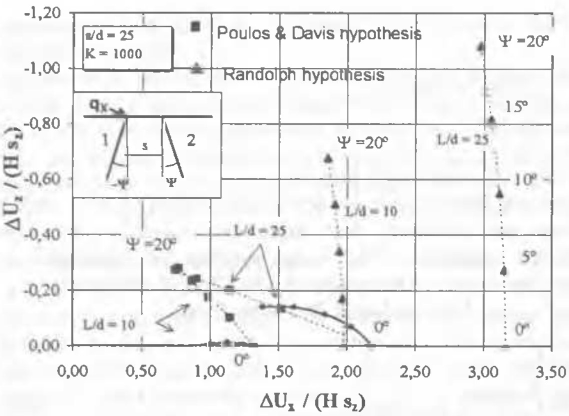

Figure 2. Comparison between the displacements, obtained with different hypotheses, at the top of pile 2 produced by a normal force acting on pile 1.

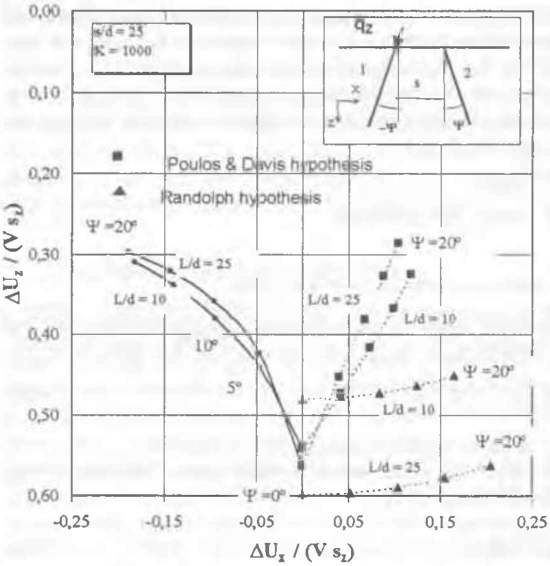

Figure 3. Comparison between the displacements, obtained with different hypotheses, at the top of pile 2 produced by an axial force acting on pile 1.

Poulos (1980), but is very significant in the authors method (Figure 2). Both methods predict similar horizontal displacements under axial load, but in opposite directions (Figure 3)

Figure 4 shows the same results obtained by the authors in Figure 2, but for $L/d = 25$, K = 1000 and different values of s/d. Figure 5 shows the influence of a variation in s/d on the interaction under axial load.

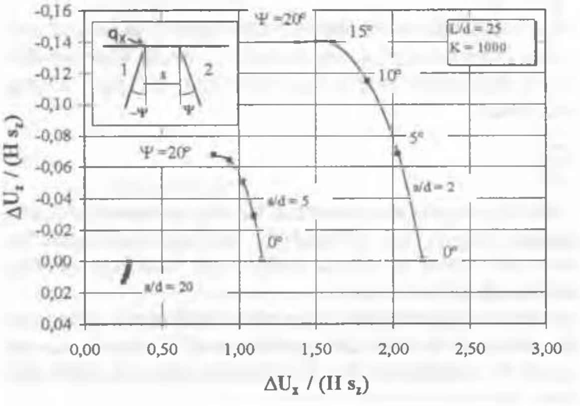

Figure 4. Displacements at the top of pile 2 produced by a normal force acting on pile 1 for different s/d values ($\nu = 0.5$)

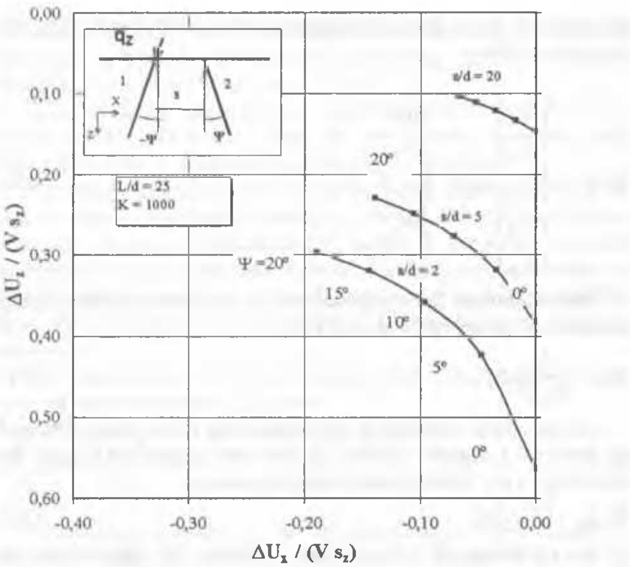

Figure 5. Displacements at the top of pile 2 produced by a axial force acting on pile 1 for different s/d values ($\nu = 0.5$)

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