Assessment of modulus of subgrade reaction of sand-sensitivity analysis

Evaluation du module de la réaction du fond de la fouille de l'analyse de la sensibilité du sable

A.M.Elmarakbi & B.B.Budkowska – *Department of Civil and Environmental Engineering, University of Windsor, Ontario, Canada*

ABSTRACT: The determination of modulus of subgrade reaction k by static equilibrium method suffers considerable inaccuracy, which can be associated with two problems. The first is attributed to an inadequacy of physical model used in simulation of soil response as the Winkler foundation. The second is connected with the fact that k depends on width of the pile. In this paper, the static equilibrium method is used for initial assessment of k for frictional soil. The models of piles of variable width embedded in homogeneous soil are laterally loaded till failure in the laboratory. The force –displacement performance curves consist of linear part combined with considerably long nonlinear part. The investigations of their full range of variability allow for determination of secant values of k. The rectification process for convergence of laboratory outcomes with numerical simulation is performed in the framework of sensitivity theory. The results show strong dependence of k on the width of the models of laterally loaded piles.

RÉSUMÉ: La détermination de la réaction du fond de la fouille k par l'équilibre stable subit une imprécision considérable, qui a rapport à deux problèmes. Le premier est attribué à une insuffisance de modèle physique employé dans la simulation de la réponse du sol comme le procédé Winkler. Le second se relie au fait que k dépend de la largeur de la pile. Dans ce document, la méthode de l'équilibre stable s'emploie pour l'évaluation initiale de k pour le sol de frottement. Les modèles de piles de largeur variable enrobées dans du sol homogène sont chargés en largeur jusqu'à la faillite au laboratoire. Les courbes de performance de force-déplacement consistent en une partie linéaire combinée à une partie non linéaire très longue. Les études de leur amplitude totale de variabilité permettent une détermination de valeurs sécantes pour k. Le procédé de rectification pour la convergence des résultats de laboratoire avec la simulation numérique s'exécute dans le cadre de la théorie de sensibilité. Les résultats montrent une forte dépendance de k sur la largeur des modèles de piles chargées en largeur.

INTRODUCTION

The pile subjected to lateral loading is usually analyzed as the beam on elastic foundation of Winkler type. This model enjoys a great popularity among engineers because of its simplicity. It is also simple to numerical analysis by e.g. the finite element method. The major difficulty the model encounters in practice is connected with determination of the reliable value of modulus of subgrade reaction k appropriate for analysis of piles.

In most applications, the response of the pile to lateral loading, treats the soil as a series of springs distributed along the length of the pile. The investigation of the modulus of subgrade reaction k reveals, that it depends not only on the soil parameters, but also on the pile stiffness. Substantial load test data show that different values of k can be obtained if several piles are tested at the same time. The site variability justifies the use of a single value of k associated with the maximum lateral displacement. Since most lateral piles are usually designed for lateral displacements at the soil surface and the pile being much stiffer than the soil, the pile flexural stiffness EI dominates so, that bending moments in the pile are little affected over a very large range of k. However the line deflections are heavily dependent on k (Canadian foundation engineering manual, (1992)).

FORMULATION OF THE PROBLEM

The differential equation of a beam on elastic foundation of Winkler type is expressed as:

$$EI\,\frac{d^4u}{dz^4} + kBu = 0 \tag{1}$$

where EI = flexural stiffness of the pile ; u = lateral displacement; k = modulus of subgrade reaction; and B = width of the beam.

The modulus of subgrade reaction k is defined as follows:

$$k = \frac{p}{\delta} \tag{2}$$

where p = pressure (kN/m^2); and δ = deflection (m).

Substitution for bending stiffness EI the following relationship

$$EI = \frac{Bh^3}{12}, \tag{3}$$

allows for modification of Equation 1, which can be written as:

$$\frac{h^3}{12}\frac{d^4u}{dz^4} + ku = 0 \tag{4}$$

where h = the cross-section's height of the beam.

It is clear, that Equation 4 is independent of the width B. In reference to laterally loaded piles, k can varies in linear fashion along the pile axis when embedded in sand. There is a reason to believe, that modulus of subgrade reaction k that is used in analysis of laterally loaded piles is width dependent.

To investigate this problem, it is worth to refer to the theory of strength of materials. In the analysis of beams, the beam is simulated as a linear element. Consequently, the concentrated force P applied to the beam can be expressed as the line load q (expressed in kN/m) acting across the width's beam B. Thus P is defined as:

$$P = q * B \tag{5}$$

For the beam of constant length l and constant height of cross-section, the deflection is proportional to P. For instance, the simply supported beam when subjected to concentrated force P in the midspan, gives the deflection under the force:

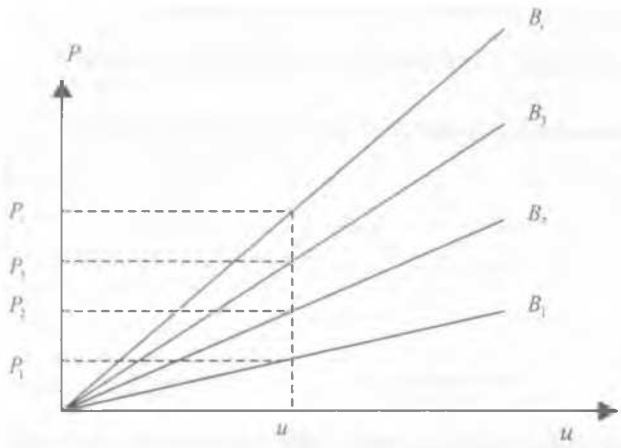

Figure 1. Example of the relationship between displacement u and force P for the beams having constant width B_i.

$$u = \frac{Pl^3}{48EI} \quad (6)$$

Substitution of relationships 5 and 3 into 6 results:

$$u = \frac{ql^3}{48E(\frac{h^3}{12})} \quad (7)$$

It is clear that for constant h, l and E, if the increase of force P is proportional to the width of the beam, then the deflections are constant. The geometrical interpretation of Equation 7 is shown in Figure 1.

It is worth noting, that if the following relationship is satisfied for arbitrary u,

$$q = \frac{P_1}{B_1} = \frac{P_2}{B_2} = \frac{P_3}{B_3} = \cdots\cdots = \frac{P_i}{B_i} \quad (8)$$

then the beams having width $B_1, B_2, \ldots\ldots, B_i$ subjected to concentrated forces $P_1, P_2, \ldots\ldots, P_i$ and having the same geometry will produce exactly the same deflections. It is obvious from Equations 7 - 8 that the deflection u is the function of q. This means, that the given q_i, for arbitrary B_i the deflection u_i is constant. This is graphically shown in Figure 2.

On the other hand, if Equation 8 is satisfied, the relationships shown in Figure 1, when presented in the coordinate system q vs. u result in a single line for all widths of the beams. This is demonstrated in Figure 3.

The presented logic can be extended to different material properties/parameters. The reference to Equation 1 indicates, that in analysis of beams on elastic foundation of Winkler type, the

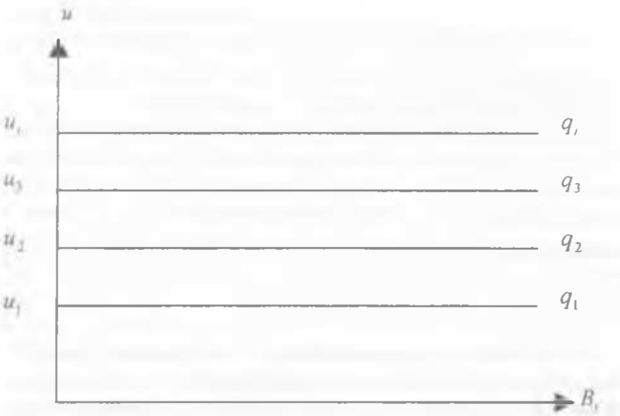

Figure 2. Independence of displacements u_i of the beams' width B_i for constant values of linear loads q_i applied across the width of the beams

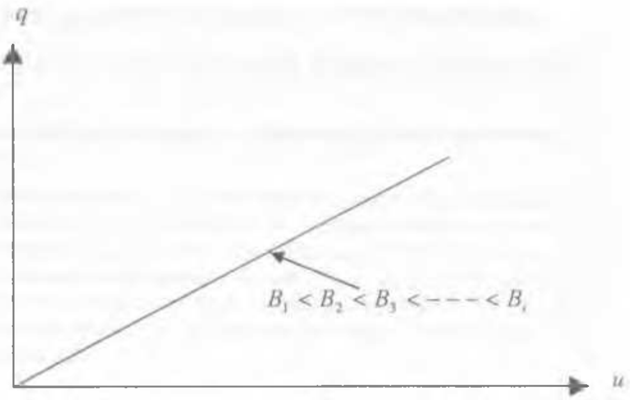

Figure 3. The relationship between displacement u vs. q for the beams having $B_1 < B_2 <$ ----$< B_i$ that satisfy condition 8.

modulus of subgrade reaction k is assumed to be constant across the beam's width for arbitrary beam.

The objective of this paper is the verification of this assumption for laterally loaded piles embedded in frictional soil. It is commonly agreed, that the modulus of subgrade reaction k for frictional soil varies in linear fashion along the pile's length. This means, that the following relationship is valid:

$$k = n_h * z \quad (9)$$

where n_h = constant of modulus of subgrade reaction; and z = spatial variable along the pile axis.

Comparison of Equation 2 and Equation 9 reveals, that n_h has units (force / length ^{4}), e.g. (kN/m^4). To answer the question if n_h is the function of width B_i, a number of laboratory tests were conducted.

LABORATORY INVESTIGATIONS

The series of tests on models of the piles have been performed in laboratory. The models were made of steel and had constant length 350 mm. Their cross-sections had the same height equal to 6.35 mm and variable width for each model. It varied from B_1 = 6.35 mm to B_8 = 50.8 mm. Each model was driven into sand soil and laterally loaded in incremental fashion till soil's failure.

The results in terms of the applied forces P_i vs. displacement u_i for various B_i are depicted in Figure 4.

INITIAL ASSESSMENT OF n_h FOR SAND

The purpose of the investigations is the determination of n_h for all discrete points located on the performance curves shown in Figure 4, that have been obtained in laboratory.

The Terzaghi's (1955) static equilibrium method proposed for this purpose is used for elastic range of deformation of the pile-soil system. This theory assumes that the applied external load generates only soil reaction, which acts perpendicular to the axis of the pile. It is easy to notice that this approach does not take into consideration other very important features of granular medium during deformation, such as the combined effect of shear and lateral earth pressure.

The soil responses generated around laterally loaded piles and their simplified models that can be used in analysis of piles subjected to bending are shown in Figure 5.

As shown in Figure 4, the behavior of each model of the pile is highly nonlinear. The static equilibrium method of determination

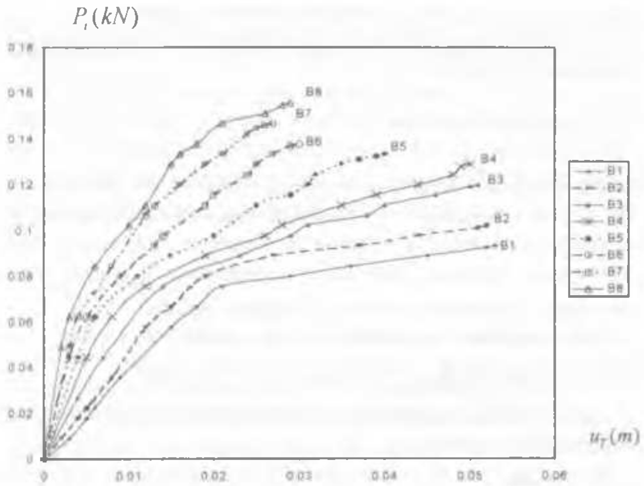

Figure 4. Laboratory results in terms of the applied force P_t vs. recorded lateral displacement u_T of the top point for models of the piles having width $B_1, B_2, \ldots, B_8$.

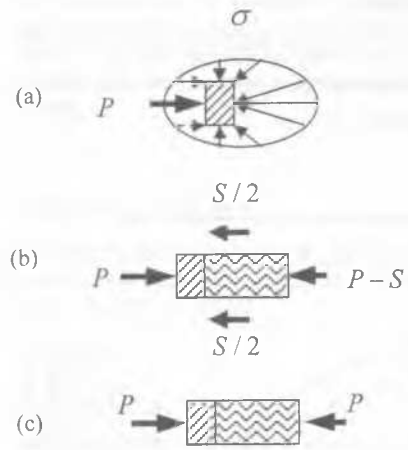

Figure 5. The cross-section of the laterally loaded model of the pile subjected to force P, a) hypothetical soil response, b) response of the soil as the Winkler foundation with shear force S, c) soil response of the Winkler foundation without shear effect.

of k proposed by Terzaghi (1955) is applied to short piles. It is based on the fact that the deformation of such piles is through rotation. It employs soil response that is shown in Figure 5 c. Taking into account the fact on nonlinear deformation of pile – soil system, the static method allows for determination of secant moduli of subgrade reaction when applied to discrete points located on the performance curves shown in Figure 4.

The validation of the model shown in Figure 5c when used in the identification of k is conducted by means of the finite element method (FEM).

The accuracy of the determined values k for each discrete point is considered as correct/accurate if the results of numerical analysis for discrete values of P_t, u_t coincide with corresponding laboratory results. In case, when the inaccuracy of the results is observed, this fact is attributed to inadequately chosen physical model of the pile-soil system. However, another approach that guarantees the convergence of the numerical and laboratory results for the model shown in Figure 5c is also possible. In other words, the improvement of k determined based on static equilibrium analysis can be performed in the framework of sensitivity theory. The sensitivity analysis is conducted with respect to initial/starting solution. The results in terms of k obtained from static equilibrium analysis serve as the initial solutions for sensitivity analysis.

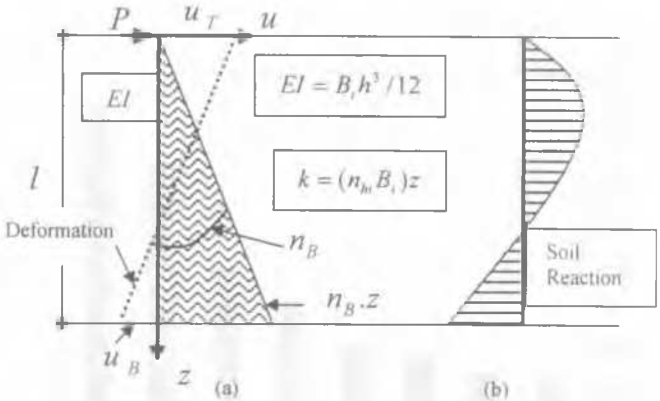

Figure 6. Laterally loaded model of the pile embedded in sand used in static equilibrium analysis. a) geometry, deformation, loading and support conditions, b) distribution of soil reaction.

The determination of initial value of k_T for each model of the pile was obtained by application of the linear regression analysis for each performance curve shown in Figure 4.

DETERMINATION OF n_h FOR SAND BASED ON STATIC EQUILIBRIUM METHOD

The characteristic parameter that is used in analysis of laterally loaded piles embedded in sand is in fact n_h defined by means of Equation 9. Its investigation for each performance curve shown in Figure 4 is conducted by means of value n_B defined as:

$$(n_B)_t = (n_h * B_t)_t \tag{10}$$

where n_B = constant of modulus of subgrade reaction associated with width B_t of the pile.

The static analysis of the laterally loaded pile embedded in sand that deforms through rotation employs the static model shown in Figure 6.

The static equilibrium equation for each model of the pile having width B_t leads to the following expression for n_B:

$$(n_B)_t = 220.286 \frac{P}{u_T} \tag{11}$$

where P = lateral force at the top of model of the pile (kN); and u_T = lateral displacement at the top of model of the pile (m).

It is worth mentioning, that Equation 11 when applied with respect to discrete points of the performance curves shown in Figure 4, results in determination of secant values of $(n_B)_t$ as shown in Figure 7.

The results presented in Figure 7, when combined with Equation 10, allow for determination of the initial values of $(n_h)_t$, which are shown in Figure 8.

RECTIFICATION OF n_B VALUES IN THE FRAMEWORK OF SENSITIVITY THEORY

The initial values $(n_B)_t$ determined by means of static equilibrium analysis when applied to numerical analysis by FEM for the discrete points on the performance curves (shown in Figure 4) did not result in the same values of the displacement as those determined in the laboratory.

The rectification of $(n_B)_t$ is performed in the framework of sensitivity theory (Dems & Mroz (1983)). The discrepancy between laboratory displacements and those obtained from

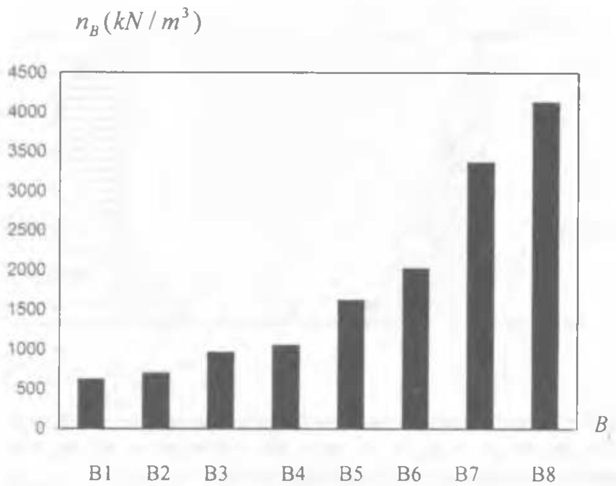

Figure 7. Initial values of $(n_B)_i$ determined based on static equilibrium analysis.

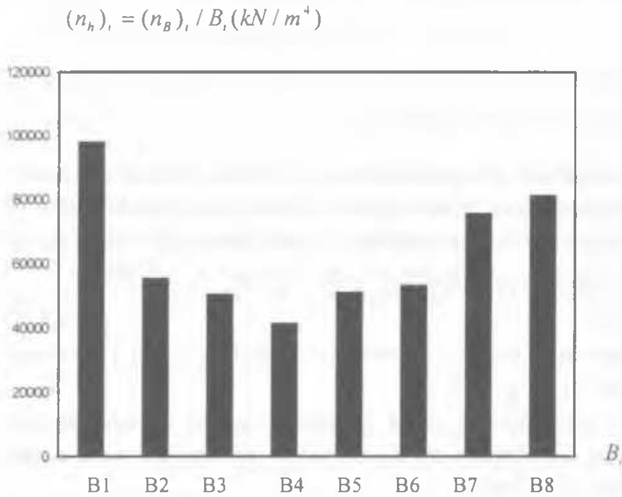

Figure 8. Initial values of $(n_h)_i$ obtained from static equilibrium analysis.

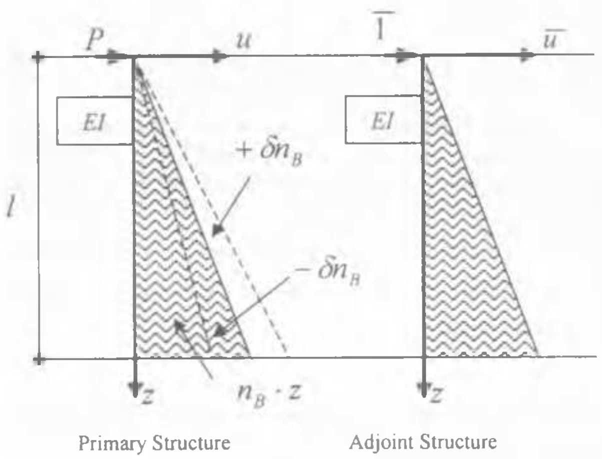

Figure 9. The geometry and load conditions for the primary and adjoint structure used in identification of $(n_B)_i$.

numerical analysis by FEM are considered as the first variations of the displacement (also called the errors/changes of the displacement) cause by the first variations of n_B (also called the changes or inaccuracies of n_B). They can be described by means of the first variations of the bending energy functional of the pile/soil system, which employs primary and adjoint system (Budkowska I, II (1997)). They are shown in Figure 9.

For laterally loaded pile, the first variation of the bending energy functional is given as:

$$\bar{1}\,\delta u = \int_0^L u\bar{u}z\,\delta n_B\,dz \qquad (12)$$

where δu = first variation of lateral displacement imposed on the primary structure; u = lateral displacement at arbitrary point of primary structure; $\bar{u}$ = lateral displacement at arbitrary point of adjoint structure; and δn_B = correction (inaccuracy/first variation) of constant modulus of subgrade reaction.

The problem discussed in the paper is focused on determination of $(\delta n_B)_i$ while $\delta u, u, \bar{u}$ are given.

Thus the final relationship for determination of the corrected values of δn_B is given as:

$$\delta n_B = \frac{1 \cdot \delta u}{\int_0^L u\bar{u}z\,dz} \qquad (13)$$

The integration required by Equation 13 is performed by means of Simpson's method.

The corrected values of $(n_B)_i$ as well as $(n_h/B_i)_i$ as the function of the lateral displacement u for constant values of B_i are presented in Figures 10 - 11.

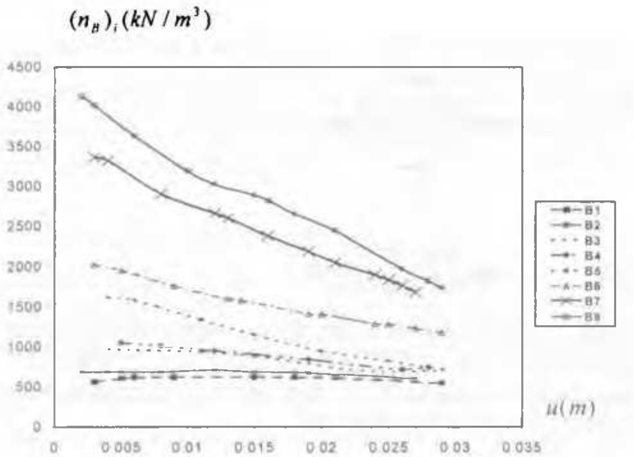

Figure 10. Distribution of secant moduli $(n_B)_i$ for laterally loaded models of the piles embedded in sand.

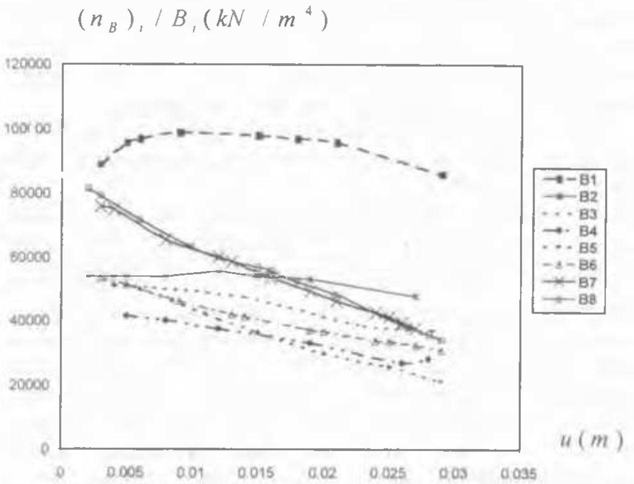

Figure 11. Distribution of secant moduli $(n_B)_i / B_i$ for laterally loaded models of the piles embedded in sand.

It is not surprising, that in the indicated range of lateral displacement u, the secant moduli $(n_B)_i$ decrease with increase of nonlinearity in the performance curves. Other interesting observations can be drawn based on inspection of Figure 11. It shows that the secant values of $(n_h)_i$ depend strongly on the width of the model of the pile. As the width of the pile increases, the corresponding secant modulus $(n_h)_i$ decreases.

CONCLUSION

The paper is focused on investigations of modulus of subgrade reaction used in analysis of laterally loaded piles when embedded in frictional soil. The relationships between the applied loads and displacements of a control point for models of the piles having variable width are determined in laboratory. They are analyzed by static method to obtain first approximation of modulus of subgrade reaction. Then they are adjusted to accurate values in the framework of sensitivity theory employing adjoint structure method. It is done by means of application of dummy load to the adjoint structure. The assessment of subgrade reaction can also be established with involvement of uniform line load applied across the width of the models of the piles. The results obtained demonstrate that secant modulus of subgrade reaction used in analysis of laterally loaded piles is width dependent.

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