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Uncertainty in analyses of rapid drawdown of reservoirs

Incertitude dans les analyses de vidange rapide de réservoirs

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ABSTRACT: Stochastic Finite Element Method (SFEM) can be used to take uncertainty on hydraulic conductivity into account in steady groundwater seepage analyses. In the paper, the techniques allowing carrying out deterministic analysis using a variational approach for solving Laplace equation by finite element method are briefly reviewed. The additional steps required to evaluate uncertainty on the results of this type of analysis using the First Order-Second Moments Method are described. Expected values, variances and covariances of hydraulic head, gradients and flow velocity are obtained. Typical results of stochastic seepage analysis in graduate materials dams for conditions of rapid drawdown of reservoir are presented.

RÉSUMÉ: La Méthode des Eléments Finis Stochastiques (SFEM) permet de tenir compte de l'incertitude sur la conductivité hydraulique dans les analyses des écoulements permanents dans les sols. Cette communication présente en premier lieu un bref rappel des techniques variationnelles qui permettent de résoudre l'équation de Laplace par la méthode des éléments finis. On indique ensuite les pas supplémentaires à suivre pour évaluer les incertitudes sur les résultats de ce type d'analyse par la méthode du Premier Ordre-Seconds Moments. Les espérances, variances et covariances des charges hydrauliques, gradients et vitesses d'écoulement sont ainsi calculées. Les résultats typiques d'une analyse stochastique des écoulements dans un barrage en terre et en enrochement soumis à des conditions de vidange rapide du réservoir sont présentés.

1 INTRODUCTION

As was shown in a previous paper (López & Auvinet 1998) it is possible to use the Stochastic Finite Element Method (SFEM) to calculate the uncertainty on the results of groundwater seepage analyses due to the uncertainty on the permeability coefficient, k, associated to lack of representative field and laboratory permeability tests, and to spatial variation of this coefficient. Such uncertainty can be modeled by means of random variables, which represent the different values that the permeability coefficient can take in distinct domains of the flow region, considered statistically homogeneous. In this paper, after a brief review of the theoretical background, this technique is illustrated by an analysis of the uncertainty in problems of rapid drawdown of reservoirs.

2 TYPICAL ANALYSIS OF GROUNDWATER SEEPAGE

As it is well known, typical analysis of two-dimensional steady-state flow can be carried out by means of the Finite Element Method (FEM). A variational approach is used to solve Laplace equation, considering that the hydraulic head varies in a simple way (i.e. linearly) within each finite element (i.e. triangles). A system of linear equations can be established:

$$[S]\{h\}=0 \tag{1}$$

where $[S]$ is a general matrix containing geometric and permeability parameters and $\{h\}$ is the vector of hydraulic heads at the nodes of all elements

Solving the previous system by Gauss Elimination Method, the hydraulic head, h, at each node of each element is determined. Hydraulic gradients and seepage forces per unit volume within each element are then easily obtained:

$$\begin{Bmatrix} i_x \\ i_y \end{Bmatrix} = \begin{Bmatrix} \partial h/\partial x \\ \partial h/\partial y \end{Bmatrix} = \frac{1}{2\Delta}[B]\{h\}^e \tag{2}$$

$$\begin{Bmatrix} j_x \\ j_y \end{Bmatrix} = -\frac{\gamma_w}{2\Delta}[B]\{h\}^e \tag{3}$$

where $[B]$ it is the geometric matrix of the element; Δ is the area of each element; $\{h\}^e$ is the vector of hydraulic heads at the nodes of each element; and γ_w is the volumetric weight of water.

From Darcy's law, seepage velocities are:

$$\begin{Bmatrix} V_x \\ V_y \end{Bmatrix} = -[R]\{i\} = -\frac{1}{2\Delta}[R][B]\{h\}^e \tag{4}$$

where $[R]$ is the permeability matrix of the element, defined as:

$$[R]=\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \tag{5}$$

The components of the discharge through the side of the triangular finite element in front of node l, Q_{lx} and Q_{ly}, in the x and y directions, respectively, are:

$$Q_{lx} = -b_l\{V_x\} \tag{6}$$

$$Q_{ly} = -c_l\{V_y\} \tag{7}$$

where b_l and c_l are terms of the geometric matrix $[B]$.

Using a similar approach, it is possible to obtain the nodal values of the stream function, ψ, to draw the flow net of the analyzed problem (Christian 1983, López & Auvinet 1998).

3 PROBABILISTIC ANALYSIS BY THE STOCHASTIC FINITE ELEMENT METHOD

Uncertainty on the results of seepage analyses performed by the Finite element Method can be estimated using the Stochastic Finite Element Method and specifically by the First Order-Second Moments Method.

3.1 *Uncertainty on the hydraulic head*

This analysis is carried out deriving the equations of System 1 with respect to the permeabilities k_x and k_y of the different materials, represented by N random variables g_p:

$$[S]\frac{\partial\{h\}}{\partial g_p} = -\frac{\partial[S]}{\partial g_p}\{h\} \tag{8}$$

Solving the previous system the vector of derivatives of the hydraulic head is obtained. The variance of h in each node and the covariance between heads in different nodes r and s can then be calculated using the following expressions based on a truncated Taylor series expansion (Benjamin & Cornell 1970):

$$\text{var}[h_r] \cong \sum_{p=1}^{N}\sum_{q=1}^{N}\left[\frac{\partial h_r}{\partial g_p}\right]_{\mu_{g_p}}\left[\frac{\partial h_r}{\partial g_q}\right]_{\mu_{g_q}} \text{cov}[g_p, g_q] \tag{9}$$

$$\text{cov}[h_r, h_s] \cong \sum_{p=1}^{N}\sum_{q=1}^{N}\left[\frac{\partial h_r}{\partial g_p}\right]_{\mu_{g_p}}\left[\frac{\partial h_s}{\partial g_p}\right]_{\mu_{g_q}} \text{cov}[g_p, g_q] \tag{10}$$

where $[\partial h_r/\partial g_p]_{\mu g_p}$ means first derivative of h_r with respect to g_p evaluated for the average value of g_p, μ_{gp}; and cov $[g_p, g_q]$ is the covariance between random variables g_p and g_q.

3.2 *Uncertainty on the hydraulic gradient*

Variance of hydraulic gradient components in the x and y directions can be calculated for each element using the following expressions:

$$\text{var}[i_x] \cong \sum_{p=1}^{3}\sum_{q=1}^{3}\left[\frac{b_p}{2\Delta}\right]\left[\frac{b_q}{2\Delta}\right]\text{cov}[h_p, h_q] \tag{11}$$

$$\text{var}[i_y] \cong \sum_{p=1}^{3}\sum_{q=1}^{3}\left[\frac{c_p}{2\Delta}\right]\left[\frac{c_q}{2\Delta}\right]\text{cov}[h_p, h_q] \tag{12}$$

In the same way, uncertainty on the *magnitude* of the hydraulic gradient vector, $\bar{i}$, can be evaluated as follows:

$$\text{var}[\bar{i}] \cong \sum_{p=1}^{2}\sum_{q=1}^{2}\left[\frac{\partial \bar{i}}{\partial i_x}\right]_{\mu_{i_x}}\left[\frac{\partial \bar{i}}{\partial i_y}\right]_{\mu_{i_y}} \text{cov}[i_x, i_y] \tag{13}$$

where:

$$\bar{i} = \sqrt{i_x^{\,2} + i_y^{\,2}} \tag{14}$$

Using a first-order development of this expression, cov $[i_x, i_y]$ can be shown to be approximately:

$$\text{cov}[i_x, i_y] = \left(\frac{1}{2\Delta}\right)^2 \{c_l b_l \,\text{var}[h_l] + c_l b_m \,\text{cov}[h_l, h_m] + c_l b_n \,\text{cov}[h_l, h_n]$$
$$+ c_m b_l \,\text{cov}[h_m, h_l] + c_m b_m \,\text{var}[h_m] + c_m b_n \,\text{cov}[h_m, h_n]$$
$$+ c_n b_l \,\text{cov}[h_n, h_l] + c_n b_m \,\text{cov}[h_n, h_m] + c_n b_n \,\text{var}[h_n]\} \tag{15}$$

3.3 *Uncertainty on seepage forces*

Since seepage forces are simply obtained by the multiplication of the hydraulic gradient by the volumetric weight of water γ_w (Eq. 3), uncertainty on these forces is also calculated with Expressions 11-13, multiplied by the constant value $(\gamma_w)^2$.

3.4 *Uncertainty on seepage velocity*

This analysis is carried out deriving the seepage velocity (Eq. 4) with respect to the permeabilities represented by g_p:

$$\frac{\partial\{V\}^e}{\partial g_p} = \frac{\partial[R][B]}{\partial g_p}\{h\}^e - [R][B]\frac{\partial\{h\}^e}{\partial g_p} \tag{16}$$

With these derivatives it is possible to calculate the variance of components of seepage velocity, in the x and y directions, for each element and the covariances between velocity components in different elements r and s, using the following expressions:

$$\text{var}[V_x] \cong \sum_{p=1}^{N}\sum_{q=1}^{N}\left[\frac{\partial V_x}{\partial g_p}\right]_{\mu_{g_p}}\left[\frac{\partial V_x}{\partial g_q}\right]_{\mu_{g_q}} \text{cov}[g_p, g_q] \tag{17}$$

$$\text{cov}[V_{x_r}, V_{x_s}] \cong \sum_{p=1}^{N}\sum_{q=1}^{N}\left[\frac{\partial V_{x_r}}{\partial g_p}\right]_{\mu_{g_p}}\left[\frac{\partial V_{x_s}}{\partial g_q}\right]_{\mu_{g_q}} \text{cov}[g_p, g_q] \tag{18}$$

For uncertainty on seepage velocity in the y direction, expressions similar to the previous ones can be used.

The uncertainty on the magnitude of the seepage velocity vector, $\bar{V}$, can be obtained as follows:

$$\text{var}[\bar{V}] \cong \sum_{p=1}^{2}\sum_{q=1}^{2}\left[\frac{\partial \bar{V}}{\partial V_x}\right]_{\mu_{V_x}}\left[\frac{\partial \bar{V}}{\partial V_y}\right]_{\mu_{V_y}} \text{cov}[V_x, V_y] \tag{19}$$

where:

$$\bar{V} = \sqrt{V_x^{\,2} + V_y^{\,2}} \tag{20}$$

And using a first-order development, cov $[V_x, V_y]$ is approximately:

$$\text{cov}[V_x, V_y] = \left(\frac{-k_x k_y}{2\Delta}\right)^2 \{c_l b_l \,\text{var}[h_l] + c_l b_m \,\text{cov}[h_l, h_m] + c_l b_n \,\text{cov}[h_l, h_n]$$
$$+ c_m b_l \,\text{cov}[h_m, h_l] + c_m b_m \,\text{var}[h_m] + c_m b_n \,\text{cov}[h_m, h_n]$$
$$+ c_n b_l \,\text{cov}[h_n, h_l] + c_n b_m \,\text{cov}[h_n, h_m] + c_n b_n \,\text{var}[h_n]\} \tag{21}$$

3.5 *Uncertainty on the discharge*

In the same form, taking into account Equations 6-7, the uncertainty on the components of the discharge is obtained as:

$$\text{var}[Q_{lx}] = b_l^{\,2}\,\text{var}[V_x] \tag{22}$$

$$\text{var}[Q_{ly}] = c_l^{\,2}\,\text{var}[V_y] \tag{23}$$

where var $[Q_{lx}]$ and var $[Q_{ly}]$ are the variances of the components of the discharge through the side of the triangular element in front of node l, in the x and y directions, respectively.

4 APPLICATION TO THE PROBLEM OF RAPID DRAWDOWN OF RESERVOIRS

4.1 *General characteristics*

It is possible to use the previous theory to analyze the problem of the seepage induced by a rapid drawdown in the external cover of a graduate materials dam (Fig. 1). It is admitted that the water level drops suddenly from Elev. 25 to Elev. 0 (López 2000). The problem is of transitory type; however, the common practice

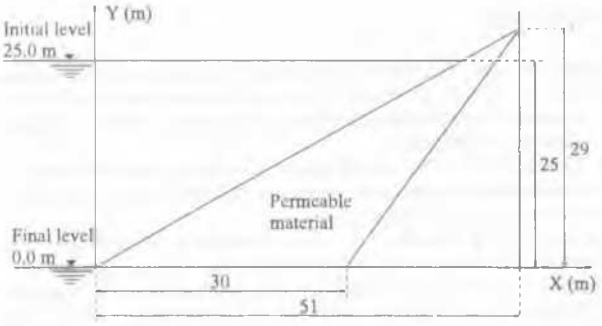

Figure 1. Permeable cover of a graduate materials dam.

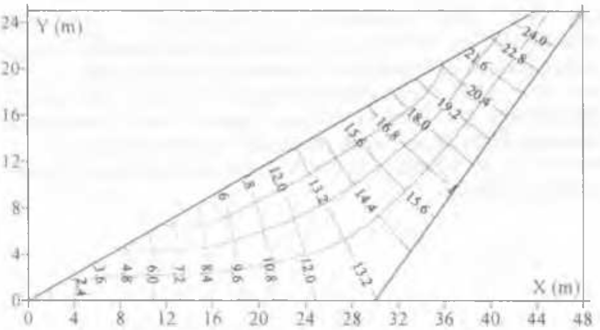

Figure 2. CASE I. Expected flow net.

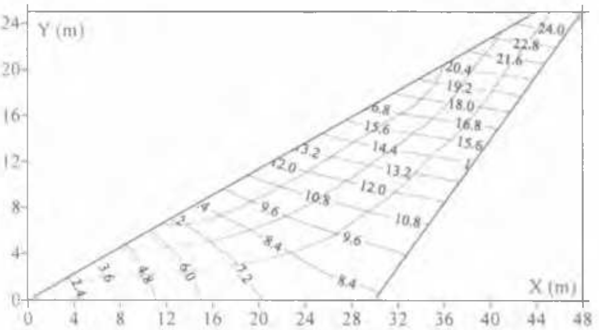

Figure 3. CASE II. Expected flow net.

(Marsal & Reséndiz 1975) for relatively permeable materials (10^{-5} m/s < k < 10^{-3} m/s), consists of admitting that drawdown is instantaneous and that steady-state flow conditions prevail. The analysis by means of the FEM is carried out within the upstream part of the earth dam, in which the material is considered incompressible and completely saturated. The soil is modeled with a net of 194 finite elements and 126 nodes. Boundary conditions imposed are: $E\{h\}$ = 25 m at all nodes located on the upper horizontal line, and $E\{h\} = z$ at all nodes located on the line defining the slope. Furthermore, the following conditions are considered:

- CASE I. Homogeneous and isotropic soil: $E\{k_x\} = E\{k_y\} = 1\times10^{-4}$ m/s.
- CASE II. Homogeneous and anisotropic soil: $E\{k_x\} = 1\times10^{-4}$ m/s; and $E\{k_y\} = 2\times10^{-5}$ m/s, where $E\{k_x\} = 5\,E\{k_y\}$.

For the probabilistic analysis, a coefficient of variation of the permeability $\sigma_k / E\{k\}$ = 50% in both soils and in both directions is assumed. In CASE I, the soil is isotropic but random, with a correlation coefficient ρ = 1 between the vertical and horizontal permeabilities; in CASE II, vertical and horizontal permeabilities are not correlated ($\rho = 0$).

4.2 *Expected values*

Expected flow nets presented in Figures 2-3, for CASES I and II, respectively, were obtained with the FLOWNETS algorithm (López 1998). The largest expected values of the hydraulic gradient, $E\{\bar{i}\}_{max}$ = 0.60 and $E\{\bar{i}\}_{max}$ = 0.84, for CASES I and II respectively, are observed in the upper part of the earth slope. The magnitude of the expected seepage forces is obtained multiplying the expected hydraulic gradient by the volumetric weight of the water, γ_w. The maximum expected values of the magnitude of the seepage velocity are $E\{V\}_{max} = 6.1\times10^{-5}$ m/s and $E\{V\}_{max} = 5.5\times10^{-5}$ m/s, for CASES I and II, respectively; in CASE I, the largest velocity is reached in the upper part of the earth slope (where the maximum expected value of the hydraulic gradient is present); and in CASE II, it is located in the lower part of the earth slope (where the flow is practically horizontal for the condition $E\{k_x\} = 5\,E\{k_y\}$). This analysis also provides the expected values of the total discharge that passes through a transverse section to the flow: $E\{Q_t\} = 4.20\times10^{-4}$ m^3/s and $E\{Q_t\} = 1.82\times10^{-4}$ m^3/s, for CASES I and II, respectively.

4.3 *Uncertainty*

The results of the probabilistic analysis performed with the MEFLUSKO algorithm (López 1998) show that:

In CASE I:

- There is no uncertainty on the hydraulic head, the hydraulic gradient and the seepage forces, since in a homogeneous and isotropic soil, the hydraulic head is independent of the permeability coefficient, k.
- The seepage velocity presents a constant uncertainty in the whole section, with a coefficient of variation that coincides with the coefficient of variation of the permeability coefficient: $\sigma_V / E\{V\} = \sigma_k / E\{k\}$ = 50%. In the same form, the variation coefficient of the discharge coincides with that of k.

In CASE II:

- Of course, there is no uncertainty on the hydraulic head in those points where the values of h were imposed as boundary conditions of the problem. The largest standard deviation, $\sigma(h)_{max}$ = 2.2 m, is observed in the lower right corner of the analyzed permeable section, in the area located at the largest distance from these boundaries. The contours of the uncertainty approximately follow the inclination of the slope (Fig. 4).
- As far as the magnitude of the hydraulic gradient is concerned, the largest standard deviation, $\sigma(\bar{i})_{max}$ = 0.18, is observed in the upper part of the earth slope. On Figure 5, the distribution of the uncertainty on the gradient presents several local irregularities, which suggest that Stochastic Finite Element Method generally requires a large number of elements to obtain an adequate accuracy.
- On the other hand, seepage forces are directly proportional to the hydraulic gradient (Eq. 3), so the distribution of its standard deviation is identical to the hydraulic gradient one. This type of uncertainty should be taken into account when evalu-

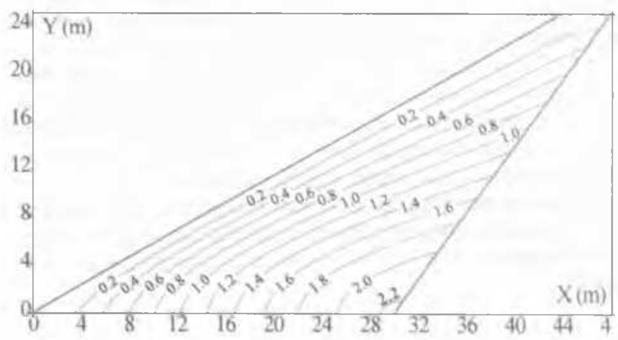

Figure 4. CASE II. Standard Deviation of hydraulic head (m).

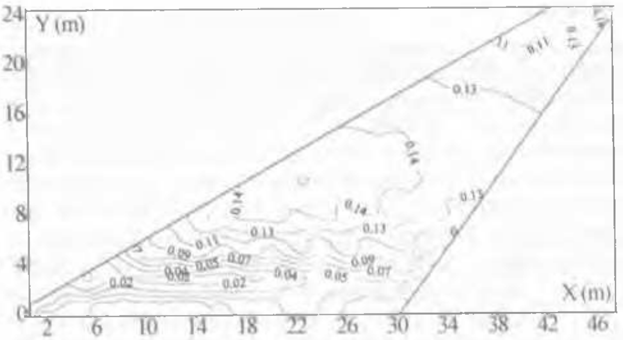

Figure 5. CASE II. Standard deviation of hydraulic gradient magnitude (dimensionless).

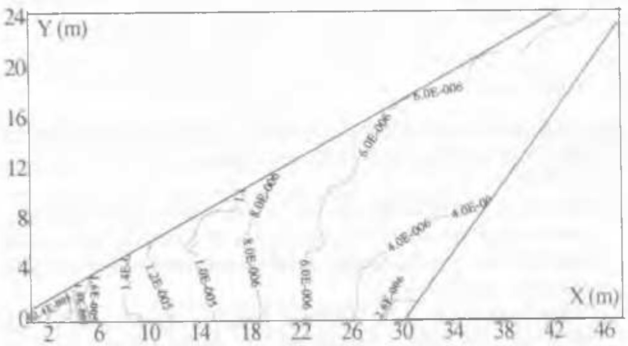

Figure 6. CASE II. Standard deviation of seepage velocity magnitude (m/s).

ating the safety factor against failure of the slope by an analysis in terms of effective stresses.

- The largest standard deviation of the seepage velocity, $\sigma(V)_{max} = 2.4\times10^{-5}$ m/s, is observed in the lower left part of the permeable region of the dam. As shown on Figure 6, this maximum value coincides with the area where the maximum expected value of the magnitude of the seepage velocity is reached.

5 CONCLUSIONS

It was shown in this paper that the random nature of the soil permeability coefficients could be taken into account in groundwater seepage analyses in a simple way using the Stochastic Finite Element Method (SFEM) and specifically the First Order-Second Moments method. The SFEM allows to calculate the expected values of hydraulic heads, gradients, seepage velocity, discharges, and seepage forces, and a to obtain a measure of the uncertainty that affect them (variance, standard deviation, covariances, etc.). When no reliable data are available, it is highly commendable to supplement the typical deterministic analyses with probabilistic evaluations, using stochastic methods such as the SFEM. It is then possible to evaluate the range of values that can be taken by the results of the seepage analyses considering the random nature of the soil permeability coefficients, instead of accepting unique values of doubtful validity. In this way it is possible to introduce a bigger degree of realism in the conventional groundwater seepage analyses. The probabilistic analysis of the seepage conditions induced in earth dams by rapid drawdown conditions presented in this paper illustrates the previous points.

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