A theoretical study of the arch effect

Étude théorétique de l'effet de la vouté

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ABSTRACT: A theoretical solution of a formula about the shape of the stable arch. The solution is based on principles and rules, which are approved in soil mechanics – Michell's solution of the tensions in the soil base from uniformly allocated load and the principle, used by Froehlich and Pusirewsky about the determination of the lineament of the plastic zones under the edges of the compressive strip foundations. By doing of underground ditches the tension of the geological load has annulled. This also gives rises to plastic zone whose external contour is the line of steady arch. The formula takes into account the important factors in arching as: variability of geological weight, strength characteristics of the soil, multilayerness, opening broad of the arch, etc. The solution presented refers to two-dimensional problems and it is applicable for the arching effect determination in both vertical and horizontal planes.

RESUME: On presente l'etablissement theorique d'une formule exprimant le contour de la voute stable dans un massif de terre. La deduction est basee sur la theorie des contraintes (solution de Michell) et l'equation de la plasticite. La methode proposee utilise et le principe de Froehlich et Pusirewsky pour la determination des sonnes plastiques sous le foundations compressives. On a tenu compte des facteurs suivants: charge geologique, l'argeur de l'orifice, caracteristiqies de resistence de sol et etc. La solution concerne le probleme planimertique et peut etre utilisee pou definir l'effet de la vote dans le plan vertical e horizontal.

1. INTRODUCTION AND GENERAL PREREQUISITES

The "arch effect" phenomenon when executing opening in soils (horizontal, vertical and sloped) is well known in the building theory and practice. It has been described and scientifically explained by a number of experts and researchers. The most common thing, which the known theoretical conclusions are based on, is the preliminary acceptance of the shape of the stable arch: triangle, wedge, circle, ellipse, parabola etc. Despite the large number of such solutions, the question of the arch effect continues to be topical in the theoretical sphere. There is still no comprehensive theoretical solution, which may give answer to the general and to a number of particular questions.

This presentation aims at explaining the basic parameters of a calculating method in a brief form, which corresponds to the established theoretical formulations of tellurian mechanics, by which the physical nature of the arch effect could be explained and the shortcoming of the preliminary acceptance of the arch shape is avoided. There are also achieved possibilities for a wider spectrum of end results.

The solution, proposed herein, is based on several fundamental initial prerequisites, namely:

- Conditions of the plain task for stresses in the soils;
- Horizontal terrain and even distribution of the geological load for each depth *h;*
- "Hydrostatic law" for the main stresses from the soils' own weight ($\sigma_1 = \sigma_3 = \gamma h$);
- Validity of the known formulas of Michell [2] for the size and directions of the main stresses in the earth base due to the action of an evenly distributed streak load;
- Validity of the equation of destruction.

Froehlich and Pousirevskiy use the same prerequisites for deducting the formulas for determining the destruction zones below the edges of foundation beds [1]. The prerequisite for the "hydrostatic law" is only initial one – within the frame of this solution it will be substituted with a different, more realistic one.

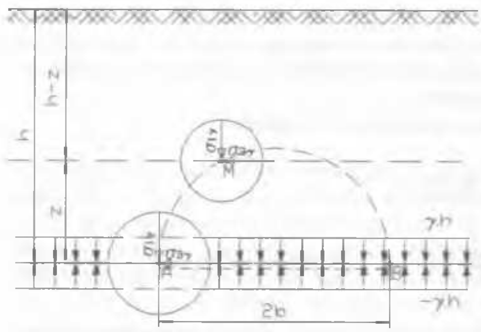

Figure 1. Tensions from geological load

2. DERIVING THE FUNDAMENTAL EQUATION

The stressed condition of the soil before and after shaping the horizontal streaked opening in the tellurian massif is examined as given in Figure 1 and Figure 2.

According to the prerequisites, at depth *h* from the terrain's surface (at the level of point *A* and point *B*) the main stresses at the beginning have the following sizes:

$$\sigma_{1\gamma} = \sigma_{3\gamma} = \gamma h \,. \quad (1)$$

For the level of the random point *M*, located at depth *h-z* they are:

$$\sigma_{1\gamma} = \sigma_{3\gamma} = \gamma(h - z) \,. \quad (2)$$

I.e. the depth *h* in the capacity of a distributed load acts on the loading, as well as its reaction - γh (Fig. 1). If at this depth a thin layer of soil is taken away (dug out, removed) and an opening is shaped with width *2b* and height *d*, the said reaction will be annulled. The active geological load γh will continue to exercise its initial action, but the massif will be unloaded from beneath, as there is lack of streak reaction – γh. Respectively, the lack of such reaction leads to a new stress condition around the opening.

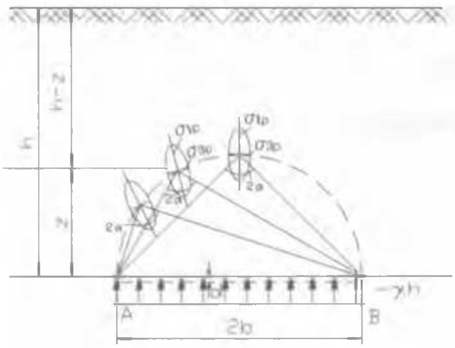

Figure 2. Tensions by Michell

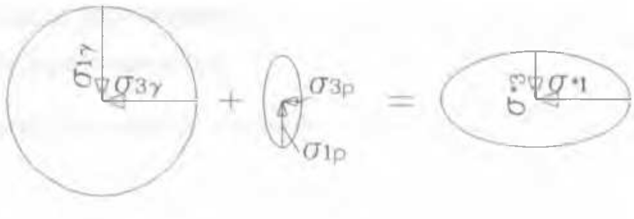

Figure 3. Total major tensions

In order to retain the massif's equilibrium, the reaction within the section *AB* will have to be replaced by the stresses, which it has (in its capacity of a constant streak load) created within the soil above *AB* – i.e. the reaction drops, but its equal value action remains (Fig. 2). The initial proportion of the main stresses in any point above the examined opening shall alter – including to a degree of creating prerequisites for destruction of the soil. This will depend on the location of the point, width of the opening *2b*, as well as the characteristics of the soil, making the massif. Immediately around the opening will be formed a zone of destructed soil. The soil at a greater distance will not fail, although alteration in the stressed condition will also occur there. As a result, above and around the zone of destructed soil the so-called arcading will occur, respectively arch effect will occur.

The redistribution of stresses, thus described, differs from the normal unloading. In this case it is expressed in reducing the one and increasing the other main stress – the same ones, which were accepted as equal according to the prerequisites. Stress sizes, by which $\sigma_{1\gamma}$ and $\sigma_{3\gamma}$ (they are denoted as σ_{1p} and σ_{3p}) will alter, are determined according to the Michell's formulas for equally distributed streaked load.

By absolute value σ_{1p} and σ_{3p} are equal to:

$$\sigma_{1p} = \frac{\gamma h(2\alpha + \sin 2\alpha)}{\pi}, \qquad (3)$$

$$\sigma_{3p} = \frac{\gamma h(2\alpha - \sin 2\alpha)}{\pi}. \qquad (3a)$$

Stresses from the expressions (3) and (3a) have the directions, known from the Michell solution – one along the bisectrix of the "angle of visibility", and the other one - perpendicular to it (Fig. 2).

When executing the described above unloading, the new main stresses settle in the tellurian massif, which we accept to write as given in Fig. 3:

$$\sigma_1{}^* = \sigma_{3\gamma} + \sigma_{3p}, \qquad (4)$$

$$\sigma_3{}^* = \sigma_{1\gamma} - \sigma_{1p}. \qquad (4a)$$

For each point of the plain – at value for the distance *z>o* the formulas (4), (4a) are as follows:

$$\sigma_1{}^* = \gamma(h-z) + \frac{\gamma h(2\alpha - \sin 2\alpha)}{\pi}, \qquad (5)$$

$$\sigma_3{}^* = \gamma(h-z) - \frac{\gamma h(2\alpha + \sin 2\alpha)}{\pi}. \qquad (5a)$$

The main stresses obtained are sufficient for outlining the relevant circle of Mohr. This circle could be up to the limit (i.e. destruction may not occur as a result of the alteration of the main stresses), but it may mean that a stressed condition up to the limit has occurred at a particular point. Whether there is or there isn't destruction depends on whether the circumference of stresses has common points with the line of cutting according to the Coulomb equation ($\tau = \sigma . \tan \varphi + c$).

If the main stresses for the random point *M* are determined according to (5) and (5a), the stress condition at this point will be defined as a result of removal of the reaction of the geological load between points *A* and *B*. Outlining the line of cutting and relevant to the stresses obtained, the circle of Mohr will show whether destruction will occur at point *M* or there will only be up to the limit redistribution of stresses. If the circle of stresses touches the line of cutting, it could be concluded that point *M* is a point of the contour of the zone of destruction, i.e. a point from the outlining of the stable arch.

Through substitution of (5) and (5a) in the equation of destruction is obtained the equation of the limiting line between the zone of bordering condition and the remaining part of the massif, located at next to bordering stress condition:

$$0.5(\sigma_1{}^* - \sigma_3{}^*) = \sin\varphi\left[0.5(\sigma_1{}^* + \sigma_3{}^*) + p_e\right] =$$
$$= \gamma h \frac{2\alpha}{\pi} = \sin\varphi\left[\gamma h - \gamma z - \gamma h(\sin\frac{2\alpha}{\pi}) + p_e\right] \qquad (6)$$

whereas $0.5(\sigma_1{}^* - \sigma_3{}^*) = \gamma h \frac{2\alpha}{\pi}$, (7)

$$0.5(\sigma_1{}^* + \sigma_3{}^*) = \gamma(h-z) - \gamma h \frac{\sin 2\alpha}{\pi}. \qquad (7a)$$

When reducing γ and solving (6) with respect to *z*, there is obtained the equation of the contour of the stable vertical arch (the destructive zone, respectively):

$$z = h - h\frac{2\alpha}{\pi}\sin\varphi - h\sin\frac{2\alpha}{\pi} + \frac{p_e}{\gamma}. \qquad (8)$$

For the incoherent soils the equation obtains the following mode:

$$z = h - h\frac{2\alpha}{\pi}\sin\varphi - h\sin\frac{2\alpha}{\pi}. \qquad (8a)$$

3. MODE OF WORK WHEN DETERMINING THE STABLE ARCH

The graphical presentation of the line, obtained with formula (8) is made analogously to outlining the zones of destruction underneath the foundations according to the Froehlich-Pousirevskiy method as follows:

- The geological section is outlined, the location of segment (A-B) is fixed and axle of symmetry is drawn;
- From random nearby centres of the axle of symmetry (O_i) are drawn circles, which pass through the edges of the opening (pt. *A* and pt. *B*).
- The "angles of visibility" for each circle are determined (measured, calculated) (Fig. 4).

$$2\alpha_i = 2\arctan\left(\frac{b}{a_i}\right). \qquad (9)$$

- According to formula (8) or (8.a) z_i is calculated for each 2α.
- According to co-ordinates ($2\alpha_i$, z_i), random number of sym-

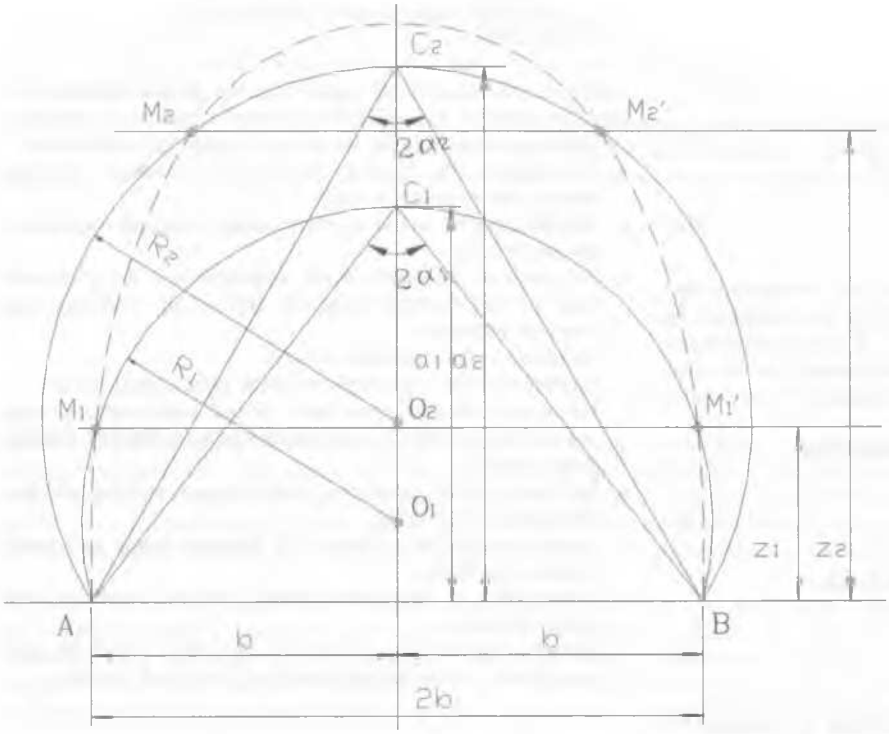

Figure 4. Determining the stable arch

metrical detailed points (M_{ii}) is determined from the line of arcading. Characteristic feature of this line is that for each point of it the tangential stresses are equal to the strength of cutting the soil. The stable arch, given in Fig. 4 with an interrupted line, is "egg-shaped", which form is know from practice.

Loading over the ceiling and the walls of a random structure (placed within the outlined arch) is the loading, ensuing only from the own weight of the destructed soil, located within the arch line contour.

4. ARCADING THE BED (COUNTER-ARCH)

Outlining the counter-arch is achieved completely analogous to arcading the opening.

The detailed points from it are determined as truncated of circles ($2\alpha_i$), for which negative values of the co-ordinate z_i are calculated according to formula (9).

5. REGARDING THE "HYDROSTATIC LAW" OF STRESSES

At the beginning of this presentation it was pointed out that the acceptance of equal in size main stresses in natural soil is a an approximation, which is rarely encountered in practice. In this meaning, the above-obtained results are also loaded with a relevant error.

The realistic connection between the main stresses, as known, is:

$$\sigma_3 = K_0\sigma_1, \tag{10}$$

whereas K_o is co-efficient of proportionality (of earth pressure at rest), which obtains value of 1.0 (as is according to the "hydrostatic law") only in very rare cases. In such case the initial for the conclusion equation (2) should be altered as follows:

$$\sigma_{1\gamma} = \gamma(h - z), \tag{11}$$

$$\sigma_3{}^* = \gamma(h - z)K_0 . \tag{11a}$$

while the stable arch equation in reporting (14) and (14a) will have the following end expression:

$$z = \frac{h\left(2p_e.\sin\frac{\varphi}{\gamma} - 2h.\frac{2\alpha}{\pi} - 2h.\sin 2\alpha.\sin 2\frac{\varphi}{\pi}\right)}{K_0 - 1 - K_0.\sin\varphi - \sin\varphi}. \tag{12}$$

The analysis shows that with formula (12) is achieved a precise outlining of the arch line at the most important point – at the "key" and a completely satisfactory precision in the upper half of the arch, because stresses σ_{1p} and σ_{3p} according to Michell in this zone turn out to be of vertical and horizontal direction, similar to the main stresses $\sigma_{1\gamma}$ and $\sigma_{3\gamma}$. This means that it is admissible for these stresses to be added up arithmetically. In its remaining part the outline has an approximation, which may be evaluated as acceptable.

6. HORIZONTAL ARCADING

Prior to the execution of a vertical excavation in the soil (borehole, spline opening), stresses in any horizontal plain are equal in all directions of the plain and may be examined as completely analogous to hydrostatic pressure. They depend of the depth h (the distance from the surface of the terrain) and on the soil's distributive properties:

$$\sigma_{2\gamma} = \sigma_{3\gamma} = K_0\sigma_{1\gamma} = K_0\gamma h , \tag{13}$$

whereas K_o is the coefficient of earth pressure at rest, giving the connection between the geological (vertical) and the lateral load.

Execution of vertical excavation works even in small quantities will provoke the described above alteration of the stress condition in the horizontal plane around the opening. The main stresses in the plane will alter.

$$\sigma_2 = K_0\gamma h + K_0\gamma h\frac{(2\alpha - \sin 2\alpha)}{\pi}, \tag{14}$$

$$\sigma_3 = K_0\gamma h + K_0\gamma h\frac{(2\alpha + \sin 2\alpha)}{\pi}. \tag{15}$$

After substituting the subtotal and the sub-difference of the main stresses in the destruction equation and rationalising the expressions, the stable horizontal arch equation is obtained.

$$K_0 h\left(\frac{2\alpha}{\pi}\sin\varphi+\frac{\sin 2\alpha}{\pi-1}\right)=\frac{p_e}{\gamma}, \qquad (16)$$

Or: $$\frac{2\alpha}{\pi}\sin\varphi+\frac{\sin 2\alpha}{\pi-1}=\frac{p_e}{K_0\gamma h}. \qquad (16a)$$

From the above equations may be drawn the conclusions that horizontal arcading for a particular depth h is determined by constants (the soil characteristics φ, p_e, γ, K_o) and is expressed only in the parameter, characterising the circle: 2α. In other words, horizontal arcading according to formula (16) or (16a) is achieved by outlining a circle.
If in the above expression (16a) it is presumed that

$$\frac{2\alpha}{\pi}\sin\varphi+\frac{\sin 2\alpha}{\pi-1}=\delta \qquad (17)$$

the formula will assume the following brief mode:

$$\delta=\frac{p_e}{K_0\gamma h}. \qquad (18)$$

Calculations and a table were made according to formula (17) and for practical use a nomogram was created. They are not given here for the purpose of briefness. From the table (the nomogram, respectively) for each δ, calculated according to formula (18) (for each depth h in fact), depending on the angle of internal friction φ could be reported 2α parameter of the circle of horizontal arcading. For plausible values of the angle 2α should be accepted only those, which are obtained within the interval 90°-180°.

In the cases of soils with no cohesion (sands, rubble), for which the $p_e=a=0$ formula (18) is simplified. It is not difficult to calculate that horizontal arcading of incoherent soils (at $\delta=0$) should be executed on circles, for which the condition $47.5° \leq 2\alpha \leq 73.5°$ is fulfilled. The radius of the circles should depend only on the angle of the inner friction of the incoherent soils. However, in view of the written above, we should draw the conclusion, that in reality there could not be any horizontal arcading of incoherent soils, because 2α always remains below 90°.

7. COEFFICIENT OF RELIABILITY

For the coefficient of reliability in this task could be used the rule of Fellenius, as the physical-and-mechanical characteristics of the soil are changed in the appropriate manner:

$$\varphi^* = \arctan\left(\frac{\tan\varphi}{F_s}\right); \quad c^*=\frac{c}{F_s}; \quad \gamma^*=\gamma F_s. \qquad (19)$$

8. CONCLUSIONS

Solving different digital examples, applying the obtained theoretical expressions give grounds for some most general conclusions regarding soils' arcading:

- Arcading is achieved at depths h of four-five times greater than the width $2b$, as established by practice.
- Arcading at smaller depths is not possible and unloading the arch (even with symmetrical removal of geological load) could cause its destruction.
- Arcading near the surface has an egg-shaped outline and with the increase of depth it gets closer to a circular one.
- The higher strength of soil parameters lead to a closer to the circular arcading and vice versa – the lower values of φ and c lead to an egg-shaped arcading.

9. DERIVATIVE RESULTS (ENSUING FROM THE SOLUTION)

Apart from the said above results, from the general solution there could be clarified and could be answered a number of questions (which are not examined in detail here), supplying solution for:

- Determining the lateral (horizontal) arcading (arcading around the vertical openings).
- Determining the lateral arcading, using tixotropic mixtures in the excavations.
- Influence of the depth factor, respectively - the geological load, as well as determining the depth from which arcading begins downwards.
- Influence of the coherence of soils.
- Influence of the anisotropy and multi-layered soils factors
- Influence of the additional loads, sloped terrain surface, vertical planning and new construction works (additional loading, respectively).
- Influence of the water level, pore pressure and water draw-down on the arch effect.
- Assessment of the influence of dynamic loads and earthquakes on arcading
- Determining a "straightened arch" (vertical slope) in horizontal arcading
- Determining the stressed state of the earth massif for each point above, below and at the side of the line of arcading.

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