Analytical solution for stress and pore pressure distribution in a cross-anisotropic and multi-layer seabed

Solution analytique pour la distribution des contraintes et de la pression interstitielle dans un massif sous-marin orthotrope et hétérogène

Behrouz Gatmiri – *Associate Professor, Tehran University, Tehran, Iran, and Senior Research Engineer, Ecole Nationale des Ponts et Chaussées, France*
Marjan Sedighi G. – *Graduate student, Tehran University, Tehran, Iran*

ABSTRACT: The wave-induced response of a porous medium plays a very important role in design of several marine structures such as buried pipelines, offshore and near-shore structures. In general, two kinds of approaches have been used to determine the response of a porous medium to wave loading, uncoupled and coupled analysis. In this paper, a new analytical solution for the coupled equations of wave-induced pore pressure and effective stresses distribution in a multi-layered and cross-anisotropic seabed with a finite thickness has been proposed.

RÉSUMÉ:-Une solution analytique pour la distribution des contraintes effectives et de la pression interstitielle dans un massif sous-marin multicouche et orthotrope avec une épaisseur finie a été présentée. Cette solution a été développée dans le cadre des approches couplées dans lesquelles les effets de l'interaction entre le squelette et le fluide interstitielle ont été pris en compte. Les différences significatives entre les résultats obtenus par cette approche et ceux d'une approche simplifiée montrent le rôle important de cette solution dans la conception des ouvrages marins et portuaires.

1 INTRODUCTION

The evaluation of wave-induced stresses and pore pressure distribution in a seabed has been the subject of various studies in marine geotechnical field.
In the uncoupled approach, no effect of the pore pressure generation on the mechanical behavior of the porous medium is assumed and the definition of effective stress after solving the equation of equilibrium is used.
In this category the solution of Fung (1965) for an infinite sea bed with horizontal surface and the solution of Gatmiri (1991) for a sloping seabed in general form of a wedge are well known.
In coupled analysis, the interaction of solid and pore fluid phases is assumed. Different authors have studied the applying the general theory of three-dimensional consolidation of Biot, which considers the elastic behavior of a porous medium.
Using this general frame, Yamamoto (1978) proposed an analytical solution for the response of a saturated soil bed of infinite thickness with isotropic permeability. Madsen (1978) has considered the same problem with anisotropic permeability, Yamamoto (1981) has extended his solution to non-homogeneous soil. Okuza (1985) proposed the analytical solution for a homogeneous unsaturated soil with infinite thickness. Gatmiri (1989) has developed a numerical solution for a nonhomogeneous seabed with finite thickness and hydraulic anisotropy. This solution has been extended to cross-anisotropic seabed by Gatmiri (1992). Jeng (1997) has developed an analytical solution for the standing wave-induced response in a cross-anisotropic seabed of infinite thickness.
In this paper, a new analytical solution for the coupled equations of wave-induced pore pressure and effective stresses distribution in a multi-layered and cross-anisotropic seabed with a finite thickness has been proposed.
This analytical solution is validated by comparison with the results of the Jeng solution (1997) for an infinite thickness case and the results of numerical solution of Gatmiri (1992) for a general case.
The wave-induced pressure on the seabed corresponding to Airy wave theory is considered as below:

$$p = \mathrm{Re}\{\frac{\gamma_w H}{2\cosh kd_w} e^{i(kx-\omega t)}\} \tag{1}$$

where the following notation are used, d_w: constant water depth, H: wave height, L: wavelength, $k=2\pi/L$: wave number, T: wave period, $\omega=2\pi/T$: radian frequency of wave motion and γ_w: unit weight of the water.

2 GOVERNING EQUATIONS

The equations for overall equilibrium of the pore elastic material, in which the total stresses are resolved in to effective stresses and pore pressure, can be written as:

$$\frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = \frac{\partial p}{\partial x} \tag{2}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma'_z}{\partial z} = \frac{\partial p}{\partial z} \tag{3}$$

where, σ_x', is the effective horizontal stress, σ_z', is the effective vertical stress, τ_{xz}, is the shear stress and, p, is the pore pressure.
If u and v shows the horizontal and vertical displacement, vo - ume strain ε can be defined as:

$$\varepsilon = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} \tag{4}$$

It is well known that behavior of anisotropic medium, could be characterized by five independent elastic parameter. These parameters are, E_x, E_z, Young's modulus in the horizontal and vertical direction, μ_{xx}, μ_{zx}, Poisson's ratio as the corresponding operators of lateral expansion due to horizontal direct stress in horizontal direction and due to vertical direct stress in horizontal direction, respectively and, G_z, modulus of shear deformation in a vertical plane. The other parameters, such as, μ_{xz}, and, Gx, are dependent. For example, these can be written as: $G_x=E_x/2(1+\mu_{xx})$ and $E_x/E_z=\mu_{xz}/\mu_{zx}$.
So for anisotropic materials in coordinate axes, x and z, the effective stress-strain relationship may be written as:

$$\sigma'_x = C_{11}\frac{\partial u}{\partial x} + C_{13}\frac{\partial v}{\partial z} \quad (5)$$

$$\sigma'_z = C_{13}\frac{\partial u}{\partial x} + C_{33}\frac{\partial v}{\partial z} \quad (6)$$

$$\tau_{xz} = G_z\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\right) \quad (7)$$

In which the, C_{ij}, can be defined as:

$$\begin{aligned} C_{11} &= E_x(1-\mu_{xx}\mu_{zx})/\Delta \\ C_{13} &= E_x(1+\mu_{xx})\mu_{zx}/\Delta \\ C_{33} &= E_z(1-\mu_{xx})/\Delta \\ \Delta &= (1+\mu_{xx})(1-\mu_{xx}-2\mu_{zx}\mu_{xz}) \end{aligned} \quad (8)$$

Substituting equations 5-7 into 2 and 3, the equations of force equilibrium are rendered as:

$$C_{11}\frac{\partial^2 u}{\partial x^2} + G_z\frac{\partial^2 u}{\partial z^2} + (C_{13}+G_z)\frac{\partial^2 v}{\partial x \partial z} = \frac{\partial p}{\partial x} \quad (9)$$

$$G_z\frac{\partial^2 v}{\partial x^2} + C_{33}\frac{\partial^2 v}{\partial z^2} + (C_{13}+G_z)\frac{\partial^2 v}{\partial x \partial z} = \frac{\partial p}{\partial z} \quad (10)$$

The consolidation equation (Biot 1941) and storage equation (Verruijt 1996), are generally accepted as the governing equation for flow of compressible pore fluid in a compressible seabed. For a hydraulically anisotrop porous bed (k_x & k_z), the equation can be expressed as:

$$\frac{k_x}{k_z}\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} - \frac{\gamma_w n\beta}{k_z}\frac{\partial p}{\partial t} = \frac{\gamma_w}{k_z}\frac{\partial \varepsilon}{\partial t} \quad (11)$$

Where, n, is the soil porosity, t, is time, and, β, is the compressibility of pore fluid. β, is related to apparent bulk modulus of the pore water, k', and the degree of saturation, S_r:

$$\beta = \frac{1}{k'} = \frac{1}{k_w} + \frac{1-S_r}{P_{w0}} \quad (12)$$

k_w, is true elastic bulk modulus of water and P_{w0}, is the absolute water pressure. For absolutely saturated soil skeleton, β is $1/k_w$.

3 ANALYTICAL SOLUTION

All of the variables depend on, x, and, t, in the form given by forcing function, $e^{i(kx-\omega t)}$. This greatly simplifies the subsequent analysis since differentiation of any variable with respect of, x, or, t reduces to the variable itself, multiplied by, ik, or, -iωt, respectively. So equation 11, can be written as:

$$\frac{\partial^2 P}{\partial z^2} - \left(\frac{k_x}{k_z}k^2 - \frac{i\omega\omega n\beta_w}{k_z}\right)P = \frac{-i\omega\omega_w}{k_z}\left(iku + \frac{\partial v}{\partial z}\right) \quad (13)$$

Also from equation 2, we obtain:

$$P = \left(\sigma'_x + \frac{1}{ik}\frac{\partial^2 \tau_{xz}}{\partial z^2}\right) \quad (14)$$

Substituting equation 14 into 13 gives:

$$\frac{\partial^2 \sigma'_x}{\partial z^2} + \frac{1}{ik}\frac{\partial^3 \tau_{xz}}{\partial z^3} - \left[\frac{k_x}{k_z}k^2 - \frac{i\omega n\beta\gamma_w}{k_z}\right]$$
$$\left[\sigma_x + \frac{1}{ik}\frac{\partial^2 \tau_{xz}}{\partial z^2}\right] = \frac{-i\omega\gamma_w}{k_z}\left[iku + \frac{\partial v}{\partial z}\right] \quad (15)$$

Introducing equation 14 into 3 gives:

$$\frac{\partial \sigma'_z}{\partial z} + ik\tau_{xz} = \frac{\partial \sigma'_x}{\partial z} + \frac{1}{ik}\frac{\partial^2 \tau_{xz}}{\partial z^2} \quad (16)$$

Finally, introduction of equations 5-7 into 16, yields after some algebraic manipulations:

$$(C_{33}-C_{13}-G_z)\frac{\partial^2 v}{\partial z^2} - k^2 G_z v = \frac{1}{ik}\left(G_z\frac{\partial^3 u}{\partial z^3} - k^2(C_{11}-C_{13}-G_z)\frac{\partial u}{\partial z}\right) \quad (17)$$

Above process, helps us to change stresses and pore pressure terms to displacements terms and it simplifies solving the parametric equations.

For more simplicity, new parameter, $P_r = k^2k_x/k_z - i\omega\gamma_w n\beta/k_z$ is defined.

Substituting equations 5-7, into 15, after some algebraic manipulations, results:

$$(C_{13}+G_z)\frac{\partial^3 v}{\partial z^3} - \frac{\partial v}{\partial z}\left(P_r(C_{13}+G_z) - \frac{i\omega\omega_w}{k_z}\right) = \frac{-\partial^4 u}{\partial z^4}\left(\frac{G_z}{ik}\right) - \frac{\partial^2 u}{\partial z^2}\left(C_{11}ik - \frac{P_r G_z}{ik}\right) + u\left(P_r C_{11}ik + \frac{k\omega\omega_w}{k_z}\right) \quad (18)$$

Differentiation of equation 17 respect to z, gives, $\partial^3 v/\partial z^3$. Introducing it into equation 18, after some manipulation, results $\partial v/\partial z$ as:

$$\frac{\partial v}{\partial z} = \left\{\frac{\partial^4 u}{\partial z^4}\left[\frac{-G_z C_{33}}{ik(C_{33}-C_{13}-G_z)}\right] + \frac{\partial^2 u}{\partial z^2}\left[-C_{11}ik + \frac{P_r G_z}{ik} - k(C_{11}-C_{13}-G_z)\frac{C_{13}+G_z}{C_{33}-C_{13}-G_z}\right] + u\left[P_r C_{11}ik + \frac{\omega k\gamma_w}{k_z}\right]\right\}$$
$$/\left\{\frac{C_{13}+G_z}{C_{33}-C_{13}-G_z}k^2 G_z - P_r(C_{13}+G_z) + \frac{i\omega\gamma_w}{k_z}\right\} \quad (19)$$

Substituting equation 19, into equation 18, next equation is obtained:

$$q_1\frac{\partial^6 u}{\partial z^6} + q_2\frac{\partial^4 u}{\partial z^4} + q_3\frac{\partial^2 u}{\partial z^2} + q_4 = 0 \quad (20)$$

In which, q_1, q_2, q_3, q_4 and D_i, are written as:

$$q_1 = \frac{-(C_{13}+G_z)G_z C_{33}}{ik(C_{33}-C_{13}-G_z)D_i} \quad (21)$$

$$q_2 = \frac{C_{13}+G_z}{D_i}\left(-C_{11}ik + \frac{P_r G_z}{ik} - ki\frac{(C_{11}-G_z-C_{13})(C_{13}+G_z)}{C_{33}-C_{13}-G_z}\right) + P_r(C_{13}+G_z) - \frac{i\omega\gamma_w}{k_z}\left(\frac{G_z C_{33}}{iD_i k(C_{33}-C_{13}-G_z)}\right) \quad (22)$$

$$q_3 = \frac{C_{13}+G_z}{D_i}\left(P_r C_{11} + \frac{k\omega\gamma_w}{k_z}\right) - \frac{P_r}{D_i}(C_{13}+G_z) - \frac{i\omega\gamma_w}{k_z} + \left(C_{11}ik + \frac{P_r G_z}{ik} - ki(C_{11}-C_{13}-G_z)\frac{C_{13}+G_z}{C_{33}-C_{13}-G_z}\right) + \left(C_{11}ik - \frac{P_r G_z}{ik}\right) \quad (23)$$

$$q_4 = -ikP_r C_{11} - \frac{\gamma_w\omega k}{k_z} - \left(P_r(C_{13}+G_z) - \frac{i\omega\gamma_w}{k_z}\right)\left(P_r C_{11}ik + \frac{\gamma_w\gamma k}{k_z}\right)/D_i \quad (24)$$

$$D_i = \frac{C_{13}+G_z}{C_{33}-C_{13}-G_z}k^2 G_z - P_r(C_{13}+G_z) + \frac{i\omega\gamma_w}{k_z} \quad (25)$$

Characteristic equation for equation 20 is, $q_1r^6+q_2r^4+q_3r^2+q_4=0$. Where r_i (i=1,2…6), the roots of the characteristic equation can be derived using an analytical software like Mathematica. Therefore, the result of the equation 20 becomes:

$$u=e^{i(kx-\omega t)}(a_1e^{r_1z}+a_2e^{-r_1z}+a_3e^{r_2z}+a_4e^{-r_2z}+a_5e^{r_3z}+a_6e^{-r_3z}) \quad (26)$$

The integration of equation 19 gives the vertical displacement function, v, with an additional unknown constant, k^*:

$$v=e^{i(kx-\omega t)}(b_1e^{r_1z}+b_2e^{-r_1z}+b_3e^{r_2z}+b_4e^{-r_2z}+b_5e^{r_3z}+b_6e^{-r_3z})+k^* \quad (27)$$

Where b_i(i=1,2…6), are given as:

$$\begin{aligned}
b_1&=\{p_1r_1^3+p_2r_1+p_3/r_1\}a_1\\
b_2&=-\{p_1r_1^3+p_2r_1+p_3/r_1\}a_2\\
b_3&=\{p_1r_2^3+p_2r_2+p_3/r_2\}a_3\\
b_4&=-\{p_1r_2^3+p_2r_2+p_3/r_2\}a_4\\
b_5&=\{p_1r_3^3+p_2r_3+p_3/r_3\}a_5\\
b_6&=-\{p_1r_3^3+p_2r_3+p_3/r_3\}a_6
\end{aligned} \quad (28)$$

And p_1, p_2 and p_3, are defined as:

$$p_1=\frac{-G_zC_{33}}{ik(C_{33}-C_{13}-G_z)D_i} \quad (29)$$

$$p_2=(-C_{11}ik+\frac{p_rG_z}{ik}-ik(C_{11}-G_z-C_{13})\frac{C_{13}+G_z}{C_{33}-C_{13}-G_z})/D_i \quad (30)$$

$$p_3=P_rC_{11}ik\frac{\gamma_w\omega k}{k_z} \quad (31)$$

Inspecting equation 17, however reveals that this equation can be satisfied only if, $k^*=0$.
Also, Substituting equations 5-7, into equation 2, pore pressure function is obtained:

$$p=C_{11}iku+\frac{\partial v}{\partial z}(C_{13}+G_z)+\frac{G_z}{ik}\frac{\partial^2u}{\partial z^2} \quad (32)$$

After some algebraic manipulation, pore pressure function can be defined as:

$$p=e^{i(kx-\omega t)}(c_1e^{r_1z}+c_2e^{-r_1z}+c_3e^{r_2z}+c_4e^{-r_2z}+c_5e^{r_3z}+c_6e^{-r_3z}) \quad (33)$$

Where c_i (i=1,2…6), are defined by ,k , c_{13}, G_z, P_1, P_2, P_3, and a_i (i=1,2…6).
However, v and P, can be defined as a function of a_i (i=1,2…6). Also effective stresses and shear stress can be obtained by equations 5-7.
So all of the variables can be defined as the function of a_i (i=1,2…6). Six appropriate boundary conditions are required for solving equation 26.

4 BOUNDARY CONDITIONS

For a porous finite seabed, the boundary conditions for impermeable horizontal bottom are, u = v = $\partial P/\partial z$=0, and for seabed surface are $\sigma'_z=\tau_{xz}$=0 and P= Re[$\gamma_w He^{i(kx-\omega t)}/2\cosh kd_w$].
If the bed thickness is infinite, all the terms including e^{iiz} (i=2,4…6), are omitted. In this condition the boundary conditions for seabed surface are enough for estimating the response of seabed.
With above boundary conditions and with the obtained

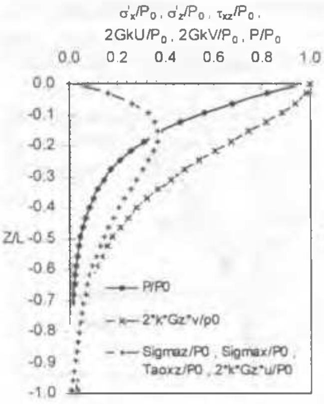

Figure 1. Comparison with Jeng and Hsu solution

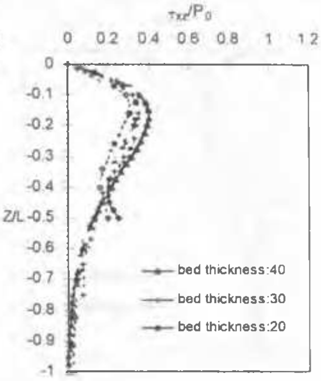

Figure 2. Comparison between presented solution and Gatmiri's numerical method for finite thickness

equations, 26, 27, 33, and the equations 5-7, the response of seabed to waves can be estimated.
For a multi-layer seabed, the governing equations can be separately written for every layer. The response of each layer are effected by its parameters. In the contact line of two layers, there are six other boundary conditions as: $u_i=u_{i+1}$, $v_i=v_{i+1}$, $\sigma'_{zi}=\sigma'_{zi+1}$, $\tau_{xzi}=\tau_{xzi+1}$, $P_i=P_{i+1}$, $k_{zi}\partial p_i/\partial z_i=k_{zi+1}\partial p_{i+1}/\partial z$

5 EXAMPLES, VERIFICATION AND DISCUSSION

First of all, the comparison between Jeng and Hsu solution and suggested method for finite problem with isotrop material is presented in Figure1. The calculation are made for Young modulus, 2.66E7 N/m^2, soil permeability, 5E-4 m/sec, and poison ratio, 0.33. Also, water depth is 4 m, wave height is 2.75 m, wave p - riod is 7 sec, wave length is 40 m and bed thickness is equal to the wave length. It can be seen that the response of seabed to ocean waves in the depth of near one wavelength tends to zero. Also, maximum effective stresses, shear stress and horizontal and vertical displacements are occured in depth of almost 0.16L. Comparison of results shows that the response of seabed in different analytical theory, cover each other exactly.Figure2 shows the comparison of shear stress distribution between new method(solid line) and Gatmiri's numerical method for finite thickness(dashed line).
As it can be seen, there is difference between the results of two last solutions. It is because of extrapolation of Gauss values or numerical solution abilities for simulating changes in impermeability coefficient in various conditions. However, this comparison shows the importance of considering the real conditions of seabed. For example, figure2 shows that the distribution of shear stress in finite thickness is not similar to the shear stress in infinite thickness and for thickness about 0.3L or less, the conditions are more critical.
Figure3 shows the comparison between response of an anisotrop multi-layer semi-finite seabed(solid lines), and an isotrop semi-finite seabed (dashed line).
Seabed parameters at the first example are written in table1.
The isotropic parameters in second example are as below:

E=2.66E7 N/m^2, G=1E7 N/m^2 and $k_x=k_z$=5E-4m/sec

Table 1. anisotrop seabed parameters

Layer No.	Ez N/m2	Ex N/m2	Gz N/m2	kz m/sec	kx m/sec	h m
1	5E7	2E7	1E7	1E-3	5E-3	10
2	4.5E7	3.6E7	2E7	2E-4	5E-4	10
3	8E7	8E7	2E7	1E-5	1E-5	20

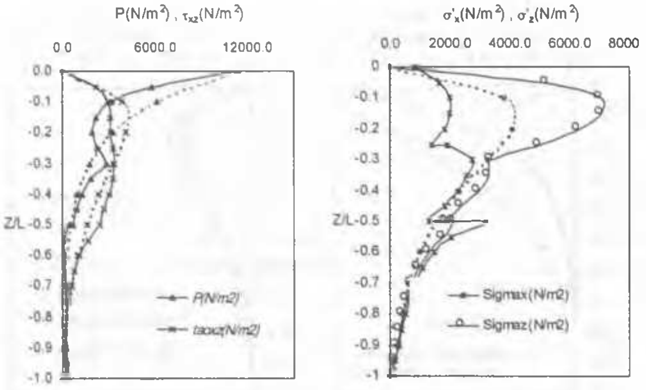

Figure 3. Comparison of effective stresses, shear stress and pore pressure in isotrop and anisotrop seabed.

In both examples, the poison ratio is equal 0.33 and the wave characteristic and water properties are, H=2.75m, T=7.0sec, L=40.0m, β=2.4E-9 N/m^2 and γ_w=10.0KN/m^2

It can be seen that considering the real characteristics of submarine seabed causes more critical conditions.

6 CONCLUSION

A general theory for calculating wave induced pore pressure, effective stresses and soil displacements in a porous seabed has been developed. The method can be generally used in seabed that both pore fluid and soil skeleton are considered compressible.

Deriving governing equations, its analytical solution and discussion of the nature of these solutions were given for the case of horizontally semi-finite and multi-layer anisotrop seabed. In this solution, the pore pressure boundary conditions can be changed in different situations. Also, it is possible to consider the effect of unsaturated seabed because of solving buried gases in pore fluid, Although the degree of saturation in most marine sediments, normally vary closed to unit.

In presented solution, considering the effect of existence of a weak layer or sliding in submarine seabed is possible using a very weak layer with very thin thickness.

However, any new investigation or construction in offshore or inshore seabed needs enough information about response of seabed to waves and it is not possible without improving the solution considering almost real conditions.

7 REFERENCE

Fung, Y.C. 1965. Foundations of solid mechanics. Englewood Cliffs, New jersy : PRENTICE-HALL, Inc.

Gatmiri, B. 1990. A simplified finite element analysis of wave induced effective stresses & pore pressure in permeable sea beds. *Geotechnique*

Gatmiri, B. 1991. Wave induced stresses & pore pressures in sloping sea bed. *International journal for numerical & analytical methods in geomechanics*

Gatmiri, B. 1992. Response of cross Anisotropyc sea bed to ocean waves. *Journal of geotechnical engineering*

Gatmiri, B.1994. Wave induced pore pressure & effective stress distributions in a non-linear sea seabed. *Thirteenth international conference in soil mechanics & foundation engineering*

Jeng, D.S. 1997. Soil response in cross - anisotropy sea bed due to stand-in waves. *Journal of geotecnical & geo-environmental engineering*

Jeng, D.S. & Hsu, J.R.C. 1996. Wave induced response in a nearly saturated seated of finite thickness. *Geotechnique*

Liu, P.L.F. 1973. Damping of water wave over porous bed, *Journal of the hydraulics division*

Madsen, O.S., 1978. Wave- induced pore pressure & effective stresses in a porous bed, *Geotechnique*

Moshagen, Y.H., 1975. Wave induced pressures in permeable sea beds, *Journal of the waterways harbors & coastal engineering division*

Okusa, S., 1985. Wave induced stresses in unsaturated submarine sediments, *Geotechnique*

Putnam, J.A., 1949. Loss of wave energy due to percolation in a permeable sea bottom, *American Geophysical union vol. 30*

Sleath, J.F.A. 1970. Wave induced pressure in beds of sand. *Journal of the hydraulics division, proceeding of the American society of civil engineers*

Yamamoto, T., 1978. Seabed instability from waves, *10th annual offshore technology conferences*

Yamamoto, T., 1981. Wave induced pore pressures & effective stresses in inhomogeneous seabed foundations, *Journal of ocean engineering*