General equations to describe the mechanical behaviour of granular soils

Equations générales pour la déscription du comportement mécanique des sols granulaires

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ABSTRACT: General equations already obtained from the principle of natural proportionality developed to describe the mechanical behaviour of clays are extended to describe the mechanical behaviour of granular soils: sands. They include: shear stress-strain equations for the pre and post-peak regions and volume change in the pre and post-peak regions as well. An analysis of the direct shear test is included.

RESUME: Des équations générales développées précédemment à partir du principe de la proportionnalité naturelle pour décrire le comportement mécanique des argiles sont étendues a la description du comportement mécanique des sables. Elles s'appliquent aux relations contraintes-déformations ainsi qu'aux variations de volume dans les domaines antérieur et postérieur au pic de résistance. Une analyse du test de cisaillement direct est également présentée.

1 INTRODUCTION

The principle of natural proportionality postulated in 1985 (Juárez-Badillo 1985) is a unifying principle from which many general equations, used to describe the mechanical behaviour of geomaterials, have emerged. A general deviatoric stress-strain theory for geomaterials has already been postulated (Juárez-Badillo 1994, 1995) and applied to clays in the pre-failure region. This time this theory is applied to a drained triaxial compression test on a coarse sand and the theory is extended to the post-failure region. A general volume change theory for clays in the triaxial tests (Juárez-Badillo 1969, 1975) is also extended to describe the volume change in the pre and post-failure regions of the same coarse sand. Finally, the principle of natural proportionality is used to describe the deviatoric behaviour of a loose sand and of a clay in the pre and post-peak regions in the direct shear test.

2 STRESS-STRAIN EQUATIONS IN THE PRE AND POST-PEAK REGIONS

Consider a sample of sand subjected to a consolidated drained compression test increasing the axial stress. Let σ_{c0} be the all around consolidation pressure and let $\sigma_1 - \sigma_3$ be the maximum principal stresses difference. Let

$$x = \frac{\sigma_1 - \sigma_3}{\sigma_{c0}} \tag{1}$$

$$x_f = \left(\frac{\sigma_1 - \sigma_3}{\sigma_{c0}}\right)_f \tag{2}$$

where x_f = maximum x. The already postulated equation for normally consolidated clays is (Juárez-Badillo 1994, 1995)

$$de_a = -\frac{1}{3}\mu \frac{dx}{\left(1 - \frac{x}{x_f}\right)^{\nu}} \tag{3}$$

where e_a = axial deviatoric natural strain and μ and ν are the shear coefficient and shear exponent respectively representing properties of the geomaterial.

Integration of Eq. (3) gives for $\nu = 1$

$$e_a = \frac{1}{3}\mu x_f \ln\left(1 - \frac{x}{x_f}\right) \tag{4}$$

while for $\nu = 2$

$$e_a = -\frac{1}{3}\mu \frac{x}{1 - \frac{x}{x_f}} \tag{5}$$

For the post-peak region the principle of natural proportionality, assuming that x and e_a are proper variables, provides: For $e_a = 0$, $x \to \infty$ while when $e_a \to \infty$, $x \to x_\infty$. e_a has a complete domain, from 0 to ∞. The proper function of x, with a complete domain, is $x - x_\infty$. Now, for $e_a = 0$, $x - x_\infty \to \infty$ while when $e_a \to \infty$, $x - x_\infty \to 0$. The principle of natural proportionality provides the equation

$$e_a \left(x - x_\infty\right)^{\nu} = \text{constant} \tag{6}$$

where ν has the same value than for the pre-peak region and x_∞ may be evaluated from the experimental data.

3 VOLUME CHANGE EQUATIONS FOR THE PRE AND POST-PEAK REGIONS

A general volume change equation in the pre-peak region for clays has already been postulated (Juárez-Badillo 1969, 1975), (Juárez-Badillo and Rico-Rodríguez 1975). It reads

$$\frac{V}{V_0} = \left\{\frac{\sigma_e + \left[\alpha\sigma_c \frac{\sigma_c}{\sigma_e} - \alpha\left(\sigma_e - \sigma_c\right)\right] y}{\sigma_{e0}}\right\}^{-\gamma} \tag{7}$$

where V = volume, V_0 = initial volume, σ_{e0} = initial equivalent consolidation pressure, σ_c and σ_e = current consolidation and equivalent consolidation pressures, y = sensitivity function given by an equation of the form

$$y = \left(\frac{x}{x_f}\right)^{\beta} \tag{8}$$

where α and β are pore pressure coefficients and γ = compressibility coefficient.

Equation (7) simply states that all volume changes may be considered as due to a dissipation of actual or virtual pore pressures. The first term (σ_e) takes into account the pore pressures due to the isotropic component of the applied stresses while the second term consist of two parts: a positive pore pressure and a negative pore pressure, both due to the shear strainning of the sample. For normally consolidated clays $\sigma_e = \sigma_c$ and there are only positive pore pressures due to the shear strainning while for highly preconsolidated clays the positive pore pressures are very small compared to the negative pore pressures.

For sands, an equation similar to (7) may be postulated, as a first step to find, in the future, a general equation. The postulated equation is:

$$\frac{V}{V_0} = \left\{\frac{\sigma_{e0} + \Delta\sigma_i - \alpha\left(\sigma_{e0} - \sigma_{c0}\right)y}{\sigma_{e0}}\right\}^{-\gamma} \tag{9}$$

$$y = \left(\frac{e_a}{e_{af}}\right)^{\beta} \tag{10}$$

where $\Delta\sigma_i$ = isotropic component of applied stresses, σ_{c0} = initial consolidation pressure. Note that Eq. (10) gives the sensitivity function with a strains formulation instead of the stresses formulation of Eq. (8), $e_{af} = e_a$ at failure.

Equation (9) considers the following assumptions: 1.- The applied stresses are very small compared to the equivalent consolidation pressure σ_{e0}. 2.- The term $\Delta\sigma_i$ represents the change in equivalent isotropic pressure plus the positive pore pressure due to the strainning of the sample.

For the case of a triaxial compression test increasing the axial stress we may write Eq. (9) as

$$\frac{V}{V_0} = \left\{1 + \frac{1}{3}\frac{\sigma_1 - \sigma_3}{\sigma_{e0}} - \alpha\frac{\sigma_{e0} - \sigma_{c0}}{\sigma_{e0}}y\right\}^{-\gamma} \tag{11}$$

This Eq. (11) may be written as a function of strains only, making use of Eq. (4). From Eq. (4) we get

$$\sigma_1 - \sigma_3 = \sigma_{c0}x_f\left(1 - e^{\frac{3e_a}{\mu x_f}}\right) \tag{12}$$

where e = base of natural logarithms. We may write therefore Eq. (11) as:

$$\frac{V}{V_0} = \left\{1 + \frac{1}{3}\frac{\sigma_{c0}}{\sigma_{e0}}x_f\left(1 - e^{\frac{3e_a}{\mu x_f}}\right) - \alpha\frac{\sigma_{e0} - \sigma_{c0}}{\sigma_{e0}}y\right\}^{-\gamma} \tag{13}$$

Equation (13) may be extended to include the post-peak region if the sensitivity function varies from 0 to 1 when e_a varies from 0 to ∞. The principle of natural proportionality gives for y the proper function $\frac{1}{y} - 1$. When $e_a = 0$, $\frac{1}{y} - 1 = \infty$ and when $e_a = \infty$, $\frac{1}{y} - 1 = 0$. Therefore the relation between them should be

$$\left(\frac{1}{y} - 1\right)e_a^{\beta} = \text{constant} \tag{14}$$

If we define the characteristic $e_a = e_a{*}$ as the e_a such that $y = \frac{1}{2}$, we therefore have

$$\left(\frac{1}{y} - 1\right)e_a^{\beta} = e_a{}^{*\beta} \tag{15}$$

and y results to be given by

$$y = \frac{1}{1 + \left(\frac{e_a}{e_a{*}}\right)^{-\beta}} \tag{16}$$

Equation (10) for the sensitivity function is to be used in Eq. (13) when dealing only with the pre-failure region, but Eq. (16) is to be used in Eq. (13) when dealing with the entire stress-strain diagram: pre and post-failure regions.

4 PRACTICAL APPLICATION

The theoretical stress-strain equations as well as the volume-change equations for the pre and post-failure regions were applied to a compression triaxial test increasing the axial stress. Fig. 1 shows "a typical plot for a dense, well-graded, coarse sand" (Lambe 1951). The volumetric and axial natural shear strain ε_v and e_a are given by the general equations (ε_a and ε_r = axial and radial natural strains):

$$\varepsilon_v = \varepsilon_a + 2\varepsilon_r \tag{17}$$

$$e_a = \varepsilon_a - \frac{\varepsilon_v}{3} \tag{18}$$

In the pre-failure region the shear and volumetric common strains at peak were ε_{ca} = -0.040 and $\varepsilon_{cv} = \frac{10}{606}$ = 0.0165 equivalent to shear and volumetric natural strains ε_a = -0.0408 and ε_v = 0.0164. In the practical application of the theoretical equations the common and natural strains were assumed to be equal due to their small values.

In the post-failure region the strains at "ultimate" were ε_{ca} = -0.150 and $\varepsilon_{cv} = \frac{36}{606}$ = 0.0594 equivalent to ε_a = -0.1625 and ε_v = 0.0577 with a corresponding axial deviatoric natural strain e_a = -0.1817. For an intermediate point ε_{ca} = -0.10, $\varepsilon_{cv} = \frac{29.6}{606}$ = 0.0488 (data from Reference) and ε_a = -0.1054, ε_v = 0.0477 with e_a = -0.1213. The differences between ε_{ca} and e_a are 0.02 for the intermediate point and 0.03 for the ultimate point. However, the author decided not to apply these modifications to the graphs since the main objective of the present paper is just to show the applicability of the theoretical equations to describe the mechanical behaviour of sands. Note that the modification in the experimental stress-strain curve, if the horizontal scale is changed to e_a, would be a stress-strain curve a "very little" higher at the ultimate strain, with practically no modification of the strain at peak.

Fig. 1 shows the experimental and theoretical curves given by Eqs. (4) and (6) with the following values of the parameters: $v = 1$, $x_f = 4.2$, $\mu = 0.007$ and $x_\infty = 2.2$, (x_1, e_{a1}) = (2.8,-0.15). The equations are, therefore, for the pre and post-failure regions:

$$e_a = \frac{1}{3}0.007 \text{ x } 4.2 \ln\left(1 - \frac{x}{4.2}\right) \tag{19}$$

$$e_a = -0.15\left(\frac{2.8 - 2.2}{x - 2.2}\right) \tag{20}$$

As the actual $x'_f = 3.96$ then we have $\dfrac{x_f}{x'_f} = \dfrac{4.2}{3.96} = 1.06$

At peak $\phi_m = 41.9°$ (from Reference) while at $e_a = \infty$, $x_\infty = 2.2$ and $\phi_\infty = 31.6°$.

Fig. 1 also shows the experimental volume changes and the theoretical curves given by Eqs. (13) and (10) for the pre-failure region and Eqs. (13) and (16) for the entire pre and post-failure regions.

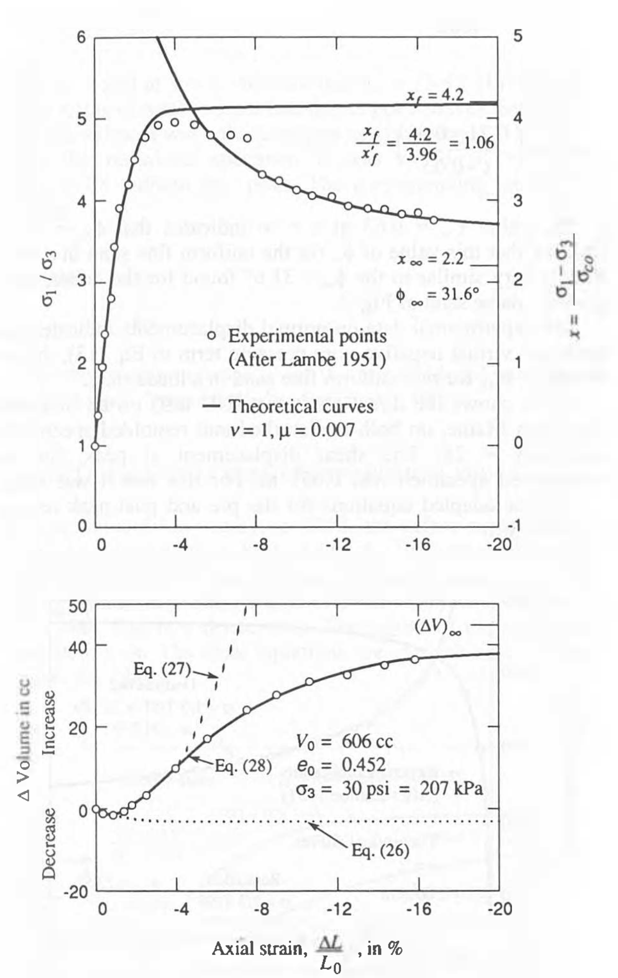

Figure 1. Triaxial compression test. Dense, well-graded, coarse sand.

The already known parameters are σ_{c0} = 30 psi = 207 kPa, $x_f = 4.2$, $e_{af} = -0.04$ and $\mu = 0.007$. The remaining parameters σ_{e0}, γ, α, $e_a{}^*$ and β were obtained as follows:

From the experimental points $\beta \geq 2$ and then the influence of the second term in Eq. (13) is small at the start of the test, say $x < \frac{1}{3} x_f$. From Eq. (11) we have that at the start of the test

$$d\varepsilon_v = \frac{dV}{V} = -\frac{1}{3}\gamma \frac{d(\sigma_1 - \sigma_3)}{\sigma_{e0}} \tag{21}$$

and

$$\frac{\gamma}{\sigma_{e0}} = -3 \frac{d\varepsilon_v}{d(\sigma_1 - \sigma_3)} \tag{22}$$

and from the experimental data it was obtained

$$\frac{\gamma}{\sigma_{e0}} = 3 \text{ x } 0.000040 = 0.000120 \frac{\text{in}^2}{\text{lb}} \tag{23}$$

For the application of Eq. (13) to Fig. 1 we need to write it as

$$\Delta V = \left\{ \left[1 + \frac{1}{3}\frac{\sigma_{c0}}{\sigma_{e0}} x_f \left(1 - e^{\frac{3e_a}{\mu x_f}} \right) - \alpha \frac{\sigma_{e0} - \sigma_{c0}}{\sigma_{e0}} y \right]^{-\gamma} - 1 \right\} V_0 \tag{24}$$

As σ_{c0} = 30 psi, $x_f = 4.2$, $\mu = 0.007$ and V_0 = 606 cc, then

$$\Delta V = \left\{ \left[1 + \frac{42}{\sigma_{e0}} \left(1 - e^{102 e_a} \right) - \alpha \frac{\sigma_{e0} - 30}{\sigma_{e0}} y \right]^{-\gamma} - 1 \right\} 606 \tag{25}$$

The component due to the first term only is

$$\Delta V = \left\{ \left[1 + \frac{42}{\sigma_{e0}} \left(1 - e^{102 e_a} \right) \right]^{-\gamma} - 1 \right\} 606 \tag{26}$$

Any pair of values of σ_{e0} and γ satisfying Eq. (23) duplicates the volume changes at the start, say $x < \frac{1}{3} x_f$, of the test. It was considered adecuate, at this time, to use σ_{e0} = 2,000 psi = 13.8 MPa and $\gamma = 0.24$. Fig. 1 shows the graph given by Eq. (26).

Fig. 1 also shows the graph given by Eqs. (25) and (10) applicable to the pre-failure region. The values of the parameters are: σ_{e0} = 2,000 psi, $\gamma = 0.24$, $e_{af} = -0.04$, $\alpha = 0.087$ and $\beta = 2$. The value of α was found from the experimental value at $e_a = e_{af}$ and the value of β from the experimental data. The final equation is

$$\Delta V = \left\{ \left[1 + \frac{42}{2{,}000} \left(1 - e^{102 e_a} \right) - 0.087 \frac{2{,}000 - 30}{2{,}000} \left(\frac{e_a}{-0.04} \right)^2 \right]^{-0.24} - 1 \right\} 606 \tag{27}$$

Fig. 1 also shows the complete theoretical curve given by Eqs. (25) and (16) for the pre and post-failure regions. The values of the parameters are σ_{e0} = 2,000 psi, $\gamma = 0.24$, $e_a{}^* = -0.06$, $\alpha = 0.27$ and $\beta = 2$. The value of $e_a{}^*$ was estimated from the complete experimental curve and the value of α was calculated from the final part of the experimental curve for the case $y = 1$. Finally the value of β was determined form the experimental data. The final equation is

$$\Delta V = \left\{ \left[1 + \frac{42}{2{,}000} \left(1 - e^{102 e_a} \right) - 0.27 \frac{2{,}000 - 30}{2{,}000} \cdot \frac{1}{1 + \left(\frac{e_a}{-0.06} \right)^{-2}} \right]^{-0.24} - 1 \right\} 606 \tag{28}$$

It should be observed that many, many values of α combined with the values of σ_{e0} and γ duplicate the experimental curves while β remains constant. This is true for Eq. (27) as well as for Eq. (28). Table 1 shows various combinations of parameters for the theoretical curves.

Table 1. Various combinations of parameters

σ_{c0}, psi	γ	β	α (Eq. 27)	α (Eq. 28)
500	0.06	2	0.34	-
1,000	0.12	2	0.17	0.50
1,500	0.18	2	0.12	0.35
2,000	0.24	2	0.087	0.27
3,000	0.36	2	-	0.185
4,000	0.48	2	-	0.145

Note: 1,000 psi = 6.9 Mpa

Note also that all these values depend on the assumptions made with respect to Eq. (9). To improve Eq. (9) tests at many other values of the initial consolidation pressure σ_{c0} as well as compression and extension tests varying only the axial stress, varying only the radial stress and with J_1 = constant, are needed.

5 DIRECT SHEAR TEST

In this section is presented the application of the principle of natural proportionality, from which all the above equations have emerged, to the direct shear test performed on a uniform fine sand in a loose state and to an inorganic clay from Maine (Lambe 1951).

Fig. 2 shows the direct shear test on a uniform, fine sand in a loose state. The shear displacement at peak was $s_f = 0.180$ in. The stress-strain equations (4), (5) and (6) adapted to the experimental data of Fig. 2 showed that for this test $v = 2$. Consequently the adapted equations, if s is the shear displacement in inches are, for the pre-failure region

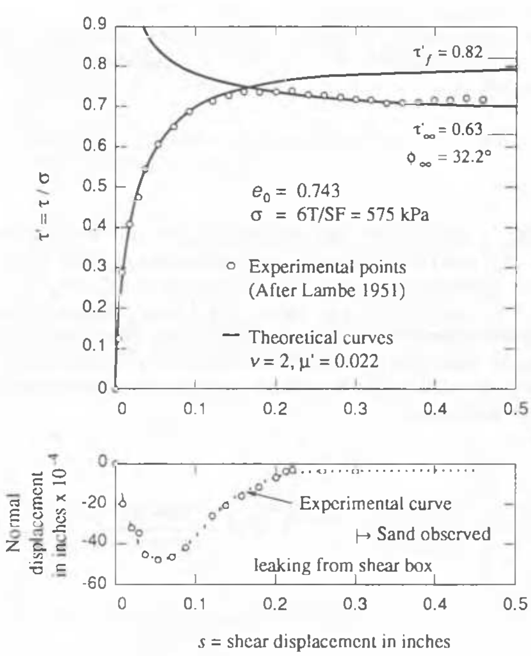

Figure 2. Direct shear test. Uniform fine sand in a loose state.

$$s = \mu' \frac{\tau'}{1 - \frac{\tau'}{\tau'_f}} \quad (29)$$

where $\tau' = \frac{\tau}{\sigma}$ and μ' is the coefficient of proportionality and, for the post-failure region

$$s\left(\tau' - \tau'_\infty\right)^2 = \text{constant} \quad (30)$$

From the experimental data it was found $\mu' = 0.022$ and $\tau'_f = 0.82$. As the actual τ''_f was 0.733 we have that $\frac{\tau'_f}{\tau''_f} = \frac{0.82}{0.733} = 1.12$. For the post-peak region it was found $\tau'_\infty = 0.63$ and the known point used was $(\tau'_1, s_1) = (0.71, 0.34)$. The corresponding equations for the pre and post-peak regions are

$$s = 0.022 \frac{\tau'}{1 - \frac{\tau'}{0.82}} \quad (31)$$

and

$$s = 0.34 \left(\frac{0.71 - 0.63}{\tau' - 0.63}\right)^2 \quad (32)$$

The value $\tau'_\infty = 0.63$ at $s = \infty$ indicates that $\phi_\infty = 32.2°$. Observe that this value of ϕ_∞ for the uniform fine sand in a loose state is very similar to the $\phi_\infty = 31.6°$ found for the dense, well-graded, coarse sand of Fig. 1.

The experimental data on normal displacements indicates that there is a virtual negative pore pressure term in Eq. (13), that is, that $\sigma_{e0} > \sigma_{c0}$, for this uniform fine sand in a loose state.

Fig. 3 shows the direct shear test (UU test) on an inorganic clay from Maine, on both undisturbed and remolded specimens, sensitivity = 28. The shear displacement at peak, for the undisturbed specimen was 0.051 in. For this test it was found $v = 2$. The adapted equations for the pre and post-peak regions are, therefore

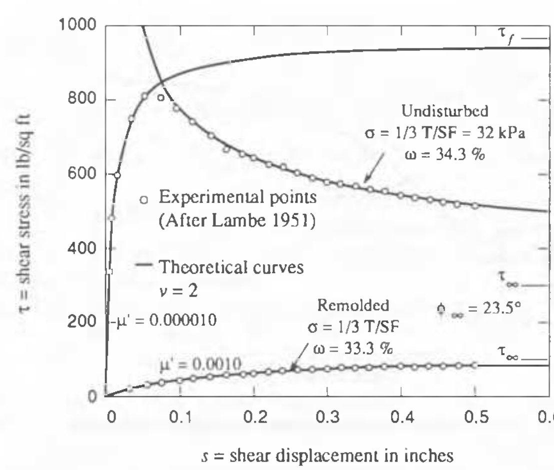

Figure 3. Direct shear test (UUtest). Inorganic clay from Maine.

$$s = \mu' \frac{\tau}{1 - \frac{\tau}{\tau_f}} \quad (33)$$

and

$$s\left(\tau - \tau_\infty\right)^2 = \text{constant} \quad (34)$$

where τ = shear stress in $\frac{\text{lb}}{\text{sq ft}}$, s = shear displacement in inches and μ' = coefficient of proportionality.

It was found, for the undisturbed specimen $\mu' = 0.000010$, $\tau_f = 970$. As the actual $\tau'_f = 808$ we have that $\frac{\tau_f}{\tau'_f} = 1.20$.

It was also found that $\tau_\infty = 290$ and $(\tau_1, s_1) = (508, 0.5)$.

The corresponding equations are, therefore:

$$s = 0.000010 \frac{\tau}{1-\frac{\tau}{970}} \tag{35}$$

and

$$s = 0.5\left(\frac{508-290}{\tau-290}\right)^2 \tag{36}$$

For $\tau_\infty = 290$ at $s = \infty$ we have that $\phi_\infty = 23.5°$. However this ϕ_∞ is in terms of total stresses and the author believes that it is not a relevant value. It was calculated just as a curiosity.

For the remolded specimen it was found $\mu' = 0.0010$, $\tau_f = \tau_\infty = 85$ without any peak. The corresponding equation is therefore

$$s = 0.0010 \frac{\tau}{1-\frac{\tau}{85}} \tag{37}$$

The relationship between μ' for the remolded and undisturbed specimens was μ' remolded/μ' undisturbed = 100.

6 ADDITIONAL TRIAXIAL COMPRESSION TESTS

While at Harvard University, in 1952, the author performed the triaxial tests presented in Figs. 4 and 5. They are included for comparison. Fig. 4 shows vacuum triaxial compression tests performed on a very uniform Franklin Falls silty sand in both, in a loose state and in a dense state. The values of the parameters appear in Fig. 4. The final equations are, for the case of loose state (see Eq. (4)).

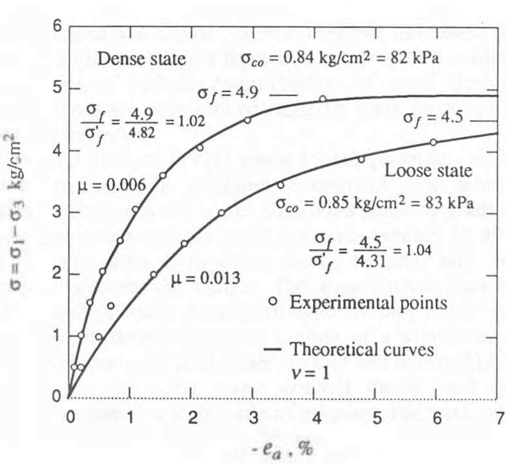

Figure 4. Vacuum triaxial compression tests. Very uniform Franklin Falls silty sand.

$$e_a = \frac{1}{3}\, 0.013\, \frac{4.5}{0.85} \ln\left(1-\frac{\sigma}{4.5}\right) \tag{38}$$

and for the case of dense state

$$e_a = \frac{1}{3}\, 0.006\, \frac{4.9}{0.84} \ln\left(1-\frac{\sigma}{4.9}\right) \tag{39}$$

Fig. 5 shows a triaxial CU compression test performed on an undisturbed sample of Boston blue clay. The final equation is (see Eq. (5)):

$$e_a = -\frac{1}{3}\, 0.010\, \frac{1}{4.94} \frac{\sigma}{1-\frac{\sigma}{3.4}} \tag{40}$$

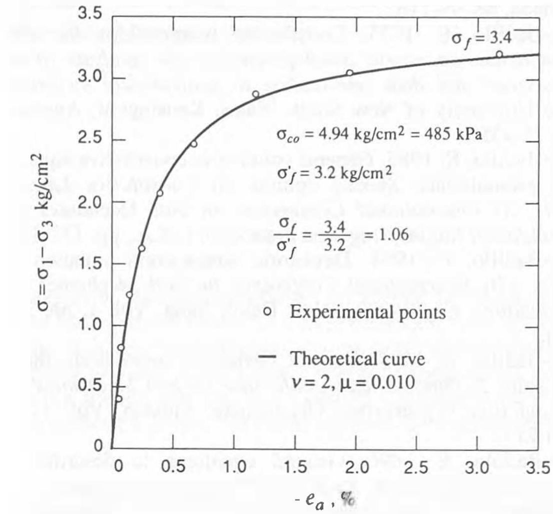

Figure 5. Triaxial CU compression test. Boston blue clay (Undisturbed).

7 FINAL COMMENTS

For sands under triaxial testing it was found $v = 1$, Figs. 1 and 4. This value of $v = 1$ has also been found for clays under drained triaxial testing (Juárez-Badillo 1994) and also for concrete made with a dense aggregate (gravel) while for concrete made with 2 lightweight aggregates (Lytag and Leca) it was found $v = 0.7$ and $v = 0.3$ (Juárez-Badillo 1996). For clays under undrained testing it was found $v = 2$ for both, triaxial, Fig. 5 and (Juárez-Badillo 1995) and direct shear tests, Fig. 3. From the above it is highly disturbant for the author the result of Fig. 2, namely, $v = 2$ for a drained direct shear test for a uniform fine sand in a loose state.

8 CONCLUSIONS

The main conclusions are:

1. It appears that the value for the shear exponent $v = 1$ is a very common value for the drained behaviour of geomaterials while the value $v = 2$ is a very common value for the undrained behaviour of geomaterials.
2. The value of the shear exponent v is the same for both, the pre and post-peak regions of the stress-strain behaviour of geomaterials.
3. To improve the general equations there is a need for a "complete spectrum" of the mechanical behaviour of geomaterials: complete stress-strain curves, including pre and post-peak regions in, say, triaxial compression and extension triaxial tests varying only the axial stress, varying only the radial stress and under J_1 = constant conditions and under many different values of the initial consolidation pressure σ_{c0}.

9 ACKNOWLEDGMENTS

The author was very happy of having discovered the very good experimental data of the complete stress-strain curves of Figs. 1, 2 and 3 in the book Soil Testing for Engineers by William T. Lambe (1951).

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