Solution of some problems in nonlinear soil mechanics

Solution de quelques problèmes de la mécanique non-linéaire du sol

S.A. Elsoufiev – *Department of Strength of Materials, Odessa State Academy of Civil Engineering and Architecture, Ukraine*

ABSTRACT: On the base of constitutive equations that take into account a nonlinear unsteady creep and damage (the development of materials' internal defects) together with criteria of maximum shearing stress and infinite elongation rate the ultimate states of structures are found. Among them wedges pressed by rigid plates or one-side vertical load are considered. A problem of cracks and plastic zones propagation near ends of a stamp are investigated too.

RESUME: Sur la base des équation reologigue en tenant compte de la non-linéarité, d'endoggement (l'évolution interne des defauts de la structure du matériel) et de critères de contraint de cisaillement maximale et de l'infinité de la vitesse de la deformation linéaire, se definissent par les états limites des constructions. Parmi eux il y a des dalles oblique pressé et incliné sous l'influence d'une charge verticale, les frissure et zones plastique qui sont des extremités stampé.

1. INTRODUCTION

As destruction and deformation often go together the first may be taken into acoount in the constitutive law. The introductiion in it true stresses and strains allows to consider the geometrical non-linearity etc. So, on the base of some mechanical ideas and tests results the following rheological law is introduced

$$\phi(\varepsilon_{eq})\varepsilon_i=\Omega(t)\sigma_{eq}^{m-1}[\beta_j(\sigma_i-\sigma_k)+\beta_k(\sigma_i-\sigma_j)].\quad(1)$$

Here orthotropy directions i, j, k coincide with main axes; the creep characteristics Ω, m, σ_{eq} are a function of time t, an exponent of a hardening law and an equivalent stress that takes into account the influence of stress state type. β_s (s=i,j,k) are anisotropy factors, for an isotropic material they are equal 0.5. Function ϕ of equivalent ε_{eq} strain takes into account a damage that induces the growth of material's volume, the third parts of creep curves, the fall of critical strains and time, and other effect. The experiments show that (1) is valid at a monotonous loading while the equivalent strain does not diminish. At stepwise or interrupted stress change law (1) may be used for the parts in which an equivalent stress is not less than on the ones before it if the time is calculated from the beginning of a new loading. The correlation of (1) with test data is better for more unsteady creep. When influence of time is negligible expression (1) turns to be the constitutive equations of plasticity theory generalized here (Elsoufiev 1982).

2. MAIN EQUATIONS

In cylindrical coordinates r, θ (Figure 1) the stresses satisfy static laws

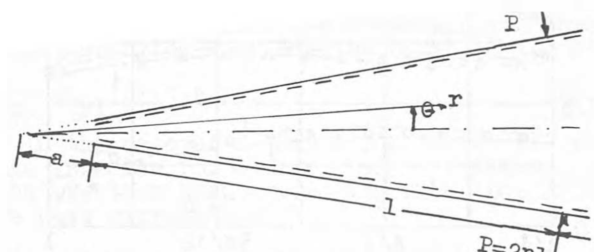

Figure 1. Pressed wedge

$$\sigma_r-\sigma_\theta+r\frac{d\sigma_r}{dr}+\frac{d\tau}{d\theta}=0,\ r\frac{d\sigma_\theta}{d\theta}+\frac{d}{dr}(r^2\tau)=0\quad(2)$$

that can be rewritten in a form

$$\frac{d^2\tau}{d\theta^2}+\frac{d}{dr}\left(r\frac{d(\sigma_r-\sigma_\theta)}{d\theta}\right)-r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r^2\tau)\right)=0\quad(3)$$

The stresses are linked with strains by the equation similar to (1) that can be given for the case of a plane deformation of a transversally isotropic body as (Elsoufiev, Nesterova 1995)

$$s=4\omega(t)\gamma_e^{\mu-1}\varepsilon_r,\ \tau=\omega(t)\gamma_e^{\mu-1}\gamma\ (\gamma_e=\sqrt{\varepsilon_r^2+\gamma^2}).\quad(4)$$

Here $\mu=m^{-1}$, $s=\sigma_r-\sigma_\theta$, $\omega(t)$ is linked with Ω and ϕ is omitted while the investigastion of stress-strain state is provided. Strains may be oomputed according to formulae

$$\varepsilon_r=\frac{du_r}{dr},\ r\varepsilon_\theta=u_r+\frac{du_\theta}{d\theta},\ \gamma=r\frac{d}{dr}\left(\frac{u_\theta}{r}\right)+\frac{du_r}{rd\theta}\quad(5)$$

The constant volume $\varepsilon_r+\varepsilon_\theta=0$ and compatibility conditions may be represented as

$$\frac{d}{dr}(ru_r)=-\frac{du_\theta}{d\theta},\ r\frac{d}{dr}\left(\frac{d}{rdr}(r^2\varepsilon_\theta)\right)-\frac{d^2\varepsilon_\theta}{d\theta^2}=\frac{d}{dr}\left(r\frac{d\gamma}{d\theta}\right).$$

It is interesting to notice an analogy between the latter expression and (3) if we replace ε_θ by τ and γ by $\sigma_r-\sigma_\theta$.

3. A WEDGE PRESSED BY INCLINED RIGID PLATES

For an ideal plastic body Nadai (1931) supposed $\tau=\tau(\theta)$ and using the yielding condition

$$\tau_e=0.5\sqrt{s^2+4\tau^2}=\tau_{yi}, \qquad (6)$$

where τ_e is maximum shearing stress, he received from (3) the first order differential equation for the problem of a mass compressed by the plates. Sokolovski (1969) introduced new variable $\psi=\psi(\theta)$ $(0<\psi<\pi/4)$ as

$$\tau=\tau_{yi}\sin 2\psi,\ s=2\tau_{yi}\sin 2\psi \qquad (7)$$

and got solution for a mass pressed through immovable plates from which using eqiation

$$\int_a^{l+a}\sigma_\theta(\lambda,r)dr=-pl \qquad (8)$$

we derive formula

$$\tau_{yi}=p(2n[(a/l+1)\ln(l/a+1)+0.5\ln\frac{n}{n-1}-1])^{-1}.$$

Diagrams $\tau_e/p=\phi(\lambda)$ at $l/a=4$, 9 are given in Figure 2 by dotted lines

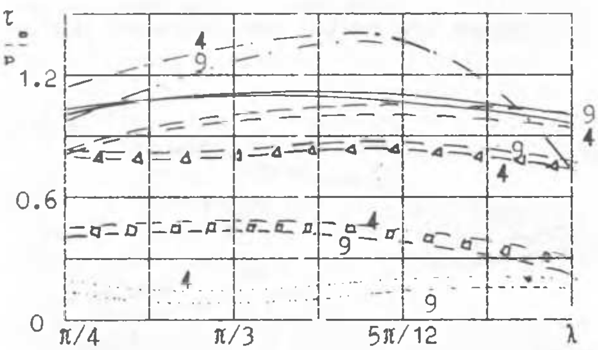

Figure 2. Diagrams $\tau_e=f(\lambda)$

Let plates move parallely to their initial position (dashed lines in Figure 1) with a displacement $V(\lambda)=V_0$. Then according to hypothesis $u_\theta=V(\theta)$

$$u_r=\frac{U(\theta)}{r}-V',\ \epsilon_\theta=-\epsilon_r=\frac{U(\theta)}{r^2},\ \gamma=\frac{U'(\theta)}{r^2}-\frac{1}{r}f(\theta), \qquad (9)$$

where $f=V''+V$, then from (4) at $\mu=1$, $\omega=G$ and the second law (2) we find the stresses. Putting them into the first static expression we get on an equality, one part of which depends on θ, and another-on r (dependence on t is hinted). So, they must be equal to a constant that should be found from condition $V(\lambda)=0$ (Sokolovski 1969). It gives $U=D(\cos 2\theta-\cos 2\lambda)$ and consequently stresses

$$\tau=G(\frac{2D}{r^2}\sin 2\theta+\frac{C}{r}\sin\theta),\ \sigma_\theta=A-2\frac{GD}{r^2}\cos 2\lambda,\ \sigma_r=\sigma_\theta-4\frac{GU}{r^2}.$$

Here constants A, C, D should be found from (8) and similar static equations

$$\int_{-\lambda}^{\lambda}\sigma_r(a,\theta)\cos\theta d\theta=0,\ \int_a^{2l+a}\tau(r,\lambda)dr=0 \qquad (10)$$

So, in the common case for the biggest τ_e value (see (6)) at $r=a$, $\theta=0$ (if $\lambda\geq\pi/4$) we compute $\tau_e=p(1-\cos 2\lambda)/B$ where

$$B=1+\frac{2}{3}\sin^2\lambda-\frac{\cos^2\lambda}{\frac{l}{a}+1}+\frac{[(\frac{2}{l}+1)\ln(1+\frac{l}{a})-(\frac{\lambda}{\sin 2\lambda}+\frac{1}{2})}{(2+\frac{a}{l})\ln(\frac{2l}{a}+1)}$$

Diagrams $\tau_e=f(\lambda)$ at $l/a=4$ and 9 are given in Figure 2 by solid lines.

Besides this common solution it is interesting to study 2 simplier options: C=0 (a compulsory flow of the material between immovable plates) and D=0 (the plates move at negligible compulsory flow) as follows

$$\max\tau_e=p(1-\cos 2\lambda)(1+\frac{2}{3}\sin^2\lambda-\frac{a\cos 2\lambda}{l+a})^{-1}, \qquad (11)$$

$$\max\tau_e=p\tan\lambda[\frac{a}{l}\ln(\frac{l}{a}+1)-\frac{1}{2}-\frac{\lambda}{\sin 2\lambda})^{-1} \qquad (12)$$

Diagrams $\max\tau_e=f(\lambda)$ at $l/a=4$, 9 are given in Figure 2 by dashed and solid with points lines respectively.

As was told before the results for these two cases coincide for an ideal plastic body (μ-0). On the other hand, they are near to the rigorous solution at $\mu=1$ and case D=0 gives safer results. It allows to seek more common solution for simpler options, and first of all for the second case.

Elsoufiev (1994) derived the solution for the movable plates at negligible compulsory flow that gives

$$\max\tau_e=\frac{p\sin 2\lambda}{m\beta}\{\frac{1}{2\sin\lambda}[\frac{\sin(\beta-1)\lambda}{\beta-1}+\frac{\sin(\beta+1)\lambda}{\beta+1}]-$$

$$-\frac{a}{l(1-\mu)}[(\frac{l}{a}+1)^{1-\mu}-1]\cos 2\lambda\}^{-1}, \qquad (13)$$

where $\beta=\sqrt{\mu(2-\mu)}$. At $\mu=1$ it reduces to (12). Diagrams $\max\tau_e=f(\lambda)$ are given in Figure 2 by lines with triangles. The analysis of the Figure shows that this option predicts more safe solution, than the option for immovable plates. These analytical results allow to find the critical strains & time according to the criterion of infinite elongation's rate (Carlsson 1965) that was generalized by Elsoufiev (1978) for quasi-brittle materials. In dangerous point $r=a$, $\theta=\lambda$ we find

$$\epsilon_*=\mu/\alpha,\ \omega(t_*)=(e\alpha/2\mu)\max\tau_e. \qquad (14)$$

Now we consider the case when only a compulsory flow takes place (a model of a volcano), and here formulae (9) are valid at V=0. From (4) we find $\gamma_e=g(\theta)/r^2$. Similar to (7) we use the represetation

$$\epsilon_r=0.5\gamma_e\cos 2\psi,\ \gamma=\gamma_e\sin 2\psi \qquad (15)$$

and put it into the compatibility equation that for the problem is $\gamma=d\epsilon_\theta/d\theta$. It gives

$$\frac{d}{d\theta}(g\cos 2\psi)+2g\sin 2\psi=0. \qquad (16)$$

From static equation (3) and following from (4) expression $\tau_e=g\,\omega(t)r$ we derive

$$\frac{d^2f}{d\theta^2}+2(1-2\mu)\frac{d}{d\theta}(g^\mu\cos 2\psi)+4\mu(1-\mu)f=0. \qquad (17)$$

where $f=g^\mu\sin 2\psi$.

It is interesting to consider some particular cases. At m=1 we have $g\sin 2\psi=D\sin 2\theta$, and we find solution (11). At $\mu=0.5$ we derive from (17) $\sqrt{g}\sin 2\psi=H\sin\theta$, and then from (16) - the differential equation

$$d\psi/d\theta=(1+\cot\theta\cot 2\psi)/(1+\cot^2 2\psi) \qquad (18)$$

that should be integrated at boundary conditions $\psi(0)=0, \psi(\lambda)=\pi/4$. Sokolovski (1969) presented the results for $0<\theta<\pi/4$. Diagrams $\psi=\psi(\theta)$ for different λ received by integration of (18) with the help of finite difference method are given in Fig.3. Using the technique above for m=1 we find

$$\tau_* = \frac{\sin\theta}{r\sin 2\psi}\left[\frac{1}{2a}\left(\frac{2J+\lambda}{\sin\lambda}+\cos\lambda\right) - \frac{\cos\lambda}{1}\ln\left(\frac{1}{a}+1\right)\right]^{-1}. \quad (19)$$

Here

$$J=\int_0^\lambda \sin 2\theta/(\sin 2\psi)d\theta.$$

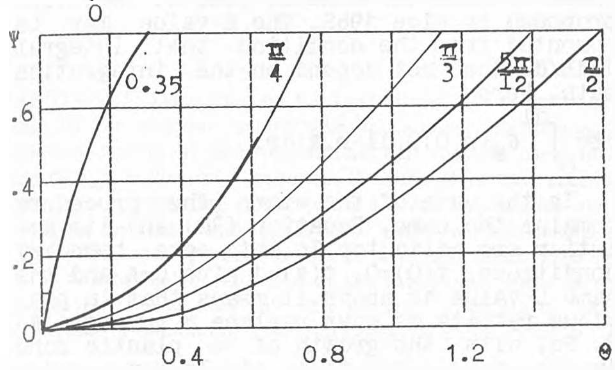

Figure 3.Diagrams $\psi=\psi(\theta)$ for $\mu=0.5$

Diagrams $\max\tau_*=\Phi(\lambda)$ are given in Figure 2 by lines with squares.

In the common case after solving together (16), (17) we get on a nonlinear differential equation of the second order that is not concretisized in Sokolovski (1969).However he gave diagrams $\psi=\psi(\theta)$ for some $\lambda<40°$ at $\mu=1/3$. After the derivation of the equation and replacement $d\theta/d\psi=\Phi$ we receive

$$\frac{\cot 2\psi d\Phi}{2\Phi\ d\psi} = \mu-1+\frac{2\mu}{\Psi} -\left(1+\frac{2\mu^2}{\Psi}\right)\Phi+\frac{\mu^2}{\Psi}\Phi^2=0 \quad (20)$$

where $\Psi=\mu+(1-\mu)\cos^2 2\psi$. The equation should be solved together with boundary condition $\Phi=1$ at $\psi=\pi/4$ which follows from (16). Then from expression $d\theta/d\psi=\Phi$ at border demands $\theta(\pi/4)=\lambda$, $\theta(0)=0$ function $\psi=\psi(\theta)$ can be determined. The second of the conditions is needed to exclude the pecularity at $\psi=\pi/4$ in (20) as follows:$0.5\cot 2\psi d\Phi/\Phi d\psi=(\Phi-1)(\mu\Phi-1-\mu)$. Now we have from (9) and (15)

$$U=-\frac{1}{2}\cos 2\psi, \frac{dU}{d\theta}=g\sin 2\psi, \frac{dU}{Ud\theta}=-2\tan 2\psi. \quad (21)$$

Integration of the last equation at boundary demand $U(\lambda)=0$ gives U,U'and $g(\theta)$. After that we find from (4) τ, s,and from (2)-σ_θ, σ_r as well as an equality

$$r^{2\mu+1}F'=-\omega(t)4\mu(1-\mu)\int_0^\theta g^\mu \sin 2\psi d\theta+g^\mu\left[\frac{\mu dg}{g d\theta}\sin 2\psi+\right.$$

$$\left.+2\left(\frac{d\psi}{d\theta}+1-2\mu\right)\cos 2\psi\right] \quad (22)$$

both parts of which must be equal to a constant that may be computed as all values in it are known. So, in function F(r) we have only one constant that can be received as well as another one (after integration of (21) from static laws (8), (10).

4.A SLOPE UNDER VERTICAL LOAD

In this case static (2) and compatibility equations are (Figire 4)

$$\frac{d\tau}{d\theta}+\sigma_r-\sigma_\theta=0, -\frac{d\sigma_\theta}{d\theta}+2\tau=0, \frac{d\varepsilon_r}{d\theta}=\gamma+C. \quad (23)$$

where C is a constant. The elastic solution of the problem is well-known, and it gives for τ_* expression

$$\tau_e=\frac{1}{2}p\sqrt{(1-2\cos 2\theta\cos 2\lambda)}/(\sin 2\lambda-2\lambda\cos 2\lambda). \quad (24)$$

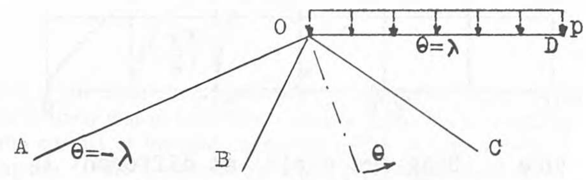

Figure 4. A slope under vertical load

If $\lambda>\pi/4$ the first residual strains appear at $\theta=0$ where according to (24)

$$p_{yi}=2\tau_{yi}(2\lambda\cos 2\lambda-\sin 2\lambda)(2\cos 2\lambda-1)^{-1}. \quad (25)$$

After that a plastic zone expands from the axis,and an ultimate state takes place when elastic angles AOB, COD are equal to $\pi/4$ at

$$p_u=2\tau_{yi}(2\lambda+1-\pi/2). \quad (26)$$

In the case of a hardening material we use formulae for strains and stresses of the previous chapter.Putting them into (23) we have expressions

$$(\gamma_e\cos 2\psi)'=2\gamma_e\sin 2\psi+C, (\gamma_*^\mu\sin 2\psi)'+2\gamma_e^\mu\cos 2\psi=0$$

after making in which the necessary operations we find the second order differential equation that is not concretusized in Sokolovski 1969.Replacing in it $d\theta/d\psi=-\Phi$ we receive the first order differential equation

$$\frac{\tan 2\psi d\theta}{\Phi\ d\psi}=2(1-\Phi)\left(\frac{\Phi-1}{\mu}+1-\frac{2}{\Psi}\right) \quad (27)$$

where $\Psi=1-(1-\mu)\sin^2\psi$.In the book mentioned results for $\mu=1/3$, $\theta\le\pi/4$ are given (see e. g.dashed line in Figure 5 for $\tau_*/p=f(\lambda)$). Solid line refers to (24) for $\theta=0$.

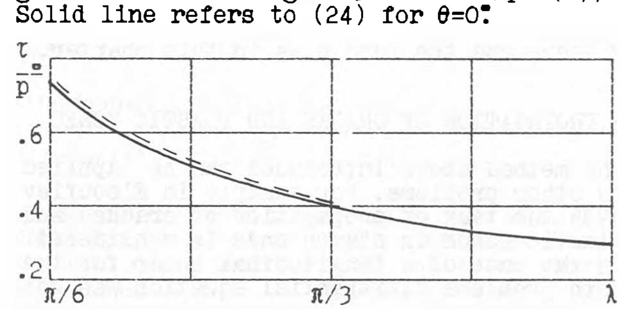

Figure 5.Diagrams $\tau_*/p=f(\lambda)$

We integrated (27) by finite difference method at boundary condition $\Phi(0)=1$. Then we integrated expression $d\theta/d\psi=-\Phi$ at border demand $\Phi(0)=\lambda$. Another condition $\theta(\pi/4)=0$ allows to choose ratio $(1-\Phi)/\tan 2\psi$ in point $\Phi(0)=1$.The results are represented in Figure 6 by dotted and dashed lines for $\mu=1/3$

and 2/3 respectively. Solid lines refer to case $\mu=1$.

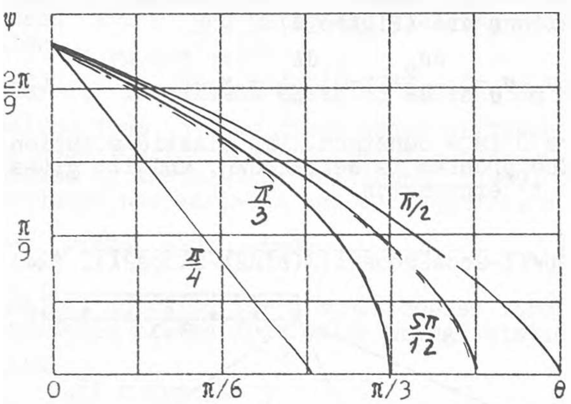

Figure 6. Diagrams $\psi=\psi(\theta)$ at different λ

When function $\psi=\psi(\theta)$ is known τ_*-value may be received from the equation following of (24) and boundary conditions $\sigma_\theta(0)=-p/2$ and $\sigma_\theta(\lambda)=-p$ as follows

$$\frac{d\tau_*}{\tau_* d\theta}+2\left(\frac{d\psi}{d\theta}+1\right)\cot 2\psi=0, \quad p=4\int_0^\lambda \tau_* \sin 2\psi d\theta \quad (28)$$

that in combination gives law for $\max\tau_*$ as

$$p=4\max\tau_*\int_0^\lambda \sin 2\psi \exp\left(-2\int_0^\theta\left(1+\frac{d\psi}{d\theta}\right)\cot 2\psi d\theta\right)d\theta. \quad (29)$$

Computations show that for $\mu=2/3$ and $\mu=1/3$ diagrams $\tau_*/p=f(\lambda)$ are near to the solid lines in Figure 5. It can be explained by the absence of μ in (29) and the vicinity of curves at the same λ in Figure 6 (for $\lambda=\pi/4$ and $\pi/2$ they coincide theoretically). It allows to use the solid line for practical calculations. In particular, using the criterion of infinite elongation rate (Elsoufiev 1978) and exponential function Φ in equations of (1) type we have according to (4) on axis $\theta=0$ $\varepsilon_\theta=\varepsilon_r=0, \gamma=2\varepsilon_1$ and (α=const)

$$\gamma_*=\mu/\alpha, \omega(t_*)=\tau_* e\alpha/2\mu.$$

In a similar way the problem of a wedge deformation under a moment in its top may be studied as well as a combination of a force P there and the load p as in this chapter.

5.PROPAGATION OF CRACKS AND PLASTIC ZONES

The method above introduced can be applied to other problems. For example in Elsoufiev 1995 the task of propagation of cracks and plastic zones in stamps ends is considered. In the case of a longitudinal shear for the both problems differential equation was got

$$f''+\left[1+m(m-1)\frac{\{(m+1)f'\}^2+f^2}{\{(m+1)f'\}^2+mf^2}\right]\frac{f}{(m+1)^2}=0 \quad (30)$$

where f is linked with displacement u and and tangential stress τ_θ (with the beginning of coordinate system in the end of the crack or stamp) by expressions

$$u=\Omega nk^m r^{1/n}\left(\left(\frac{m}{n}f\right)^2+f'^2\right)^{\frac{m-1}{2}}f', \tau_\theta=K\frac{m}{n}r^{-1/n}f. \quad (31)$$

Here n=m+1, and the rigorous solution is

$$f=\frac{2^{(m+2)/n}nm^{-2m/n}\rho D}{\{(n^2+4\rho^2+n\nu)^n+(n^2+1)(n^2+4\rho)+2m^2n\nu\}^{1/2n}},$$

where $\nu=\sqrt{n^2+4\rho^2}, \rho=\tan(C-\pi)$.

In the crack case $f(\pi)=0$, $f(0)=1$, and $C=0$, $D=m^{m/n}/n$. The analysis shows that with the growth of m the component τ_r tends to 0 and condition τ_*=const is near to a circle, touching the crack's end outside it. All these confirm the elastic-plastic solution proposed by Rice 1968. The K-value may be computed from the condition that integral J=dE/dl does not depend on the integration path. Here

$$dE=\int_0^{dl}\sigma_\theta(r,0)u(dl-r,\pi)dr.$$

In the case of the stamp the procedure remains the same. Equation (30) and its solution are valid too. In this case boundary conditions $f(0)=0$, $f(\pi)=1$ give $C=\pi$ and the same D-value as above. It means that in previous results we must replace θ by angle $\pi-\theta$. So, with the growth of m plastic zone in the form near to a circle moves under the stamp and τ_θ - component tends to zero. All that confirms the solution derived in Elsoufiev 1995.

In the same manner the problem of crack and plastic zone propagation in tension-compression and transversal shear may be in-investigated.

6.CONCLUSION

It is hardly possible to find a single constitutive law for all the soils. Here we take into account the main mechanical feature of an earth-its non-linearity that may be revealed almost from the beginning of a loading. As we use the hypothesis of the incompressibility we consider such soils as clays, frozen masses etc.

REFERENCES

Carlsson, R.1965. Creep induced tensile instability. *J.of Mech.Eng Sci*.7,2.218-229.

Elsoufiev,S. 1978. On one scheme of determination of ultimate creep strains. *Str. of Materials.(USA)*.10.

Elsoufiev, S.1982.The use of simplified rules for describing the processes of deformation and fracture of materials. *Sov. Materials Sci*.13, 1, 62-65.

Elsoufiev, S. 1994. Solution of some problems in non-linear soil mechanics. *Proc. of X Congr. Bras.de Mec.dos Solos e Eng. de Fund*.V.3.726-730.

Elsoufiev S. 1995. To the solution of some nonlinear problems in geomechanics. *Proc. of 10th Danube-Europ.Conf.on SMFE*.553-8.

Elsoufiev, S; Nesterova, L.1995.A method of strength computation in plasticity. *Proc. of XLI Konf.Nauk.Komitetu PAN*.V.3.89-94.

Rice, J.1968.Mathematical analysis in the mechanics of fracture. *Fracture*.V.2.N.-Y-London.

Sokolovski, V. *Theory of Plasticity*. 1969, 606p.(In Russian).