A multiscale theory for consolidation of swelling clay soils

Une théorie à plusieurs échelles pour la consolidation de sols argileux conflants

M.A. Murad – *LNCC/CNPq, Rio de Janeiro, Brazil*
L.S. Bennethum – *Center for Comp. Math., University of Colorado-Denver, Colo., USA*
John H. Cushman – *Center for Applied Math, Purdue University, USA*

Abstract; A thermodynamical theory of swelling clay soils which incorporates physico-chemical effects and adsorbed water flow is proposed. At the macroscale the clay platelets and the adsorbed water are homogenized in the spirit of hybrid mixture theory (HMT). This framework yields a rigorous derivation of some modified effective stress principles for clays which account for physico-chemical effects. A modified form of Darcy's law for the adsorbed water flow is also derived.

1 Introduction

Swelling clay soils consisting of an assemblage of clusters of hydrous alluminium and magnesium silicates with an expanding layer lattice are widely distributed in the earth's crust. Their behavior is of paramount importance in almost all aspects of life, where they are responsible for many reactions and processes. For example, compacted clays such as bentonite have been extensively used to impede the movement of water through cracks and fissures. They play a critical role in various high waste isolation scenerios such as barriers for commercial land fills. Swelling clays also play a critical role in the consolidation and failure of foundations, highways and runways.

Since Terzaghi [16] and Biot [2] first proposed linear poroelastic models for deformable media, the criterion for rupture and failure of soils has been based on the concept of effective stress. It has been experimentally verified that the classical Terzaghi effective stress principle accurately describes coarse-grained soils such as sands, silts and low and medium plastic clays such as kaolinite or illite. However, in its classical form it has been found to be inadequate for explaining deformation of swelling clays, in particular active plastic clays such as bentonite and montmorillonite [10],[15]. The reason is that the classical effective stress principle assumes no other forces except the effective stress and pore pressure are present. The existence of physico-chemical forces within and between the clay particles are excluded. It has been shown that interparticle forces arising from physico-chemical mechanisms are of paramount importance for active clays. Researchers have heuristically modified Terzaghi's effective stress principle to account for the physico-chemical forces and consequently different mechanistic pictures of the various stresses have been derived [14],[10]).

The nature of the physico-chemical forces remains controversial. In contrast to the effective stress, net attractive(A)-repulsive(R) forces between the clay particles do not depend upon direct contact. They have at least three components: the Van der Walls attraction, electrostatic (or osmotic) repulsion and surface hydration (a structural component). The electrostatic component arises from the electro negativity of natural smectites. The hydrophilic structure of the platelets manifest short range hydration forces between the minerals and water. In the case of clay and many other hydrophilic colloidal particles, hydration forces are believed to arise from the hydrophilic character of the mineral surfaces [9]. It has been argued by Low and co-workers [11], Derjaguin and co-workers [5] and Israelachvili and co-workers [9] that for intersticies smaller than 30Å, the structural hydration forces play a crucial rule in swelling. More specifically, these authors have advocated that for small interlamellar spaces the diffuse double layer forces play a negligible role in swelling and are too weak to explain the anomalous behavior of the adsorbed water.

Hybrid mixture theory, HMT [7] consists of classical mixture theory applied to a multiphase system with volume averaged balance equations. An average value for each phase property is established at every point in the mixture, forming M coexisting continua at each point. Constitutive equations are developed on the averaged scale and are subject to constraints placed by the entropy inequality [4]. In earlier papers [1, 12, 13] the authors have extensively applied HMT to improve the understanding of flow and deformation in swelling systems such as smectitic clays. Within the framework of the HMT, our goal is to provide a natural thermodynamical definition for physico-chemical forces within the clay particles. This is accomplished by adopting a proper theory of constitution which includes appropriate internal variables needed to capture the swelling character of the system. In particular, the approach developed herein provides a thermodynamical basis for the role hydration forces play on the consolidation of a swelling clay soil and also explains some modified effective stress principles which account for hydration forces discussed in [14],[10]. In addition by treating the adsorbed water as a phase different from the clay minerals we get a novel form of Darcy's law, which governs the averaged adsorbed water flow. This form involves an additional interaction potential gradient accounting for the adsorptive character of the clay platelets. Within the current framework, we can overcome some limitations of the works of Lambe [10] and Hueckel [8] where the adsorbed water is often considered "immobile" water and considered part of the solid phase with only one total particle stress assumed equal in the platelets and adsorbed water

2 Constitutive Relations for the Swelling Clay Particles

In this section we present the constitutive assumptions and constitutive theory for a two-phase system composed of clay-platelets and adsorbed water (clay particles). The average balance laws can be found in [7]. At this scale the clay particles are viewed as two liquid-solid coexisting continua. The clay systems we have in mind are smectite swelling clays such as montmorillonite. This system may swell under hydration and shrink under desiccation. We assume the macroscopic medium is non-heat conducting and the macroscopic fluid is non-viscous. The behavior of the system is then dictated by the following independent variables:

$$T,\ \rho_l,\ \rho_s,\ \boldsymbol{E}_s,\ \phi_l,\ \boldsymbol{\nabla}\phi_l,\ \boldsymbol{v}_{l,s} \tag{2.1}$$

where T is the temperature, ρ_l and ρ_s are the averaged densities of the adsorbed water and clay minerals, ϕ_l the volume fraction, $\boldsymbol{v}_{l,s}$ is the mass-average velocity of the adsorbed water relative to the solid phase and $\boldsymbol{E}_s$ is the averaged strain tensor of the solid phase given by $2\boldsymbol{E}_s = \boldsymbol{F}_s^T\boldsymbol{F}_s - \boldsymbol{I}$ with $\boldsymbol{F}_s$ denoting the deformation gradient (Eringen [6]). The novelty in the above set of independent variables is the inclusion of ϕ_l and $\boldsymbol{\nabla}\phi_l$ which allows for the medium to swell at the averaged scale [13],[12]).

It is usually postulated that the Helmholtz free energies of the phases A_α ($\alpha = l, s$ denotes the liquid and solid phase, respectively) depend only on a subset of the set of independent variables. For the system under consideration we postulate the following dependence of the Helmholtz free energies as

$$\begin{aligned} A_s &= A_s(T, \rho_s, \boldsymbol{E}_s) & (2.2)\\ A_l &= A_l(T, \rho_l, \phi_l). & (2.3) \end{aligned}$$

Note that A_l depends on ϕ_l in (2.3). This dependence is motivated by the experimental observations of Low [11] relating the behavior of the adsorbed water to the platelet separation.

For simplicity, consider the isothermal case where temperature gradients and heat fluxes are absent. Let $\boldsymbol{d}_\alpha$, $\boldsymbol{t}_\alpha$ and η_α $(\alpha = l, s)$ denote, respectively the symmetric part of the gradient of velocity, averaged stress tensor and entropy of the α-phase. To obtain a constitutive theory for the swelling clay particles we exploit the entropy inequality [7],[12]

$$T\Lambda = \sum_{\alpha=l,s} -\phi_\alpha\rho_\alpha\left(\frac{D_\alpha A_\alpha}{Dt} + \eta_\alpha\frac{D_\alpha T}{Dt}\right)$$

$$+\phi_\alpha \mathrm{tr}(\boldsymbol{t}_\alpha\boldsymbol{d}_\alpha) - \boldsymbol{v}_{l,s}\cdot\widehat{\boldsymbol{T}}_l \geq 0$$

where Λ is the rate of net entropy production, D_α/Dt denotes the material time derivative following the α-phase and $\widehat{\boldsymbol{T}}_l$ represents the net gain of momentum of the adsorbed water from the solid phase arising in the momentum equation,

$$\phi_\alpha\rho_\alpha\frac{D_\alpha\boldsymbol{v}_\alpha}{Dt} - \mathrm{div}(\phi_\alpha\boldsymbol{t}_\alpha) = \widehat{\boldsymbol{T}}_\alpha \quad (\alpha = l, s) \tag{2.4}$$

where gravity effects have been neglected. To exploit the restrictions placed by the entropy inequality on the constitutive theory we apply the Coleman and Noll method [4]. Within this framework the total derivatives of the free energies are rewritten in terms of partial derivatives using the chain rule and the functional forms postulated in (2.2)-(2.3). Denoting the volume fraction of the solid phase by $\phi_s = 1 - \phi_l$ and the thermodynamic pressure of the α-phase by $p_\alpha = \rho_\alpha^2(\partial A_\alpha/\partial\rho_\alpha)$ and using the relation $D_s\boldsymbol{E}_s/Dt = \boldsymbol{F}_s^T\boldsymbol{d}_s\boldsymbol{F}_s$ (Eringen [6]), the entropy inequality can be rewritten as [12]

$$\begin{aligned} T\Lambda &= \sum_{\alpha=l,s} -\phi_\alpha\rho_\alpha\left(\frac{\partial A_\alpha}{\partial T} + \eta_\alpha\right)\frac{D_s T}{Dt}\\ &+ \phi_l \mathrm{tr}\left((\boldsymbol{t}_l + p_l\boldsymbol{I})\boldsymbol{d}_l\right)\\ &+ \phi_s \mathrm{tr}\left((\boldsymbol{t}_s + p_s\boldsymbol{I} - \boldsymbol{t}_s^e)\boldsymbol{d}_s\right)\\ &- \boldsymbol{v}_{l,s}\cdot(\phi_l p_*\boldsymbol{\nabla}\phi_l - p_l\boldsymbol{\nabla}\phi_l + \widehat{\boldsymbol{T}}_l)\\ &- \frac{D_s\phi_l}{Dt}(\phi_l p_* - p_l + p_s)\\ &\geq 0 \end{aligned}$$

where

$$\boldsymbol{t}_s^e = \rho_s\boldsymbol{F}_s\frac{\partial A_s}{\partial\boldsymbol{E}_s}\boldsymbol{F}_s^T, \qquad p_* = \rho_l\frac{\partial A_l}{\partial\phi_l} \tag{2.5}$$

denote respectively the effective stress tensor and the hydration pressure [12]. The above relation provides an alternative thermodynamical definition for the effective stress tensor $\boldsymbol{t}_s^e$. For example, for a linear isotropic elastic solid with a given pair of Lame constants $\{\lambda_s, \mu_s\}$ the Helmholtz free energy of the solid phase has the quadratic form

$$\rho_s A_s = \frac{\lambda_s}{2}(\mathrm{tr}\boldsymbol{E}_s)^2 + \mu_s\mathrm{tr}\boldsymbol{E}_s^2,$$

which when combined with (2.5) implies that $\boldsymbol{t}_s^e$ is given as in the classical Biot theory [2]

$$\boldsymbol{t}_s^e = \lambda_s\mathrm{tr}\boldsymbol{E}_s + 2\mu_s\mathrm{tr}\boldsymbol{E}_s.$$

The new quantity inherent to swelling porous media is the hydration pressure p_* [12],[13]. It appears because we have postulated the dependency of A_l on ϕ_l. This dependency is the key point of this work since it distinguishes our theory from other thermodynamical theories for granular nonswelling porous media [7].

We now linearize the entropy inequality about equilibrium to derive near equilibrium results. We choose to linearize only about the one variable which gives a positive quadratic form in the entropy inequality. So for example, if z is a variable of the set $\{\boldsymbol{v}_{l,s}, D_s\phi_l/Dt\}$ which vanishes at equilibrium and f is the coefficient of z within the entropy inequality, the linearization procedure gives an approximation for the near-equilibrium value of f as, $f_{\mathrm{neq}} \approx f_{\mathrm{eq}} + Cz$, where C is the linearization constant. In addition, Λ is a linear function of the rates of change $\{D_sT/Dt,\ \boldsymbol{d}_l$ and $\boldsymbol{d}_s\}$ which are neither dependent (constitutive) nor independent. Hence to satisfy the entropy inequality for all possible processes, the coefficients of these variables must be identically zero. Note that $D_s\phi_l/Dt$ is not included in this latter set but assumed to be a constitutive variable. In order to have the same number of equations as unknowns an additional equation must be added to the system. This is a typical closure issue discussed in [3]. We postulate that $D_s\phi_l/Dt$ is a dependent variable with dependence given in terms of a volume fraction topological law [3]. In applying the above procedure, the entropy inequality yields the following relations

$$\sum_{\alpha=l,s}\varepsilon_\alpha\rho_\alpha\left(\frac{\partial A_\alpha}{\partial T} + \eta_\alpha\right) = 0 \tag{2.6}$$

$$\phi_l\boldsymbol{t}_l = -\phi_l p_l\boldsymbol{I} \tag{2.7}$$

$$\phi_s\boldsymbol{t}_s = -\phi_s p_s\boldsymbol{I} + \boldsymbol{t}_s^e \tag{2.8}$$

$$\phi_l p_*\boldsymbol{\nabla}\phi_l - p_l\boldsymbol{\nabla}\phi_l + \widehat{\boldsymbol{T}}_l = -\boldsymbol{R}_l\boldsymbol{v}_{l,s} \tag{2.9}$$

$$p_l - p_s = \phi_l p_* + \mu_*\frac{D_s\phi_l}{Dt} \tag{2.10}$$

where R_l and μ_* are material coefficients arising from the linearization procedure.

3 Modified Effective Stress Principle

Equation (2.6) is a classical result stating that entropy and temperature are dual variables. Equation (2.10) is crucial in the present formulation. To exploit its physical significance let us introduce the total particle stress tensor $\boldsymbol{t} = \phi_s\boldsymbol{t}_s + \phi_l\boldsymbol{t}_l$. By adding (2.7) and (2.8) and using (2.10) we obtain

$$\boldsymbol{t} + p_l + \mu_*\phi_s\frac{D_s\phi_l}{Dt}\boldsymbol{I} = \boldsymbol{t}_s^e + p_*\phi_l\phi_s\boldsymbol{I}. \tag{3.1}$$

The above result provides important information related to the stress analysis of the swelling particles. For simplicity consider the equilibrium case ($D_s\phi_l/Dt = 0$). First note that if we postulate that the free energy of the adsorbed water A_l is independent of ϕ_l, as in the case of a nonswelling granular medium, then $p_* = 0$, and at equilibrium (3.1) reduces to the form

$$\boldsymbol{t} + p_l\boldsymbol{I} = \boldsymbol{t}_s^e. \tag{3.2}$$

In classical soil mechanics (3.2) is recognized as the Terzaghi's effective stress principle for non-swelling granular media with p_l and $\boldsymbol{t}_s^e$ normally being referred to as pore pressure (or bulk phase pressure) and the effective stress tensor. The modified effective stress principle (3.1) for swelling media has the additional term, $p_* \phi_l \phi_s \boldsymbol{I}$. In contrast with coarse-grained soils where stress mechanisms are primarily controlled by the contact stresses $\boldsymbol{t}_s^e$, for swelling clays such as montmorillonite the additional stress component $p_* \phi_l \phi_s \boldsymbol{I}$ governs the deformation of swelling particles. Clearly this additional intra-particle stress resulting from the presence of adsorbed water within the particles is of physico-chemical nature and can be viewed as a stress structural component arising from surface hydration. Eq (3.1) resembles in form some modified effective stress principles for clays [14] or [10]. Historically, physico-chemical forces have heuristically been modeled at the macroscale through the addition of a term in the Terzaghi's principle which incorporates the net repulsive ($R\boldsymbol{I}$) and attractive ($A\boldsymbol{I}$) forces between particles. This stress is commonly denoted by $(R-A)\boldsymbol{I}$ (see [14],[10]). Equation (3.1) is the first rational attempt to extend the modified Terzaghi's principle and to the partition of particle stress into its adsorbed water, effective matrix, and physico-chemical stress components.

4 Modified Darcy's Law for the Adsorbed Water Flow

Defining the permeability tensor of the clay particles as $\boldsymbol{K}_l = \phi_l^2 \boldsymbol{R}_l^{-1}$ then eliminating $\widehat{\boldsymbol{T}}_l$ in (2.9) using the momentum equation (2.4), using (2.7) and neglecting inertial effects we obtain Darcy's law for the adsorbed water

$$\phi_l \boldsymbol{v}_{l,s} = -\boldsymbol{K}_l \left(\nabla p_l + p_* \nabla \phi_l \right) , \tag{4.1}$$

In addition to a pressure gradient, the above form of Darcy's law contains a gradient of a generalized interaction potential which accounts for swelling. The appearance of this additional term is consistent with the fact that volume fraction gradients provide a potential for adsorbed water flow in a swelling medium. Note that if we postulate that A_l is independent of ϕ_l, then $p_* = 0$ and (4.1) reduces to the classical form of Darcy's law.

5 Viscoelastic Behavior

The modified effective stress principle (3.1) also provides important information regarding the constitutive behavior of the swelling particles near equilibrium. Note that though the solid is considered elastic, the appearance of the retardation term $\mu_* \phi_s D_s \phi_l / Dt$ in (3.1) leads to viscoelastic behavior for the volumetric stresses. The coefficient μ_* may be thought of as a retardation factor which among other effects, accounts for the re-ordering of the adsorbed water molecules as they are disturbed, i.e. an entropic effect. If this is the only source of retardation, then it follows that for a granular media, $\mu_* \approx 0$, since there is very little ordering of the liquid phase in such a medium.

6 Conclusions

A hybrid mixture theory for swelling clay soils was presented. Clay minerals and adsorbed water were treated as distinct phases in order to capture swelling at the particle level and physico-chemical effects. This physico-chemical component has a thermodynamical definition in terms of change of the free energy of the adsorbed fluid with respect to the volume fraction. Some notable consequences of the theory developed herein are that it provides a rational basis for some heuristically modified effective stress principles for smectite clays. In addition a modified form of Darcy's law was derived for the adsorbed water flow.

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