MICROSTRUCTURE AND MECHANICAL PROPERTIES OF DRAINED SANDS

MICROSTRUCTURE ET PROPRIETES MECHANIQUES DES SABLES DRAINES

Anil Misra

Assistant Professor, Department of Civil Engineering
University of Missouri-Kansas City/Columbia
Independence, MO, U.S.A.

SYNOPSIS: The stress-strain behavior of sands is modelled using a micro-mechanical approach. Sands are perceived to be collection of particles. Based upon inter-particle interactions, an expression is derived for the stress-strain behavior of sands under large strain conditions which include inter-particle sliding and separation. The representation of micro-structure is the primary concern in the development of the present model. Under small strain conditions, the fabric tensor is found to adequately represent the granular micro-structure. At large strains, in order to account for heterogeneity, more detailed descriptions of micro-structure are required.

INTRODUCTION

The mechanical properties of sands are significantly influenced by the geometric arrangement of particles, viz. fabric, and the particle mechanical properties, viz. particle elasticity and inter-particle friction. In recent years, based upon the micro-mechanical considerations of inter-particle interactions, efforts have been made to account for the above mentioned factors directly in mathematical models describing the deformation behavior of sands. These micro-mechanical methods, which model sands as a collection of particles, may be classified into two types, namely the discrete approach and the equivalent continuum approach, cf. Misra (1991). In the discrete approach, the deformation behavior of a collection of particles is obtained by considering the equilibrium of each particle in terms of forces developed at particle contacts. Thus each particle is followed during a loading process. In the equivalent continuum approach, the effort is upon approximating the behavior by representing the geometric structure in terms of appropriate statistical measures while preserving the discreteness of inter-particle interaction. The focus of this paper will be upon the equivalent continuum approach for modelling mechanical properties of sands accounting for the effect of micro-structure. The parameters required for a model predictions based upon the equivalent continuum approach are inter-particle friction, elastic properties of particle, and parameters defining the sand micro-structure.

Within the framework of equivalent continuum approach, various approximations of the deformation behavior can be derived that require a varying detail of micro-structure representation. Under very small strain regimes ($\sim 10^{-4}\%$), wherein the strains can be assumed to be fairly uniform in an element of sand, closed form relationships of stress-strain behavior can be derived. In these relationships, the sand micro-structure is represented by the void ratio, average number of contacts per particle, and the directional distribution of contacts (Chang and Misra 1990a). At large strains accompanied by particle slidings and separation, the heterogeneity of granular media dominates such that the uniform strain assumption is not representative and is found to over-predict the stress-strain curve, (Chang and Misra 1990b). To account for the effect of non-homogeneity, effective medium theories, such as the 'self consistent' averaging method, are employed (Hill 1967). In this paper, we describe a methodology, which uses the general framework of 'self consistent' method in conjunction with micro-mechanics based modelling scheme, to develop an effective stress-strain relationship of granular solids accounting for heterogeneity. The micro-mechanics scheme is useful for describing stress-strain behavior at a local level for a micro-element, while the 'self consistent' method provides a tool for describing the effect of interactions among the micro-elements. In the development presented here, it is found that micro-structure representation used in models based upon uniform strain theory is inadequate. Additional statistical descriptors of micro-structure modelling the nearest neighbors for each particle are required.

MICRO-MECHANICAL MODELLING OF SANDS

We consider a micro-element of sand consisting of particles arranged randomly in space supporting imposed loads at the boundary through resistance at inter-particle contacts. The inter-particle interactions are represented through the Hertz-Mindlin model of two non-conforming elastic bodies in contact (Mindlin and Deresiewicz 1953). The model idealizes the elastic contacting bodies to be rigid connected via springs which account for the deformations at particle contacts. The relative movement causes the springs to deform. For example, the relative displacement δ_i^{nm} between particles m and n, is given by

$$\delta_i^{nm}=u_i^m-u_i^n+e_{ijk}(\omega_j^m r_k^m-\omega_j^n r_k^n) \qquad (1)$$

where u_i = the particle displacement, ω_k = the particle rotation, r_j is the vector joining the centroid of a particle to the contact point, the superscripts, n and m, refer to the particles and e_{ijk} is the permutation symbol. And the contact force f_i and the relative displacement δ_i^{nm} are related via the spring stiffness K_{ij} as follows:

$$f_i = K_{ij}\delta_j \tag{2}$$

where K_{ij} is also referred to as the contact stiffness tensor. Very commonly (cf. Chang and Misra 1990b), the contact stiffness tensor is represented by the following simple form

$$K_{ij} = K_n n_i n_j + K_s(s_i s_j + t_i t_j) \tag{3}$$

where K_n and K_s are the contact stiffnesses along the normal and tangential direction of the contact surface respectively. The unit vector **n** is normal to the contact surface and vectors **s** and **t** are arbitrarily chosen such that **nst** forms a local cartesian coordinate system.

For the idealized granular system considered here, two alternative mathematical representations may be envisaged, namely: (1) a discrete description; and (2) an equivalent continuum description. The focus, herein, is on the equivalent continuum representation of the discrete granular system.

EQUIVALENT CONTINUUM APPROACH

For the purposes of continuum description, the granular media is conceptually viewed to be composed of continuum cells. To each continuum cell, we assign a local stiffness tensor relating the local stress σ_{ij}^n and local strain ε_{kl}^n , such that for the n-th cell

$$\sigma_{ij}^n = C_{ijkl}^n \varepsilon_{kl}^n \tag{4}$$

where the superscript n refers to the cell and C_{ijkl}^n is the local stiffness tensor. For convenience, the continuum cells are considered to be 'Voronoi' polyhedra constructed of a single particle and associated void space. The size of these cells is given by $V^n = V_s^n(1+e)$ where V^n is the volume of the n-th cell, V_s^n is the volume of the particle in the cell and e is the void ratio of the granular media. The 'Voronoi' polyhedron is found to be specifically useful in capturing the heterogeneity at each location of granular system.

To preserve the discrete nature of sands, the local stress is related to contact forces and the local strain is related to relative movements between particles in accordance with the micro-mechanical modelling scheme (see Chang and Misra 1990). The local stress σ_{ij}^n for the n-th cell is obtained in terms of the contact force f_j^{nm} generated as a result of the interaction between the particle in the n-th cell and its m-th neighbor, given by (Christoffersen et al. 1981)

$$\sigma_{ij}^n = \frac{1}{2V^n}\sum_m l_i^{nm} f_j^{nm} \tag{5}$$

where the superscript nm refers to the m-th contact of the n-th particle, V^n is the volume associated with the nth particle, and $l_i^{nm} = X_i^m - X_i^n$ is the branch vector joining the centroid of the particle in n-th cell with its neighbor at m-th contact. The average strain of the n-th cell is represented by ε_{kl}^n defined to include the effect of particle rotation as (Chang and Liao 1990)

$$\varepsilon_{ji}^n = u_{i,j}^n + e_{ijk}\omega_k^n \tag{6}$$

where $u_{i,j}^n$ is the local displacement gradient, ω_k^n is the uniform rotation of the particle in the cell and its neighbors.

The local stiffness tensor C_{ijkl}^n is derived in terms of the contact stiffnesses and the relative position of the neighboring particles. The derivation is facilitated by considering: (a) the interaction of two particles; (b) the relationship of local stress and contact forces; and (c) the relationship between local strain and relative movement of the particle in the cell with respect to neighboring particles. The local stiffness tensor C_{ijkl}^n is obtained to be (Misra 1991)

$$C_{ijkl}^n = \frac{1}{2V^n}\sum_m l_i^{nm} K_{jl}^{nm} l_k^{nm} - \frac{1}{2V^n}\sum_m l_i^{nm} K_{jr}^{nm} \Gamma_{rkl}^n \tag{7}$$

where $\Gamma_{rkl} = (\sum_\alpha K_{rs}^{n\alpha})^{-1}(\sum_\alpha K_{sl}^{n\alpha} l_k^{m\alpha})$. The stiffness tensor thus obtained is a function of the packing structure measures l_i and V^n, and contact stiffnesses K_n and K_s. Recall that the contact stiffness depends upon the particle elastic properties and inter-particle friction.

In equivalent continuum approach the overall stress-strain behavior of an element of granular media containing several particles is required. The 'self consistent' averaging method offers a powerful approach for obtaining the overall stress-strain behavior of a collection of particles. In this regard, the 'self consistent' method similar to that used for polycrystals, albeit in a more generalized setting, is utilized here to obtain the overall stiffness tensor for the granular media.

For an element of sand, the overall nominal stress $\bar{\sigma}_{ij}$ and strain $\bar{\varepsilon}_{ij}$ are obtained based on the volume averaging criteria as

$$\bar{\sigma}_{ij} = \frac{1}{V}\sum_n V^n \sigma_{ij}^n \quad and \quad \bar{\varepsilon}_{ij} = \frac{1}{V}\sum_n V^n \varepsilon_{ij}^n \tag{8}$$

The incremental strain at the particle level ε_{ij}^n is estimated in terms of the overall strain $\bar{\varepsilon}_{ij}$ as

$$\varepsilon_{pq}^n = H_{pqkl}\bar{\varepsilon}_{kl} \tag{9}$$

where H_{pqkl}^n is a 'concentration' tensor conceptually similar to Eshelby's 'transformation' tensor. The 'concentration' tensor is obtained by considering each particle to be equivalent to an ellipsoidal inhomogeneity of stiffness C^n_{ijkl} embedded in an infinitely extended homogeneous media of equivalent stiffness C_{ijkl} (see Misra 1991). The equivalent stiffness tensor C_{ijkl} is given by (from Eqs. 4, 8 and 9)

$$C_{ijkl} = \frac{1}{V}\sum_n V^n C^n_{ijpq} H^n_{pqkl} \tag{10}$$

Since the 'concentration' tensor H^n_{pqkl} is a function of the equivalent stiffness tensor C_{ijkl} an iterative numerical effort is required to obtain the equivalent stiffness tensor at each load increment.

EXAMPLE APPLICATIONS

To demonstrate the applicability of the stress-strain law we perform two calculations: (1) we simplify the stress-strain law under uniform strains to study the initial moduli of sands, and (2) we study complete stress-strain curves of computer generated particle assemblies.

Initial Moduli

Under uniform strains the stress-strain law can be simplified to

$$C_{ijkl} = \frac{N}{2V}\int_\Omega l_i(\Omega)K_{jl}(\Omega)l_k(\Omega)\xi(\Omega)d\Omega \tag{11}$$

where $\int_\Omega (\ )d\Omega = \int_0^{2\pi}\int_0^{\pi}(\ )\sin\gamma\, d\gamma\, d\beta$. In Eq. 11, $\xi(\Omega)$ is a density function of directional distribution of contacts. For fairly uniform sized particles, the stress-strain relationship is further simplified to

$$C_{ijkl} = \frac{r^2N}{2\pi V}\int_0^{2\pi}\int_0^{\pi} K_{jl}F_{rs}n_i n_k n_r n_s \sin\gamma\, d\gamma\, d\beta \tag{12}$$

where F_{rs} is a fabric tensor which accounts for the inherent anisotropy of the sand deposit (see Chang and Misra 1990a). The above relationships are fairly general and can be numerically evaluated for various loading conditions by appropriately accounting for force dependent contact stiffnesses, such as Hertz-Mindlin contact stiffnesses.

For example, in the case of an inherently isotropic sand element subject to an initial isotropic confining stress, the shear modulus and Poisson's ratio of the sand element can be written in a closed form as

$$G = \frac{1}{10}\left(\frac{\bar{n}}{4\pi(1+e)}\right)^{1-\alpha} r^{2\alpha-1}C_1(2+C_2)\sigma_c^\alpha \tag{13}$$

and

$$\nu = \frac{1-C_2}{4+C_2} \tag{14}$$

In obtaining Eq. 13 and 14, the contact stiffnesses were taken to be $K_n = C_1 f_n^\alpha$ and $K_s = C_2 K_n$. The factors C_1 and C_2 can be obtained from Hertz-Mindlin contact theory in terms of the particle elastic properties. The exponent α is quoted to vary between 0.33 to 0.6. It is interesting to note that the expression for shear modulus in Eq. 13, which was obtained through purely theoretical considerations bears a close resemblance to the empirical expressions proposed by Hardin and Drnevich (1972) and Chung et al (1984). These expressions are also found to agree well with experimental results (Chang and Misra 1989).

Stress-Strain Curve

While under uniform strains the stress-strain relationship can be considerably simplified and the geometric structure represented by an overall fabric tensor, the representation of geometric structure becomes complex in problems involving non-uniform strains as is the case for stress-strain behavior at larger strain levels. This may be seen from the expression for the overall stress-strain law in Eq. 10 which explicitly involves a local stress-strain law valid at a particle level. In order to obtain the local stiffness tensor, the required geometric information includes the number of nearest neighbors and the local fabric tensor. The statistical characterization of this information is still an unsolved problem. Nevertheless, the stress-strain curve of computer generated random assemblies of particles can be analyzed using the derived theory. An example is presented here to illustrate the model capability. The computer generated random packing of particles used in the example is shown in Fig. 1. The inter-particle friction angle is taken to be 10° and the contact stiffnesses are taken to be $K_n = K_s = 20kN/m$.

The stress-strain behavior is investigated under a compressive biaxial loading condition. Prior to the beginning of the loading path, the assembly is loaded to an isotropic confining condition. During the incremental loading the horizontal stress is held constant at the initial confining stress, i.e. $\bar{\sigma}_{xx}=\sigma_c$ and $\Delta\bar{\sigma}_{xx}=0$, while a strain controlled compressive load is applied in the vertical direction, i.e. $\Delta\bar{\varepsilon}_{yy}>0$. The stress-strain curve obtained for the disk assembly in terms of the vertical strain $\bar{\varepsilon}_{yy}$ and stress ratio $\bar{\sigma}_{yy}/\bar{\sigma}_{xx}$ is shown in Fig. 2 for two values of confining stresses (σ_c = 130 KPa and 300 KPa). Under a low confining stress the behavior is seen to be brittle accompanied by a large amount of strain softening subsequent to a peak strength. Under a high confining stress the behavior is more ductile exhibiting almost no strain softening. The volume change behavior of the assembly is shown in Fig. 3. The behavior is dilative at a low confining stress and compressive at higher confining stress. Experimental results on sands exhibit similar stress-strain and volume change behavior under varying confining stresses.

CONCLUSIONS

A micro-mechanics based stress-strain model is presented for sands under drained conditions. The unique aspect of the model is that it explicitly accounts for the geometric structure of sands as well as the particle properties. It is found that the representation of geometric structure by an overall fabric tensor is useful under uniform strain assumption, such as for estimating initial moduli. However, to obtain the complete stress-strain curve for sands, local fabric tensors and distributions of particle nearest neighbors are required. Although this a daunting task, the present formulation is encouraging since it captures the salient features of sand behavior, namely hardening, softening, stress dependency and dilation-contraction. In addition, the present approach provides a methodology by which a discrete granular system can be modelled as an equivalent continuum system which is mathematically and computationally more tractable. This approach will be useful for comprehensive modelling of granular materials under complex loading conditions.

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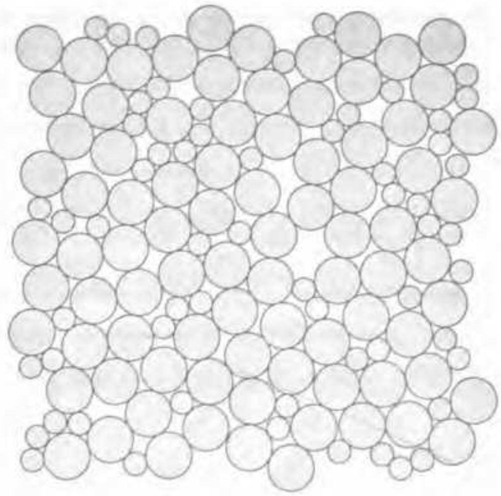

Fig. 1 Particle packing used in the example.

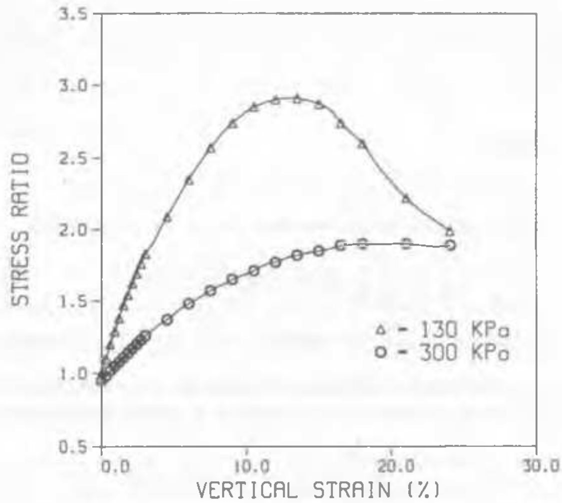

Fig. 2 Stress-strain behavior.

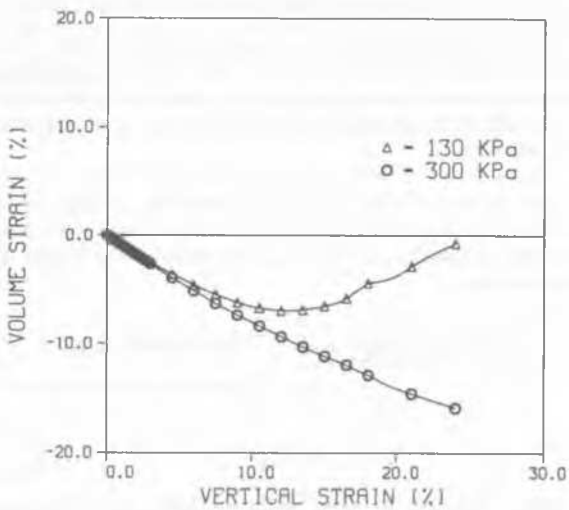

Fig. 3 Volume change behavior.