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THE MOTION OF PLATEY PARTICLES UNDER A SIMPLE SHEAR AND PURE SHEAR DEFORMATION OF THE SURROUNDING FLUID

LE MOUVEMENT DE PARTICULES PLANES SOUS DEFORMATION D'EFFORT TRANCHANT SIMPLE ET EFFORT TRANCHANT PUR DU FLUIDE ENVIRONNANT

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SYNOPSIS: The particular case of a disc-shaped or perfectly platey particle orienting under the influence of shear deformation of the fluid surrounding the particle, has been investigated as an extension of the more general theory of motion of ellipsoidal particles immersed in a viscous fluid. The motion has been studied for simple shear and pure shear deformation of the fluid. It has been shown how, under finite simple shear strain, the particle will tend to orient itself parallel to zero extension lines, i.e. the direction of straining. During pure shear deformation the particle will tend to orient itself perpendicular to the direction of maximum compression, i.e. the particle will tend to lie parallel to the planes of maximum elongation. This result agrees with experimental, very well known evidence.

INTRODUCTION.

The motion of ellipsoidal rigid particles immersed in a matrix of viscous fluid during simple shear deformation, has been treated into full detail by Jeffery (1923) who extended the works of Einstein (1896) (1911), Oberbeck (1876) and Edwards (1892). Jeffery found out that the motion of an ellipsoidal particle in a simple shear deformation field is periodic, with its axis of revolution describing a cone about the perpendicular to the plane of the undisturbed motion, to which its inclination varies between limiting values. In the degenerate case of a disc, Jeffery quotes that it can take any position in which its faces are entirely composed of stream-lines of the undisturbed motion of the fluid, moving through the fluid edge-on with its axis at any inclination to the fluid shear.

Later on Jaeger (1969), discussing the work of Jeffery, comments that if an ellipsoidal particle tends to be disc-shaped (excentricity=1) the particle tends to set itself parallel to the planes of shearing.

The motion of ellipsoidal rigid particles immersed in a matrix of viscous fluid during pure shear deformation has been thoroughly treated by Gay (1968), while trying to justify that the development of preferred orientation in geological environnments is best described by a pure shear type of flow. Gay treated the problem deriving his equations from Jeffery's general equations of motion for ellipsoidal particles, and concluded that if the ellipsoid is originally aligned with its axes parallel to either of the strain axes, it will remain stationary in these stable positions. However, if its major axis makes some angle with the direction of shortening, the ellipsoid will rotate towards the direction of elongation, becoming parallel to it after an infinite amount of strain. The rate of this rotation increases with increasing excentricity of the ellipsoid.

While, as quoted by Jaeger (1969), the theory of finite strain is concerned only with the geometry of two sets of positions of the particles of the body, i.e. the initial and final state, it does not consider the way in which the particles move from their initial positions to their final positions, or the way in which the parameters describing the strain vary during this movement.

Jaeger (1969) further emphasizes the importance of the study of the movement of marker points (such as veins or outlines of fossils) in geological materials, pointing out that such a study is of direct practical significance.

On the other hand, the development of preferred orientation of platey particles during shear of geologically weak materials (soils) has been thoroughly demostrated in the literature, when studying the post-peak behaviour of clay-like materials and the development of residual strength, differentiating, in the way, the "turbulent" versus the "sliding" soil behaviour; see for instance Skempton (1964); Lupini (1981), and Lupini et al. (1981). Therefore a tool for predicting the development of such orientation could be useful in theory. In practice one may correctly argue wether a theory based on laminar motion of a viscous fluid could be used for predicting quantitatively the behaviour of actual materials on an engineering scale.

While this could place a severe limitation to the use of such a tool, it does not contravene its qualitative predictive capacity which is in full agreement with experimental and field observations. Figures 1 to 4, which follow, show the orientation of platey particles after being sherared at very large strains in the Imperial College Bishop's design ring shear apparatus. Lupini (1981).

Figure 1 is a scanning electron micrograph of the principal shear surface after Taren ring shear test. View is normal to the surface. The direction of movement is clear. A silt-size particle which has degraded to a "planar" shape shows in the middle of the plate. This large particle has not rotated in the plane of shear as the grooves have the same direction as the striations (slickensides).

Figure 2 is a scanning electron micrograph of the principal shear surface formed after a ring shear test on a high liquid limit Folkestone Gault clay sample. View is in the direction of shear (in and out of the plane of the photograph). The shear surface is normal to the plane of the photograph and goes from the bottom left to the top right of the photograph. The good orientation of the platey minerals can be seen. A few plates protrude from the shear plane probably as a result of the separation process employed to show the surface parallel to the plane of the photograph. Sliding behaviour results.

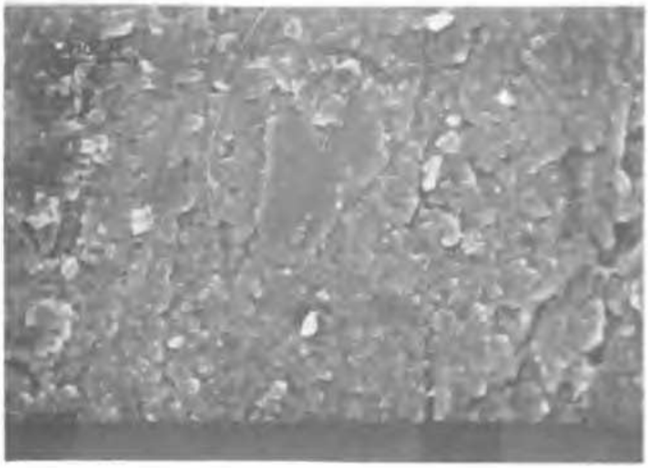

Figure 1. Scanning electron micrograph of the principal shear surface after Taren ring shear test. Lupini (1981).

Figure 2. Scanning electron micrograph of the principal shear surface formed after a ring shear test on a high liquid limit Folkestone Gault clay sample. Lupini (1981).

Figure 3 is a scanning electron micrograph of a London clay sample after ring shear test. View is normal to the principal shear surface and shows the tight arrangement of clay plates. Striations indicate the direction of movement. Note that there are no "gaps" between clay plates. Typical sliding behaviour results with low residual strength.

Figure 4 is a scanning electron micrograph of a London clay sample after ring shear test. It is a view of the shear surface. The shear direction is from left to right and right to left. Note the near perfect particle orientation of clay particles.

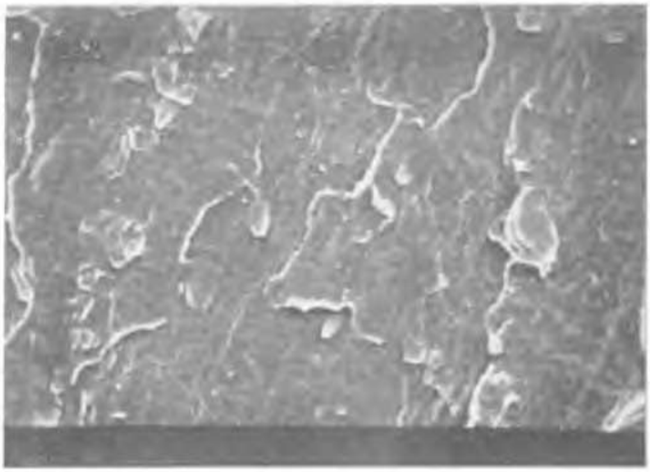

Figure 3. Scanning electron micrograph of a London clay sample after ring shear test. Lupini (1981).

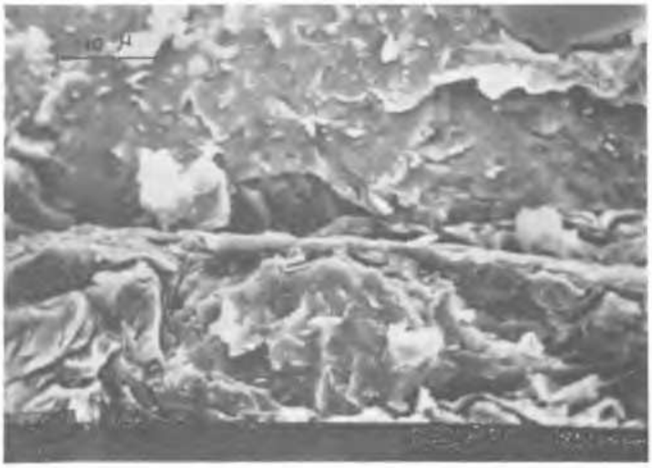

Figure 4. Scanning electron micrograph of a London clay sample after ring shear test. Lupini (1981).

DEVELOPMENT OF PREFERRED ORIENTATION.

Development Of Preferred Orientation Of Platey Particles Under Simple Shear Deformation Of The Surrounding Fluid.

The assumptions for this case follow from the work of Jeffery (1922) in which the flow of material around the particle is of a viscous kind and where acceleration effects can be neglected because of the slow rate of motion. Also, the particle is rigid and its shape is not influenced by the applied stresses which can only produce orientation of the particle in the space coordinate system. He considered the slow motion of a rigid ellipsoidal particle of semiaxes A,B,C embedded in an incompressible fluid of viscosity μ. He found that the equations that govern the motion at the boundary of the particle, which is solidary with the fluid motion at that boundary (no slip at the interface) are (using Jaeger notation):

$$(B^2 + C^2)\,\omega_1 = B^2\,(p+f) + C^2\,(p-f) \quad \text{......} (1)$$

$$(C^2 + A^2)\,\omega_2 = C^2\,(q+g) + A^2\,(q-g) \quad \text{......} (2)$$

$$(A^2 + B^2)\,\omega_3 = A^2\,(r+h) + B^2\,(r-h) \quad \text{......} (3)$$

where A,B,C are the semiaxial lengths of the ellipsoid and ω_1, ω_2, ω_3 are the spins or rates of rotation of the ellipsoid about its own x,y,z axes.

On the other hand, p, f, q, g, r and h are the constants that describe the rate of rotation according to the equation that describes the undisturbed motion of the fluid in the neighbourhood of the particle:

$$\dot{u} = ax + hy + gz + \quad qz - ry \quad \text{......} (4)$$

$$\dot{v} = hx + by + fz + \quad rx - pz \quad \text{......} (5)$$

$$\dot{w} = gx + fy + cz + \quad py - qx \quad \text{......} (6)$$

components for the rate of strain — **components for the rate of rotation**

where also: $\dot{u} = dx/dt$, $\dot{v} = dy/dt$, $\dot{w} = dz/dt$.

If one defines a fixed coordinate system x, y, z in order to locate the particle in space such as in Figure 5.

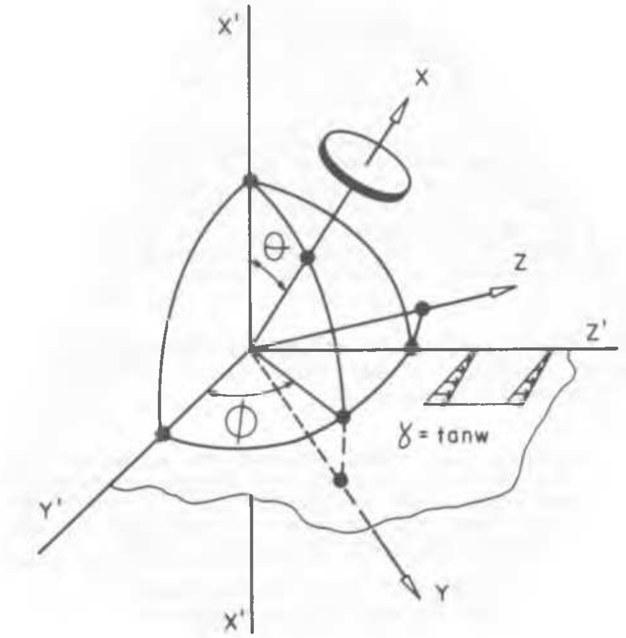

Figure 5. Eulerian reference system for the motion of a disc-shaped particle induced by a simple shear or laminar motion on the y′z′ plane of the fluid surrounding the particle.

In the particular case of simple shear in the y′z′ plane, the solution of system of equations **(1)**, **(2)**, **(3)** was found by Jeffery (1922) to be for an ellipsoid of revolution with B = C ≠ A.

$$(A^2 + C^2)\,\dot{\phi} = \dot{\gamma}\,(A^2\cos^2\phi + C^2\sin^2\phi) \quad \text{......} (7)$$

$$(A^2 + C^2)\,\dot{\theta} = (1/4)\,\dot{\gamma}\,(A^2 - C^2)\sin 2\theta \sin 2\phi \quad \text{......} (8)$$

If one is dealing with disc-shaped particles therefore $A \Rightarrow 0$ which means that equations **(7)** and **(8)** are reduced to:

$$\dot{\phi} = \dot{\gamma}\sin^2\phi \quad \text{......} (9)$$

$$\dot{\theta} = -\,(1/4)\,\dot{\gamma}\sin 2\theta \sin 2\phi \quad \text{......} (10)$$

Solving first for ϕ one has:

$\dot{\phi} = \dot{\gamma}\sin^2\phi$, or: $(\dot{\phi} / \sin^2\phi) = \dot{\gamma}$, therefore:

$[(d\phi / dt) / \sin^2\phi] = d\gamma / dt$,

also: $$\int_{\phi_i}^{\phi} (1/\sin^2\phi)\,d\phi = \int_{\gamma_i}^{\gamma} d\gamma \quad \text{......} (11)$$

If one defines the initial value of shear such as $\gamma_l = 0$ then one obtains (ϕ_l may have any value) upon integration of **(11)**:

$$\int_{\phi_i}^{\phi} \operatorname{cosec}^2\phi\, d\phi = -\cot\phi \Big]_{\phi_i}^{\phi} = \gamma \quad \text{......} (12)$$

obtaining:

$$\cot\phi_i - \cot\phi_f = \gamma \quad \text{......} (13)$$

when the shear reaches the value of $\cot\phi_i$:

$\gamma = \cot\phi_i$, therefore $\cot\phi_f = 0$; or

$\phi_f = (\pi/2 \;\; ; 3\pi/4)$

We shall prove that this value corresponds to the minimum or the maximum value of the angle θ during the particle movement. The final value of angle ϕ_f after infinite amount of strain will be:

when $\gamma \rightarrow \infty$; then $\phi_f \rightarrow \pi$

That is, whichever value takes ϕ_i the final value will tend to be π, which means that the particle will orient so that the xy plane will coincide with the x′y′plane after infinite amount of strain. Solving now for θ, using equations **(9)** and **(10)**

$$\dot{\phi} = \dot{\gamma}\sin^2\phi \quad \text{......} (9)$$

$$\dot{\theta} = -\,(1/4)\,\dot{\gamma}\sin 2\theta \sin 2\phi \quad \text{......} (10)$$

dividing equation **(10)** by equation **(9)** it follows that:

$\dot{\theta} / \dot{\phi} = -(1/2) \sin 2\theta \, (\cos\phi / \sin\phi)$

which may be rearranged as:

$\dot{\phi} \cot\phi = - 2\dot{\theta}/\sin 2\theta$

which can be expressed as in equation (14).

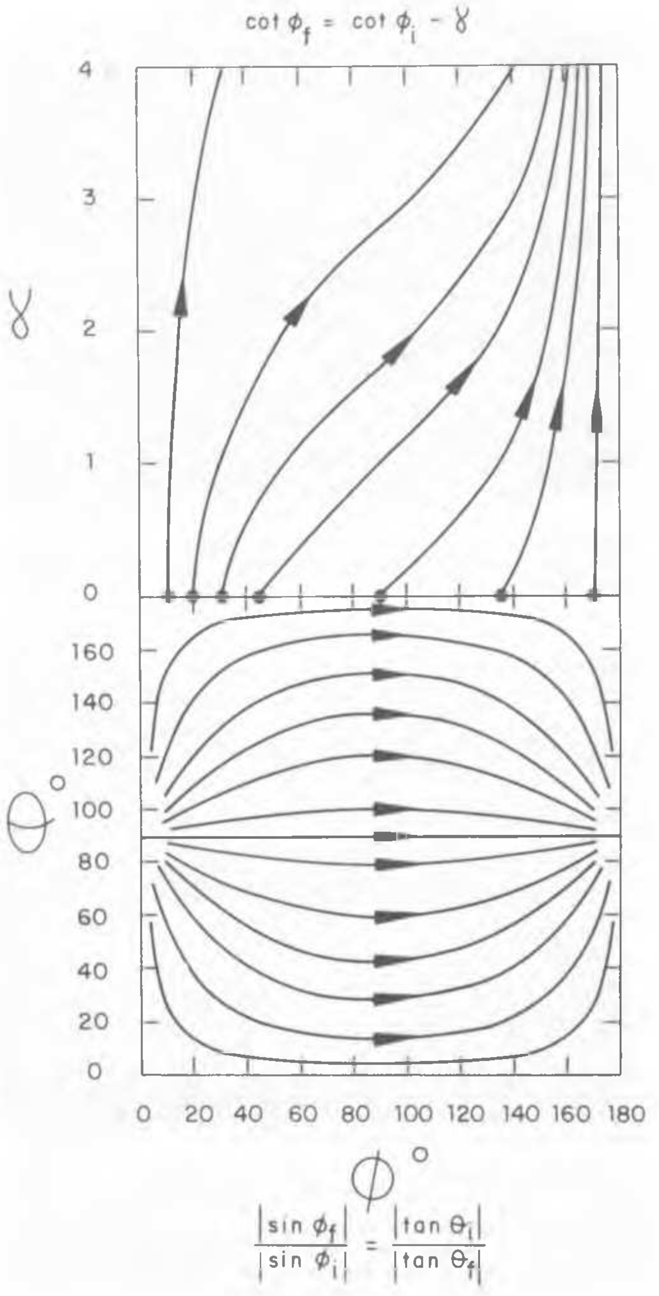

Figure 6. Orientation of a platey or disc-shaped particle under a simple shear deformation of the surrounding fluid.

$$\int_{\phi_i}^{\phi_f} \cot\phi \, d\phi = - \int_{\theta_i}^{\theta_f} (2/\sin 2\theta) d\theta \qquad ......(14)$$

and is readily integrated to give:

$$\ln|\sin\phi| \Big]_{\phi_i}^{\phi_f} = -\ln|\tan\theta| \Big]_{\theta_i}^{\theta_f} \qquad ....(15)$$

or:

$$\ln|\sin\phi| \Big]_{\phi_i}^{\phi_f} = \ln|\cot\theta| \Big]_{\theta_i}^{\theta_f} \qquad ....(16)$$

therefore:

$$\ln|\sin\phi_f| - \ln|\sin\phi_i| = \ln|\cot\theta_f| - \ln|\cot\theta_i|$$

or:

$$\ln\{|\sin\phi_f| / |\sin\phi_i|\} = \ln\{|\cot\theta_f| / |\cot\theta_i|\} ....(17)$$

One can now proceed to plot the meaning of equations **(13)** and **(17)** which relate the strain γ with ϕ_f, and ϕ_f with θ_f respectively. If this is done one obtains the results of figure 6 which relates γ with θ and ϕ.

From this figure one can easily conclude that, independently from the initial values of θ and ϕ, for infinite amount of strain $\theta \to \pi/2$ and $\phi \to \pi$; that is the particle will orient itself to be perpendicular to the plane y'z' and parallel to the z' direction which means that it will set itself parallel to the planes of shearing such as Jaeger (1969) pointed out.

This orientation or reorientation of platey particles during shear has been observed by many authors in the case for instance of mica and clay particles which are disc-shaped by their layer-type of structure. Figure 6 allows us to study trends and to have a qualitative appreciation of the reorientation mechanism of platey particles subjected to shear deformation.

The Development Of Preferred Orientation Of Platey Particles Under Pure Shear Deformation Of The Surrounding Fluid.

For this case Gay (1968) obtained the equations that govern the motion of an ellipsoidal particle embedded in a viscous fluid which is subjected to a pure shear deformation. Gay's work is based on equations **(1)**, **(2)**, **(3)**, which were put forward by Jeffery (1922).

The reader is referred to the original paper for a full discussion of the derivation of the equations.

Here one will define again, as in the previous simple shear case, a fixed coordinate system x'y'z' in order to locate the particle in space such as follows in figure 7.

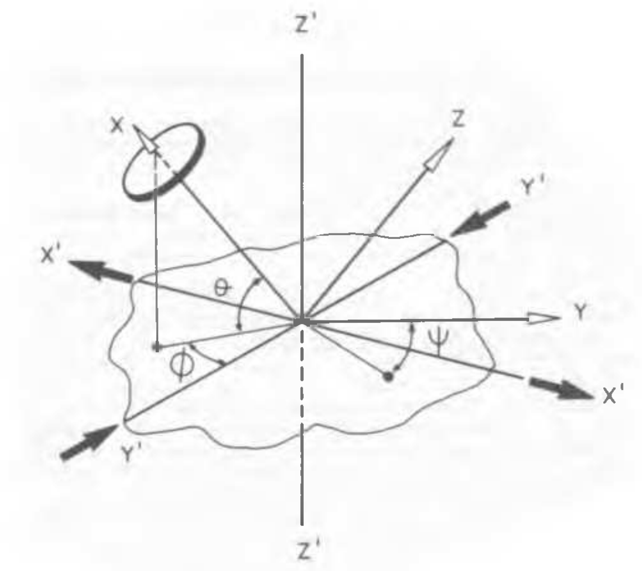

Figure 7. Eulerian reference system for the motion of a disc-shaped particle induced by a pure shear motion of the x′y′ plane of the fluid surrounding the particle.

Gay (1968) found that the equation that relates strain with the Eulerian angle φ is:

$$\ln(\cot\phi_f)=\ln(\cot\phi_i) + [(A^2-B^2)/ (A^2+B^2)]\ln(\lambda_2/\lambda_1)^{1/2} \quad .....(18)$$

where $\lambda_1{}^{1/2}$ and $\lambda_2{}^{1/2}$ are the major and minor semi axial lengths for the strain ellipse and are related to the natural strain by the following equation:

$$2\bar{\varepsilon} = \ln (\lambda_1 / \lambda_2)^{1/2} \quad ....(19)$$

which in turn is related to the shear strain γ, not taking into account the rotational component by:

$$2\bar{\varepsilon} = 2 \sinh^{-1} [(1/2)\tan\gamma] \quad ....(20)$$

The equation that relates the Eulerian angle θ to φ was found by Gay (1968) to be:

$$\cot\theta_f = \cot\theta_i [(\sin 2\phi_i) / (\sin 2\phi_f)]^{1/2} \quad ....(21)$$

Developing equation **(18)** one has for the case of a disc (B=C ; A=0):

$$\ln \cot\phi_f = \ln \cot\phi_i + \ln (\lambda_1/\lambda_2)^{1/2}$$

or:

$$\cot \phi f = e^{[\ln \cot\phi_i + \ln (\lambda_1/\lambda_2)^{1/2}]} \quad ....(22)$$

which can be also expressed as follows:

$$\tan \phi f = e^{-[\ln \cot\phi_i + \ln (\lambda_1/\lambda_2)^{1/2}]} \quad ....(23)$$

therefore obtaining ϕ_f as a function of ϕ_i and the strain:

$$\phi f = \tan^{-1} \{e^{-[\ln \cot\phi_i + \ln (\lambda_1/\lambda_2)^{1/2}]}\} \quad ....(24)$$

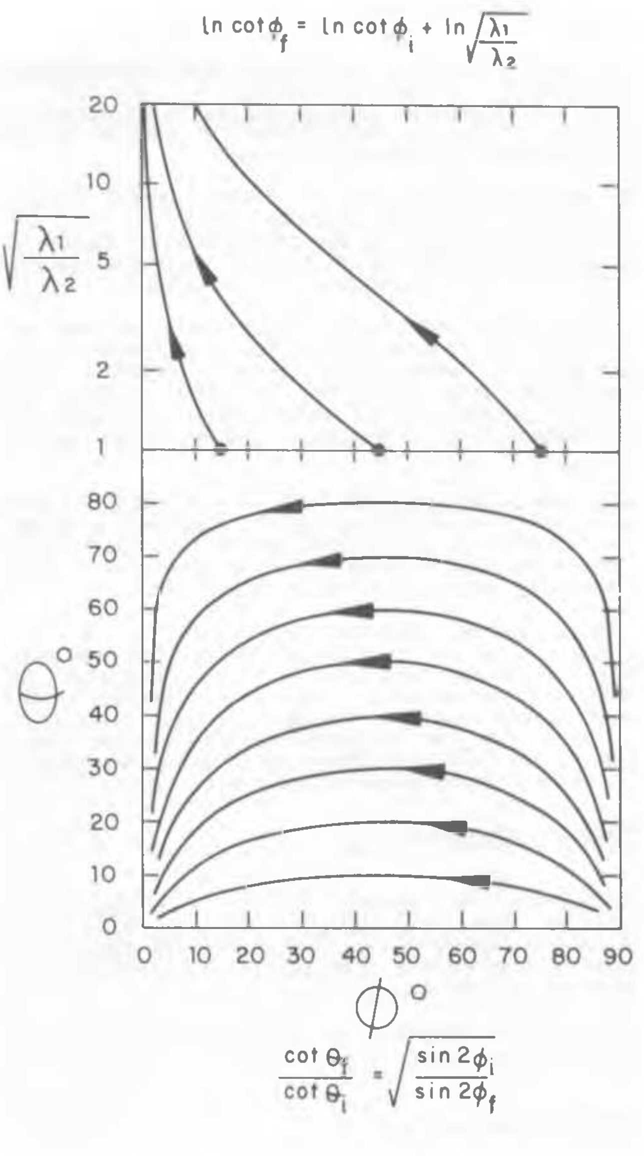

Figure 8. Orientation of a platey or disc-shaped particle under a pure shear deformation of the surrounding fluid.

On the other hand one can investigate the variation of θ with φ by means of equation **(21)** so that a plot relating the strain λ_1/λ_2 in pure shear with ϕ_f and θ_f can be obtained to give the result of figure **8**.

From this figure one can easily conclude that, independently from the initial values of θ and φ for infinite amount of strain θ → 0° and φ → 0°, which means that the particle will orient itself to be perpendicular to the direction of maximum compression (y′ axis), this also means that it will set itself parallel to the maximum extension planes, this result being equivalent to the previous analysis for the simple shear deformation mode.

CONCLUDING REMARKS.

It has been shown how, a platey or disc-shaped particle surrounded by a viscous deforming fluid will tend to orient, in a progressive manner, parallel to the planes of shearing, in case of simple shear, or parallel to the extension planes, in case of pure shear.

The post-peak shape of the shear strength versus deformation curve of clay-like materials may be qualitatively explained by the progressive orientation of disc-shaped particles which tend to align themselves parallel to the shear surface or shear surfaces.

It can be shown that, during drained ring shear tests on clay materials, the decay of shear resistance to the residual state, when platey particles reach a near perfect orientation parallel to the shear surface (or surfaces), generally occurs before 100 mm to 1000 mm displacement. See for instance Figure 4 in Lupini et al. (1981).

The shape of the post-peak curve is a function of many factors such as, to mention a few, thickness of shear zone, presence of cementation bonds, ratio of massive to platey-like clay minerals in the soil composition and, as shown by Lemos et al (1985), rate of strain.

In a simplified, approximate, modelling of soil behaviour, it may be possible to correlate the post-peak decay of shear strength with an "average" or statistically determinated varying interference angle, measured with respect to the shear surface orientation. So that, when full residual conditions have been reached, full sliding frictional behaviour predominates and the interference angle will tend to be zero.

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This paper is dedicated to the memory of the late Professor A.W. Bishop.

SYMBOLS AND NOTATION.

A, B, C	semiaxial lengths of an ellipsoid.
a,f,g,h	constants for rate of strain.
p,q,r	constants for rate of rotation.
e	Napierian number.
sin	trigonometrical sine function.
sinh	hyperbolic sine function.
cos	trigonometrical cosine function.
tan	trigonometrical tangent function.
cot	trigonometrical cotangent function.
$\dot{u},\dot{v},\dot{w}$	components of velocity in the undisturbed motion of the fluid.

Greek:

ϕ	Eulerian angle.
$\omega_1,\omega_2,\omega_3$	spins.
ε	natural strain.
π	180°
γ	shear deformation.
θ	Eulerian angle.
$\lambda_1^{1/2}$, $\lambda_2^{1/2}$	major and minor semi axial lengths for the strain ellipse
μ	viscosity.

REFERENCES.

Edwards (1892). As cited by Jeffery (1922). *Quart. Jour. Math, Vol.26.*

Einstein (1896). As cited by Jeffery (1922). Eine neue Bestimmung der Moleküldimensionen. *Ann. d. Physik, vol.19, p.289, with a correction in vol.34, p.591 (1911).*

Gay, N.C. (1968). The Motion of Rigid Particles Embedded in a Viscous Fluid during Pure Shear Deformation of the Fluid. *Tectonophysics, 5, pp. 81-88.*

Jaeger, J.C. (1969). Elasticity , Fracture and Flow with Engineering and Geological Applications. *Third Edition, Methuen & Co. LTD.*

Jeffery, G.B. (1922). The Motion of Ellipsoidal Particles Immersed in a Viscous Fluid. *Proc. Roy. Soc. of London A(102). pp 161-179.*

Lemos, L. ; Skempton, A.W.; Vaughan, P.R. (1985). Earthquake loading of shear surfaces in slopes. *Proc. XI Int. Conf. on Soil Mech. and Foundation Engng., San Francisco vol 4 pp. 1955-1958.*

Lupini, J.F. (1981). The residual strength of soils. *PhD thesis, University of London.*

Lupini, J.F ; Skinner, A . E ; Vaughan, P . R. (1981). The drained residual strength of cohesive soils. *Géotechnique 31, Nº 2, 181-213.*

Oberbeck (1876). As cited by Jeffery (1922). *Crelle, vol.81.*

Skempton, A.W. (1964). Long-term stability of clay slopes. *Géotechnique 14, Nº 2, 77-102.*