XIII ICSMFE, 1994, New Delhi, India / XIII CIMSTF, 1994, New Delhi, Inde

A CONSTITUTIVE LAW FROM FRICTIONAL TO COHESIVE MATERIALS

UNE LOI CONSTITUTIVE POUR MATERIAUX FROTTANTS OU ADHERENTS

H. Matsuoka[1] **D.A. Sun**[2] **T. Konda**[2]

[1]Professor, [2]Graduate Student
Department of Civil Engineering
Nagoya Institute of Technology, Nagoya, Japan

SYNOPSIS: A unified constitutive law from frictional to cohesive materials is presented on the basis of "Extended Spatially Mobilized Plane (Extended SMP)" with a parameter of "bonding stress σ_0". The "Extended SMP" includes the SMP applicable to frictional materials such as granular materials ($\sigma_0=0$) and the octahedral plane applicable to cohesive materials such as metals ($\sigma_0\to\infty$) at the both ends. The constitutive law is checked by true triaxial tests ($\sigma_1>\sigma_2>\sigma_3$), triaxial compression tests ($\sigma_1>\sigma_2=\sigma_3$) and triaxial extension tests ($\sigma_1=\sigma_2>\sigma_3$) on a cement-treated sand selected as an intermediate material with both friction and cohesion. These test results are analyzed by an elasto-plastic model derived from the unified constitutive law on the "Extended SMP".

INTRODUCTION

It is an interesting problem to present a unified constitutive law including a failure criterion for a wide range of engineering materials from frictional materials such as granular materials without bond between grains, to cohesive materials such as metals with strong bond due to crystalline structure. One of the authors has already proposed the Matsuoka-Nakai criterion and unique stress-strain relations for granular materials such as soils on the basis of "Spatially Mobilized Plane (SMP)" (Matsuoka and Nakai,1974 and 1985).

In order to extend the Matsuoka-Nakai criterion and constitutive relation on the "SMP" for frictional materials to those for frictional and cohesive materials, the idea of "Extended Spatially Mobilized Plane (Extended SMP)" with a parameter of "bonding stress σ_0" will be introduced. The "Extended SMP" becomes the "SMP" when $\sigma_0=0$ and the octahedral plane when $\sigma_0\to\infty$ as the two extremities. The proposed failure criterion and constitutive relation on the "Extended SMP" are reduced to the Matsuoka-Nakai criterion and stress-strain relation on the "SMP" in the case of $\sigma_0=0$ and the Mises criterion and stress-strain relation on the octahedral plane in the case of $\sigma_0\to\infty$.

To investigate the constitutive law based on the "Extended SMP", a cement-treated sand is selected as an intermediate material with friction and cohesion, and true triaxial ($\sigma_1>\sigma_2>\sigma_3$), triaxial compression ($\sigma_1>\sigma_2=\sigma_3$) and triaxial extension ($\sigma_1=\sigma_2>\sigma_3$) tests are carried out on the cement-treated sand. The arrangement of test results of the cement-treated sand on the "Extended SMP" will be tried here, in order to check whether unique stress-strain relations are obtained under the three-dimensional stress conditions or not. Based on the "Extended SMP", an elasto-plastic model for frictional and cohesive materials is developed, which will be used to predict the test results.

TEST PROCEDURE

A cement-treated Toyoura sand (mixing ratio by weight; Toyoura sand:cement:water=15:1:3, curing period=about 3 months) is used for tests. The physical properties of Toyoura sand are as follows: D_{50} =0.2mm, U_c=1.3, G_s=2.65, e_{max}=0.95 and e_{min}=0.58. The size of the sample for true triaxial tests is 10cm×10cm×10cm and for triaxial compression and extension tests is 5cm in diameter and 10cm in height. In the true triaxial apparatus, three principal stresses (σ_1, σ_2 and σ_3) are independently applied to the sample by three pairs of rigid loading plates. In order to reduce the friction between the sample and the loading plates, a pair of loading plates in the vertical direction is counterbalanced, as shown in Fig.1. All tests are conducted under constant mean effective principal stresses σ_m ($=(\sigma_1+\sigma_2+\sigma_3)/3$).

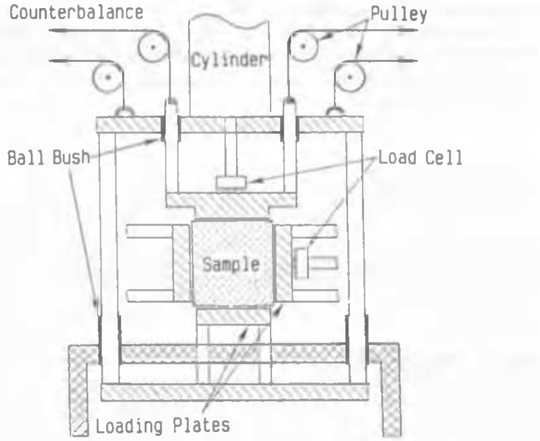

Fig. 1 Side view of true triaxial apparatus

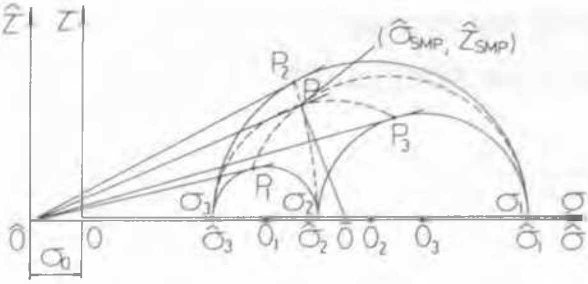

Fig. 2 Normal and shear stresses on "Extended SMP" in Mohr's plane

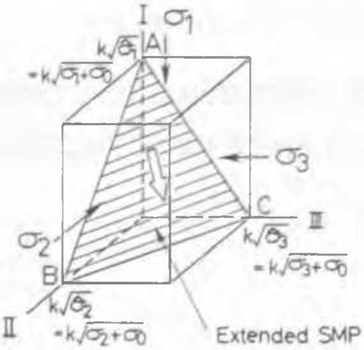

Fig. 3 "Extended SMP" in physical space

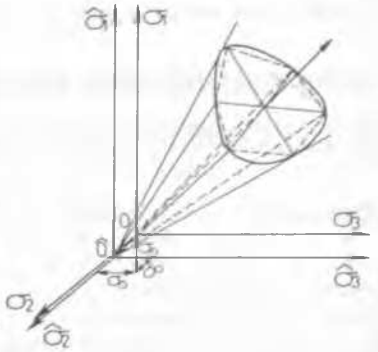

Fig. 4 Failure criterion based on "Extended SMP" in principal stress space

A UNIFIED CONSTITUTIVE LAW FOR FRICTIONAL AND COHESIVE MATERIALS BASED ON "EXTENDED SMP"

As shown in Fig.2, by introducing a parameter σ_o ($=c\cdot\cot\phi$) and taking the new $\hat{\sigma}(=\sigma+\sigma_o)$ vs. $\hat{\tau}(=\tau)$ coordinates, the same form of equations as that for frictional materials ($\sigma_o=0$) can be developed for frictional and cohesive materials ($\sigma_o>0$). Using the principal stresses $\hat{\sigma}_i$ ($=\sigma_i+\sigma_o$, $i=1,2$ and 3) and the stress invariants $\hat{J}_1$ ($=\hat{\sigma}_1+\hat{\sigma}_2+\hat{\sigma}_3$), $\hat{J}_2(=\hat{\sigma}_1\hat{\sigma}_2+\hat{\sigma}_2\hat{\sigma}_3+\hat{\sigma}_3\hat{\sigma}_1)$ and $\hat{J}_3(=\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3)$ in the new coordinate system, the direction cosines $\hat{a}_i$ of the normal of the "Extended SMP" (see Fig.3) are expressed in the same form of the original "SMP" as follows (Ohmaki,1979):

$$\hat{a}_i=\sqrt{\hat{J}_3/(\hat{\sigma}_i\hat{J}_2)} \qquad (i=1,2 \text{ and } 3) \qquad (1)$$

It should be noted in Eq.(1) that when $\sigma_o=0$, $\hat{a}_i$ becomes $a_i=\sqrt{J_3/(\sigma_i J_2)}$, the direction cosines of the normal of the original "SMP", which is successfully applicable to granular materials, and when $\sigma_o\to\infty$, $\hat{a}_i$ becomes $1/\sqrt{3}$, the direction cosines of the normal of the octahedral plane, which is successfully applicable to metals. The normal stress $\hat{\sigma}_{SMP}$ and the shear stress $\hat{\tau}_{SMP}$ on the "Extended SMP", and the normal component $d\hat{\varepsilon}^*_{SMP}$ and the parallel component $d\hat{\gamma}^*_{SMP}$ of the principal strain increment vector to the "Extended SMP" in the principal strain increment space are expressed in the same form of the original "SMP" (Nakai, 1989) as follows:

$$\hat{\sigma}_{SMP}=\hat{\sigma}_1\hat{a}_1^2+\hat{\sigma}_2\hat{a}_2^2+\hat{\sigma}_3\hat{a}_3^2 \qquad (2)$$

$$\hat{\tau}_{SMP}=\sqrt{(\hat{\sigma}_1-\hat{\sigma}_2)^2\hat{a}_1^2\hat{a}_2^2+(\hat{\sigma}_2-\hat{\sigma}_3)^2\hat{a}_2^2\hat{a}_3^2+(\hat{\sigma}_3-\hat{\sigma}_1)^2\hat{a}_3^2\hat{a}_1^2} \qquad (3)$$

$$d\hat{\varepsilon}^*_{SMP}=d\varepsilon_1\hat{a}_1+d\varepsilon_2\hat{a}_2+d\varepsilon_3\hat{a}_3 \qquad (4)$$

$$d\hat{\gamma}^*_{SMP}=\sqrt{(d\varepsilon_1\hat{a}_2-d\varepsilon_2\hat{a}_1)^2+(d\varepsilon_2\hat{a}_3-d\varepsilon_3\hat{a}_2)^2+(d\varepsilon_3\hat{a}_1-d\varepsilon_1\hat{a}_3)^2} \qquad (5)$$

If frictional and cohesive materials fail when the shear-normal stress ratio $\hat{\tau}_{SMP}/\hat{\sigma}_{SMP}$ on the "Extended SMP" reaches a limiting value, a new failure criterion is expressed by the following equation (Hashiguchi, 1975).

$$\frac{\hat{\tau}_{SMP}}{\hat{\sigma}_{SMP}}=\sqrt{\frac{\hat{J}_1\hat{J}_2-9\hat{J}_3}{9\hat{J}_3}}$$

$$=\frac{2}{3}\sqrt{\frac{(\sigma_1-\sigma_2)^2}{4(\sigma_1+\sigma_0)(\sigma_2+\sigma_0)}+\frac{(\sigma_2-\sigma_3)^2}{4(\sigma_2+\sigma_0)(\sigma_3+\sigma_0)}+\frac{(\sigma_3-\sigma_1)^2}{4(\sigma_3+\sigma_0)(\sigma_1+\sigma_0)}}$$

$$=\text{const.} \qquad (6)$$

It is noteworthy that Eq.(6) becomes the Matsuoka-Nakai criterion ($\tau_{SMP}/\sigma_{SMP}=$const.) for granular materials when $\sigma_o=0$ and the Mises criterion ($\tau_{oct}=$const.) for metals when $\sigma_o\to\infty$, which can be proved by putting const.$=(2\sqrt{2}/3)(c/\sigma_o)$ in Eq.(6). Fig.4 shows the shape of the proposed failure criterion expressed by Eq.(6) in the principal stress space. It is interesting to know that Figs. 3 and 4 are reduced to the "SMP" and the Matsuoka-Nakai criterion when $\sigma_o=0$ ($c=0$), and to the octahedral plane and the Mises criterion when $\sigma_o\to\infty$ ($\phi=0$), respectively.

Fig. 5 shows the results of the preceding true triaxial tests (Fig. 1) on the cement-treated Toyoura sand under $\sigma_m=800$kPa and $\theta=$const., in which $\theta=\tan^{-1}[\sqrt{3}(\sigma_2-\sigma_3)/\{(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3\}]$ (see Figs. 8 and 9). In Fig. 5, $\hat{\sigma}_1/\hat{\sigma}_3=(\sigma_1+\sigma_o)/(\sigma_3+\sigma_o)$. The value of σ_o is determined from Fig. 6 as $\sigma_o=350$kPa. Fig. 6 shows Mohr's stress circles at failure obtained by triaxial compression and extension tests on the same cement-treated sand.

Fig. 7(a) shows the stress ratio $\hat{\tau}_{SMP}/\hat{\sigma}_{SMP}$ vs. strain increment ratio $-d\hat{\varepsilon}^*_{SMP}/d\hat{\gamma}^*_{SMP}$ relation arranged on the "Extended SMP". The same straight line can be drawn for all the different radial stress paths ($\theta=0°$, $15°$ and $30°$). Fig. 7(b) shows $\hat{\tau}_{SMP}/\hat{\sigma}_{SMP}$ vs. $\hat{\gamma}^*_{SMP}$ relation and $\hat{\varepsilon}^*_{SMP}$ vs. $\hat{\gamma}^*_{SMP}$ relation

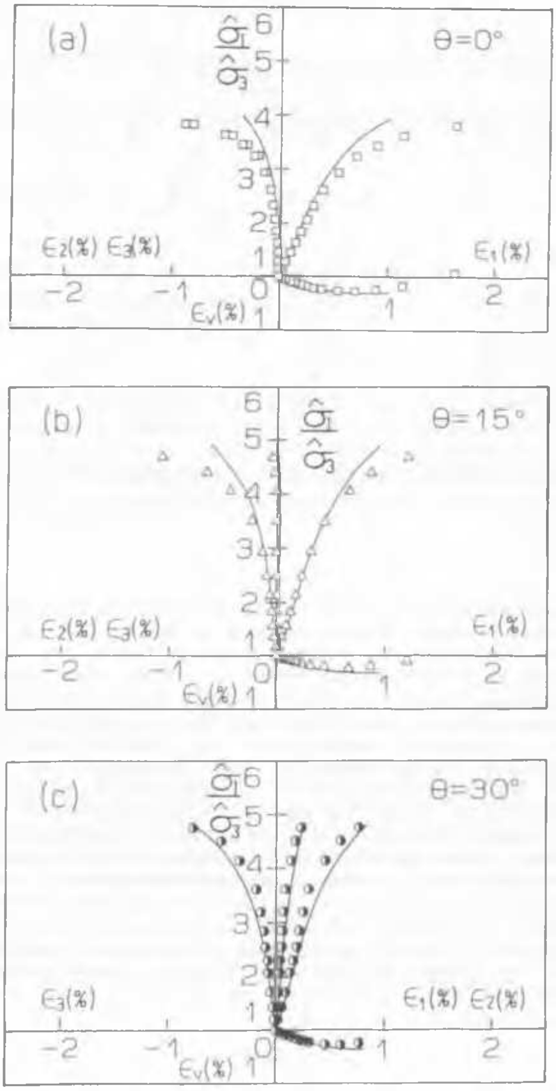

Fig. 5 Stress-strain relation obtained by true triaxial tests under σ_m=800kPa

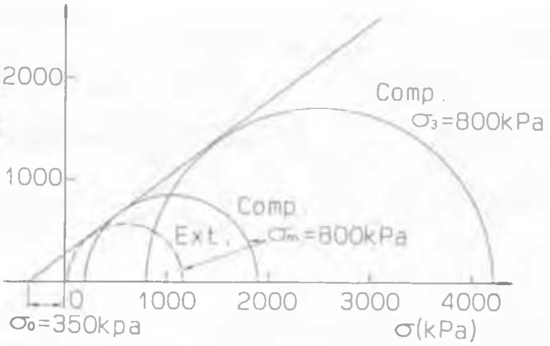

Fig. 6 Determination of σ_0 from Mohr's stress circles at failure

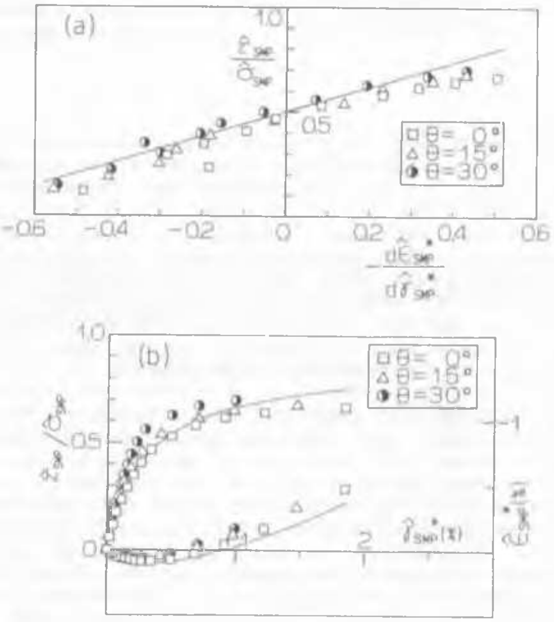

Fig. 7 Stress-strain performance based on "Extended SMP"

arranged on the "Extended SMP". It is seen from Figs. 7(a) and (b) that the test results under three different principal stresses are uniquely arranged on the "Extended SMP". It has been also checked that the test results in triaxial compression and triaxial extension are uniquely arranged on the "Extended SMP" (Matsuoka and Sun, 1991). Such unique arrangement has been already obtained when the test results of sands and normally consolidated clays are arranged on the "SMP" (σ_0=0) (Matsuoka and Nakai, 1974).

An elasto-plastic model for frictional and cohesive materials (σ_0>0) can be developed from such unique arrangement as shown in Fig. 7, just in the same way as the model for sands (σ_0=0) based on the "SMP" (Nakai, 1989). The plastic potential function ψ and the yield function f of the proposed model is written as follows:

$$\psi=f=\ln\hat{\sigma}_{SMP}+\{-\alpha/(1-\alpha)\}\ln\left|1-(1-\alpha)\hat{\tau}_{SMP}/(M^*\hat{\sigma}_{SMP})\right|$$

$$-\{1/(m+1)\}\ln(\hat{W}^{*p}/K_1+\hat{\sigma}^{m+1}_{SMPo})=0 \qquad (7)$$

The hardening parameter $\hat{W}^{*p}$ is expressed as $\hat{W}^{*p}=\int(\hat{\sigma}_{SMP}d\hat{\varepsilon}^{*p}_{SMP}+\hat{\tau}_{SMP}d\hat{\gamma}^{*p}_{SMP})$. The parameters in Eq. (7) are determined for the cement-treated Toyoura sand as follows; α=0.6, M^*=0.5, m=0.8 and K_1=0.015%. The curves in Figs. 5(a), (b) and (c) represent the predicted values by this elasto-plastic model.

Fig. 8 shows the comparison between the stress states at failure on the octahedral plane obtained by true triaxial, triaxial compression and triaxial extension tests on the cement-treated sand, and the proposed failure criterion ($\hat{\tau}_{SMP}/\hat{\sigma}_{SMP}$=0.73) expressed by Eq. (6). Fig. 9(a) shows observed shear strain increment vectors on the

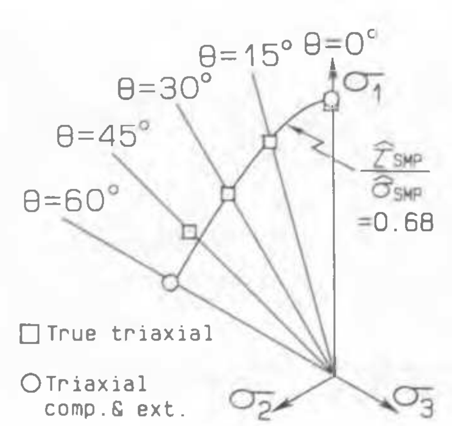

Fig. 8 Proposed failure criterion and test results on octahedral plane

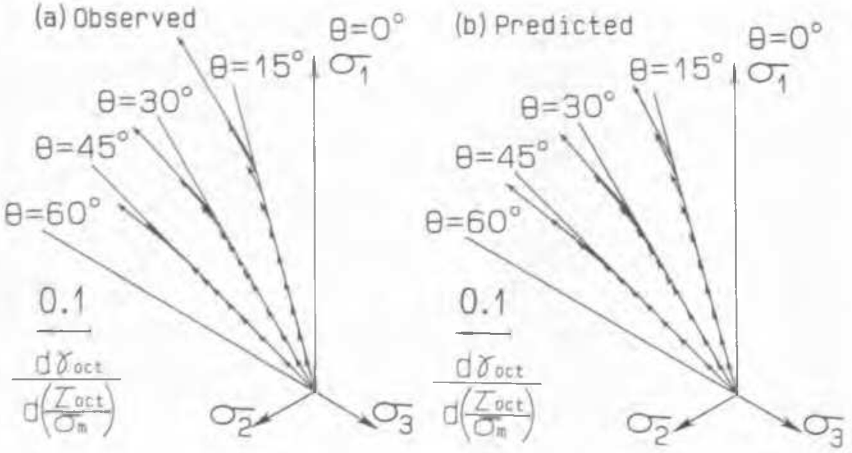

Fig. 9 Strain increment vectors on octahedral plane

octahedral plane, by true triaxial tests and Fig. 9(b) shows predicted ones, by the elasto-plastic model using the same parameters mentioned before. It is seen from Fig. 9(a) that the direction of shear strain increment vectors deviates from the direction of shear stress increment vectors on the octahedral plane under high stress ratios, which is similar to the case of sands. As seen from Figs. 9(a) and (b), the proposed model can predict well the deviation of shear strain increments from shear stress increments in direction.

It is noteworthy that the model based on Eq.(7) for frictional and cohesive materials is reduced to the model for frictional materials such as sands (Nakai,1989) when σ_o=0. On the other hand, materials with infinite bond ($\sigma_o \to \infty$) are considered to be perfectly cohesive materials such as metals, so no dilatancy occurs in such materials (volumetric strain increment $d\varepsilon_v = 0$). From these conditions of $\sigma_o \to \infty$ and $d\varepsilon_v = 0$, the Mises yield condition τ_{oct}=const., which is often used for metals, can be derived. Therefore, the proposed constitutive model for frictional and cohesive materials can explain deformation and failure of both perfectly frictional materials like granular materials (σ_o=0) and perfectly cohesive materials like metals ($\sigma_o \to \infty$) as the two extremities.

CONCLUSIONS

The main results are summarized as follows:

(1) By the introduction of a parameter, i.e., the "bonding stress σ_o" and the concept of "Extended SMP", the Matsuoka-Nakai criterion and constitutive relation on the "SMP" for frictional materials can be extended to frictional and cohesive materials. The proposed failure criterion and constitutive relation on the "Extended SMP" are coincident with the Matsuoka-Nakai criterion and stress-strain relation on the "SMP" when σ_o=0, and with the Mises criterion and stress-strain relation on the octahedral plane when $\sigma_o \to \infty$. The value of σ_o can be estimated by plotting an envelope of Mohr's stress circles at failure.

(2) From the arrangement on the "Extended SMP" of the test results of a cement-treated Toyoura sand selected as a material with friction and cohesion, unique stress-strain relations up to failure have been obtained by true triaxial tests ($\sigma_1>\sigma_2>\sigma_3$), triaxial compression tests ($\sigma_1>\sigma_2=\sigma_3$) and triaxial extension tests ($\sigma_1=\sigma_2>\sigma_3$). This suggests that the stress-strain behaviour and failure of frictional and cohesive materials in three-dimensional stresses are governed by the "Extended SMP".

(3) An elasto-plastic constitutive model for frictional and cohesive materials has been developed from such unique stress-strain relations on the "Extended SMP". The stress-strain behaviour and failure under three-dimensional stress conditions are well predicted by the proposed model. The model can also explain deformation and failure of both perfectly frictional materials such as granular materials (σ_o=0) and perfectly cohesive materials such as metals ($\sigma_o \to \infty$) as the two extremities.

REFERENCES

Hashiguchi, H. (1975). Constitutive Equations for a Plastic Soil in the Strain Hardening and Softening States. Dr. Eng. Thesis, Tokyo Institute of Technology, 164-184 (in Japanese).

Matsuoka, H. and Nakai, T. (1974). Stress-deformation and Strength Characteristics of Soil under Three Different Principal Stresses. Proc. of Japan Soc. of Civil Engs., 232, 59-70.

Matsuoka, H. and Nakai, T. (1985). Relationship among Tresca, Mises, Mohr-Coulomb and Matsuoka-Nakai Failure Criteria. Soils and Foundations 25(4): 123-128.

Matsuoka, H. and Sun, D. A. (1991). A Constitutive Law Covering Granular Materials to Metals. Proc. of 9th Asian Regional Conf. on Soil Mech. and Found. Eng., Bangkok, Vol. 1, 55-58.

Nakai, T. (1989). An Isotropic Hardening Elastoplastic Model for Sand Considering the Stress Path Dependency in Three-Dimensional Stresses. Soils and Foundations, 29(1):119-137.

Ohmaki, S. (1979). Strength and Deformation Characteristics of Overconsolidated Cohesive Soil. Proc. of 3rd Int. Conf. on Numerical Methods in Geomechanics, Aachen, 465-474.