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NONLINEAR VISCO-ELASTIC CONSTITUTIVE MODEL FOR TIME DEPENDENT BEHAVIOUR OF CLAYS

MODELE CONSTITUTIF VISCO-ELASTIQUE NON-LINEAIRE POUR COMPORTEMENT D'ARGILES EN RELATION AU TEMPS

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SYNOPSIS : The mechanics of interaction between structure and foundation is governed by the mechanical response of the compressible subsoil. If the soil response is time dependent, it must be expressed by a constitutive relation which includes time as a variable in addition to stress and strain. In order to predict the settlement of a structure and its foundation, it is essential to have an appropriate stress-strain-time relationship for the sub-soil. In this study, an attempt has been made to present a non-linear viscoelastic constitutive model for clays alongwith its convenient form for in-corporation into finite element analysis for soil-structure interaction studies. The paper also describes the experimental procedure for laboratory determination of the rheological constants from a series of triaxial tests.

INTRODUCTION

In soil-structure interaction studies, the main problem is how to predict deformations,particularly when structure, foundation and soil mass are treated as one integral compatible unit. These deformations can be realistically predicted when an appropriate constitutive law is used in the interaction analysis. When these deformations are time dependent, a stress-strain-time relationship is required using which the visco-elastic constants can be computed both as functions of strain and time. A glance of the earlier work reveals that several rheological models have been proposed by various investigators. Some of these models take into account primary consolidation and creep whereas others also account for three dimensional effects. The rheological constants of these models can not be used directly in soil-structure interaction studies.

In the present investigation, nonlinear viscoelastic stress-strain-time relationship is derived for a three dimensional state of stress using a non-linear Kelvin model for fully saturated clay. In general, any stress tensor can be split up into hydro-static and deviatoric stress components. Non-linear Kelvin model constants have therefore been derived for both the stress components. The paper presents the experimental procedure and the methodology for determination of these constants from laboratory tests from three different series of constant stress triaxial tests. A convenient form of this constitutive relationship is also presented in matrix form, for both two and three dimensional problems, for application in soil-structure interaction studies using the finite element method.

CONSTITUTIVE LAW FOR LINEAR VISCOELASTIC MATERIAL

(i) Uniaxial Stress Situation

The clay soil when considered to behave in a linear visco-elastic manner can be idealised as a Kelvin body (Fig. 1). The time-dependent stress-strain relationship for this model under uniaxial stress situation is commonly written (Flugge, 1967 and Findley, 1976) as :

$$\sigma = R\varepsilon + \eta\dot{\varepsilon} = [R + \eta(\partial/\partial t)]\,\varepsilon = (R + \eta D)\,\varepsilon \qquad (1)$$

where, R and η are the spring and dashpot viscosity constants, $\dot{\varepsilon}$, the strain rate and D = ∂ / ∂ t, the differential operator.

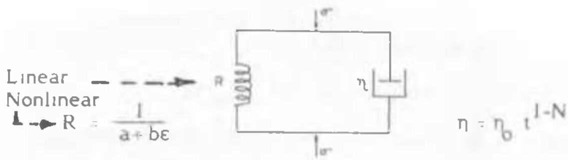

Fig. 1. Kelvin Body

(ii) Multiaxial Stress Situation

For a multiaxial stress situation, the state of stress at any point within a loaded body can be decomposed into its hydro-static and deviatoric components. In visco-elastic analysis, constitutive law for each component of the stress in differen-tial form is used separately (Flugge, 1967 and Findley, 1976).

(a) Deviatoric case - The differential form of the constitutive law for deviatoric component is expressed as :

$$P_1 S_{ij}(t) = Q_1 d_{ij}(t) \qquad (2)$$

where, P_1 and Q_1 are time operators of the form

$$P_1 = P'_o + \sum_{n=1}^{\infty} P'_n \frac{\partial^n}{\partial t^n} \quad \text{and} \qquad (3)$$

$$Q_1 = Q'_o + \sum_{n=1}^{\infty} Q'_n \frac{\partial^n}{\partial t^n} \qquad (4)$$

If the material is isotropic and homogeneous, the stress-strain relation for Kelvin body under deviatoric stress situation can be written as :

$$\sigma'_{ij}(t) = [R' + \eta'(\partial/\partial t)]\,\varepsilon'_{ij}(t) \qquad (5)$$

Comparison of Eq. 2 with Eq. 5, gives -

$$P_1 = 1 ; \quad Q_1 = [R' + \eta'(\partial/\partial t)] = (R' + \eta' D) \tag{6}$$

where R' and η' are the spring and dashpot constants for Kelvin body under deviatoric stress condition.

(b) Hydrostatic case – The constitutive law for hydrostatic component, in differential form can also be written as

$$P_2\, \sigma_{ii}(t) = Q_2 d_{ii}(t) \tag{7}$$

where P_2 and Q_2 are again time operators of the form - as Eqs. 3 and 4.

The stress-strain relationship for Kelvin body under hydrostatic stress situation, similar to deviatoric case, can be expressed as :

$$\sigma''_{ii}(t) = [R'' + \eta''(\partial/\partial t)]\ \varepsilon''_{ii}(t) \tag{8}$$

Comparison of Eq. 7 with Eq. 8 gives,

$$P_2 = 1; \quad Q_2 = [R'' + \eta''(\partial/\partial t)] = (P'' + \eta'' D) \tag{9}$$

where R'' and η'' are the spring and dashpot constants for Kelvin body under hydrostatic stress condition.

(iii) Visco-Elastic Constants

The stress-strain relations for an elastic body are :

$$S_{ij} = 2G\, d_{ij} \quad \text{(for deviatoric case)} \tag{10a}$$

$$\sigma_{ii} = 3K\, \varepsilon_{ii} \quad \text{(for hydrostatic case)} \tag{10b}$$

By comparing Eq. 2 with Eq. 10a and Eq. 7 with Eq. 10b, the corresponding elastic constants are modified in terms of time operators as given below :

$$G = Q_1/2P_1 \quad \text{and} \quad K = Q_2/3P_2 \tag{11}$$

where G and K are shear and bulk modulii of the material.

The modulus of elasticity, E and Poisson's ratio, can be related to K anf G as follows :

$$E = 9KG/(3K+G) \quad \text{and} \quad \nu = (3K-2G)/(6K+2G) \tag{12}$$

Substituting the values of G and K in terms of time operators from Eqs. 11, into Eq. 12, the time dependent Young's modulus, E_v (t) and Poisson's ratio, ν_v (t) for visco-elastic material can be obtained as -

$$E_v(t) = \frac{3Q_1 Q_2}{P_2Q_1 + 2P_1Q_2} \quad \text{and} \quad \nu_v(t) = \frac{P_1Q_2 - P_2Q_1}{P_2Q_1 + 2P_1Q_2} \tag{13}$$

(iv) Visco-Elasticity Matrix

Knowing the time-dependent Young's modulus and Poisson's ratio, the stress-strain relation for visco-elastic material at any time, t can be expressed in analogous form as the elastic material.

(a) Three dimensional stress situation - Analogous to the elastic stress-strain relationship for a three dimensional stress state, viz. $\{\sigma\} = [D]\{\varepsilon\}$,

Viscoelastic stress-strain relationship will be given by

$$\{\sigma\} = [V]\ \{\varepsilon\} \tag{14}$$

where the viscoelasticity matrix is given by -

$$[V] = \frac{E_v(t)}{(1+\nu_v(t))(1-2\nu_v(t))} \begin{bmatrix} D11 & D12 & D13 & 0 & 0 & 0 \\ D21 & D22 & D23 & 0 & 0 & 0 \\ D31 & D32 & D33 & 0 & 0 & 0 \\ 0 & 0 & 0 & D44 & 0 & 0 \\ 0 & 0 & 0 & 0 & D55 & 0 \\ 0 & 0 & 0 & 0 & 0 & D66 \end{bmatrix} \tag{15}$$

where,

$$\begin{aligned} D11 &= D22 = D33 = (1-\nu_v(t)) \\ D12 &= D21 = D13 = D31 = D23 = D32 = \nu_v(t) \\ D44 &= D55 = D66 = [1 - 2\nu_v(t)] / 2 \end{aligned} \tag{15a}$$

Substituting the value of $E_v(t)$ and $\nu_v(t)$ from Eq. 13 into Eq. 15 and rearranging the terms, the visco-elasticity matrix in terms of time operator is :

$$[V] = \frac{1}{3} \cdot \begin{bmatrix} V11 & V12 & V13 & 0 & 0 & 0 \\ V21 & V22 & V23 & 0 & 0 & 0 \\ V31 & V32 & V33 & 0 & 0 & 0 \\ 0 & 0 & 0 & V44 & 0 & 0 \\ 0 & 0 & 0 & 0 & V55 & 0 \\ 0 & 0 & 0 & 0 & 0 & V66 \end{bmatrix} \tag{16a}$$

where,

$$\begin{aligned} V11 &= V22 = V23 = (2P_2Q_1 + P_1Q_2) \\ V12 &= V21 = V13 = V31 = V23 = V32 = (P_1Q_2 - P_2Q_1) \\ V44 &= V55 = V66 = (3/2)\ P_2Q_1 \end{aligned} \tag{16b}$$

This is a general form of the visco-elasticity matrix and can be expressed in terms of spring and dashpot constants for Kelvin body by substituting Eqs. 6 and 9 into Eq. 16 to get-

$$[V] = \frac{1}{3} \cdot \begin{bmatrix} a11 & a12 & a13 & 0 & 0 & 0 \\ a21 & a22 & a23 & 0 & 0 & 0 \\ a31 & a32 & a33 & 0 & 0 & 0 \\ 0 & 0 & 0 & a44 & 0 & 0 \\ 0 & 0 & 0 & 0 & a55 & 0 \\ 0 & 0 & 0 & 0 & 0 & a66 \end{bmatrix} \tag{17a}$$

where,

$$\begin{aligned} a11 &= a22 = a33 = (2R' + R'') + (2\eta' + \eta'')D \\ a12 &= a21 = a13 = a31 = a23 = a32 = (R'' - R') + (\eta'' - \eta')D \\ a44 &= a55 = a66 = (3/2)\ (R' + \eta' D) \end{aligned} \tag{17b}$$

Above visco-elastic matrix can be split up into two parts, with one consisting of only R' and R'' terms and the other, only η' and η'' terms.

This matrix can be written as :

$$[V] = ([\bar{R}] + [\bar{\eta}]D) \tag{18}$$

where the elements of matrices $[\bar{R}]$ and $[\bar{\eta}]$ are the spring and dashpot constants of Kelvin body under hydrostatic and deviatoric stress conditions.

Using visco-elastic matrix, the stress-strain relation for visco-elastic material can be written as :

$$\{\sigma\} = ([\bar{R}] + [\bar{\eta}]\, D)\ \{\varepsilon\} \tag{19}$$

where $\{\sigma\}$ and $\{\varepsilon\}$ consist of six stress and strain components.

(b) Two dimensional plane strain situation - It is possible to write down the visco-elasticity matrix for plane strain case by eliminating the third, fifth and sixth rows and columns from Eqn. 17a.

NONLINEAR VISCOELASTICITY CONSTITUTIVE LAW

For the determination of the rheological constants from triaxial tests, a nonlinear Kelvin model consisting of a Hookean element with spring constant, R and dashpot with constant, η , both connected in parallel (Fig. 1) is proposed.

and follows the same stress-strain law (Eq. 1).

As the strain versus time response is nonlinear and has been idealised by a rectangular hyperbola, the rheological constants can be defined as :

$$R = 1/(a + b\,\varepsilon) \quad \text{and} \quad \eta = \eta_o\, t^{\,1-N} \tag{20}$$

where a, b, η_o and N constants.

Substituting the values of R and η in Eq. 1 and rearranging gives the governing differential equation for the proposed nonlinear Kelvin model -

$$\frac{d\,\varepsilon}{(\sigma - \frac{\varepsilon}{a + b\,\varepsilon})} = \frac{t^{N-1}}{\eta_o}\, dt \tag{21}$$

The solution of Eq. 21 can be written as :

$$\frac{t^N}{N \cdot \eta_o} = \frac{b}{f}\,(\varepsilon - \varepsilon_o) - \frac{a}{f^2}\,\log_e\left(\frac{a\sigma + f\,\varepsilon}{a\sigma + f\,\varepsilon_o}\right) \tag{22a}$$

where $f = b\sigma - 1$ and ε_o is the initial strain at time, t=0.

When the initial strain is zero, Eq. 23a simplifies to -

$$\frac{t^N}{N \cdot \eta_o} = \frac{b\,\varepsilon}{f} - \frac{a}{f^2}\,\log_e\left(1 + \frac{f\,\varepsilon}{a\sigma}\right) \tag{22b}$$

Equation 22b contains four constants a,b, η_o and N. These constants can be determined under both the hydrostatic and deviatoric stress conditions separately from triaxial tests on soil samples.

For the determination of the constants appearing in Eq. 22b, it is essential to know the experimental strain-time curve under constant stress condition. It is easier to obtain this curve for hydrostatic stress situation but it can not be directly obtained for the deviatoric condition. The concept of computing the deviatoric strains from the strains for total stress situation by substracting the hydrostatic strains, has been adopted for obtaining the experimental strain versus time plots for deviatoric stress situation.

LABORATORY TEST PROGRAM

Test Procedure

Triaxial tests were conducted on samples of clay of high compressibility with three different sets of principal stresses (Sharma, 1989). The magnitude of each set of principal stresses was kept more than the overburden pressure so that appreciable volumetric changes in the samples could be observed. The three sets of principal stresses adopted during the triaxial tests along with their hydrostatic and deviatoric components are given below :

Total Stress (KN/m^2) — Hydrostatic Stress (KN/m^2) — Deviatoric Stress (KN/m^2)

1st Set

$$\begin{bmatrix} 220 & 0 & 0 \\ 0 & 160 & 0 \\ 0 & 0 & 160 \end{bmatrix} = \begin{bmatrix} 180 & 0 & 0 \\ 0 & 180 & 0 \\ 0 & 0 & 180 \end{bmatrix} + \begin{bmatrix} 40 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -20 \end{bmatrix} \tag{23a}$$

2nd Set

$$\begin{bmatrix} 275 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 200 \end{bmatrix} = \begin{bmatrix} 225 & 0 & 0 \\ 0 & 225 & 0 \\ 0 & 0 & 225 \end{bmatrix} + \begin{bmatrix} 50 & 0 & 0 \\ 0 & -25 & 0 \\ 0 & 0 & -25 \end{bmatrix} \tag{23b}$$

3rd Set

$$\begin{bmatrix} 330 & 0 & 0 \\ 0 & 240 & 0 \\ 0 & 0 & 240 \end{bmatrix} = \begin{bmatrix} 270 & 0 & 0 \\ 0 & 270 & 0 \\ 0 & 0 & 270 \end{bmatrix} + \begin{bmatrix} 60 & 0 & 0 \\ 0 & -30 & 0 \\ 0 & 0 & -30 \end{bmatrix} \tag{23c}$$

For each stress system, two tests were performed. The first test was conducted under total stress system and the second test under hydrostatic stress situation.

Drained Triaxial Tests Under Constant Total Stress Condition

These tests were conducted with -

σ_3 = 160.0 KN/m^2, 200.0 KN/m^2, 240.0 KN/m^2 respectively as per Eqs. 23 a,b,c.

Required axial stress, $(\sigma_1 - \sigma_3)$ was applied instantaneously on the sample = 60.0 KN/m^2, 75.0 KN/m^2 and 90.0 KN/m^2 as per Eqs. 23a,b,c respectively). The drainage valves were opened immediately after application of axial stress. The vertical displacements and volumetric changes of sample in the form of water drained in burette were observed with respect to time. The observations were taken till the steady state was reached.

Triaxial Tests Under Constant Hydrostatic Stress Situation

After applying the required chamber pressure, (σ_3 = 180,225 and 270 KN/m^2 respectively), the drainage valve was opened and volumetric changes were noted with respect to time till the changes in volume became negligible i.e. steady state reached.

ANALYSIS OF TEST DATA

Data obtained from different tests were analysed to estimate the spring and dashpot constants namely R" and η" and R' and η' for hydrostatic and deviatoric stress conditions respectively. The axial strain due to total stress condition were calculated by dividing the observed axial displacements by the original length of the soil sample. The volumetric strains were computed by dividing the colume change (observed in the form of drained water from soil sample in burette) by the original volume of the sample. The strain vs time plot (Fig. 2) was made for the hydrostatic stress condition, considering one third of the volumetric strain as axial strain.

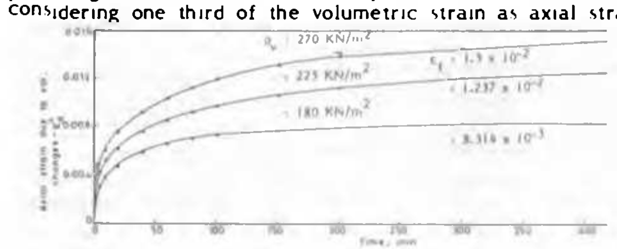

Fig. 2. Axial strain Vs time plot for hydrostatic stress condition

The deviatoric strains were computed by substracting the one third of the volumetric strains from the axial strains obtained under total stress condition and the strain vs time curve (Fig. 3) for deviatoric stress condition was also plotted.

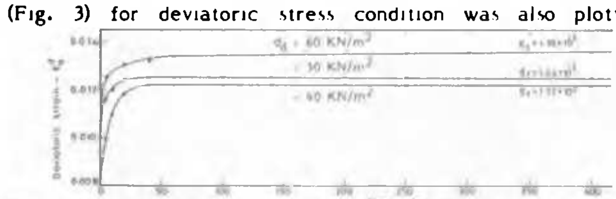

Fig. 3. Deviatoric strain Vs time plot for deviatoric stress condition

Figures 2 and 3 show axial strain vs time plots for all the three sets of the tests corresponding to three different stress conditions. It can be seen that axial strain vs time curves for both the stress conditions tend to become asymptotic when the rate of strain tends to be zero. The stress-strain law for the nonlinear Kelvin model is -

$$\sigma = \varepsilon/(a + b\varepsilon) + \eta_o\, t^{\,1-N}\, \frac{d\varepsilon}{dt} \tag{24}$$

when the strain rate, $(d\epsilon/dt) \rightarrow 0$ as $\epsilon \rightarrow \epsilon_f$ (the final strain), the above equation becomes -

$$\epsilon_f/\sigma = a + b\ \epsilon_f \quad (25)$$

The final strain, ϵ can be obntained from strain-time curves (Figs. 2 and 3). Equation 25 suggests that if the ratio, (ϵ_f/σ) is plotted against ϵ_f, a linear relationship would be obtained such that 'a' is the intercept on ϵ_f/σ axis 'b' is the slope of the straight line. Figures 4 and 5 show these plots and the values of parameters 'a' and 'b' obtained for both the hydrostatic and deviatoric stress conditions from these plots are presented in Table 1.

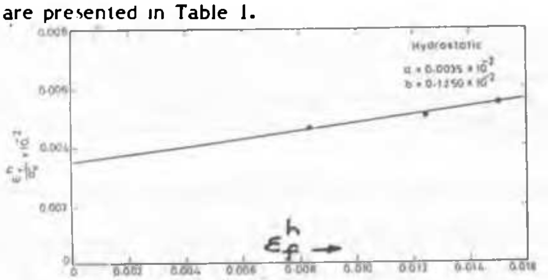

Fig. 4. Plot of (ϵ_f^h/σ) Vs (ϵ_f^h) for hydrostatic stress condition

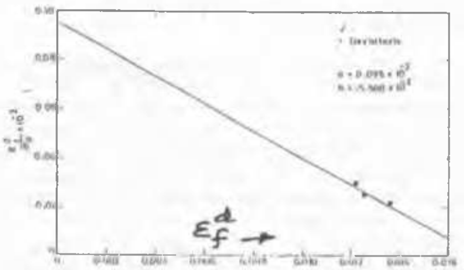

Fig. 5. Plot of (ϵ_f^d/σ_d) Vs (ϵ_f^d) for deviatoric stress condition

Table 1. Computations of a and b

Stress Condition	Magnitude of Stress (KN/m^2)	Magnitude of Final Axial Strain	Parameters a	Parameters b
Hydrostatic	180.0	8.314×10^{-3}	0.0035×10^{-2}	0.1250×10^{-2}
	225.0	1.237×10^{-2}		
	270.0	1.500×10^{-2}		
Deviatoric	40.0	1.220×10^{-2}	0.0950×10^{-2}	-5.500×10^{-2}
	50.0	1.240×10^{-2}		
	60.0	1.360×10^{-2}		

The constants η_o and N can be computed as follows using Eq. 22b. The Eq. 22b can be written as -

$$f/a\sigma = -(1/\epsilon_f) \quad (26)$$

Substituting the values of $f(= b\sigma - 1)$ and $f/a\sigma$ (Eq. 26) in Eq. 22b and rearranging the terms yields -

$$F(t) = \frac{t^N}{N\eta_o} = -\frac{a}{(1-b_o)^2} \cdot \log_e\left(1 - \frac{\epsilon}{\epsilon_f}\right) - \left(\frac{b\epsilon}{1-b\sigma}\right) \quad (27)$$

where F(t) is function of time, t.

If F(t) is plotted against time on logarithmic plot, the constants N and η_o can be computed. The values of N and η_o were determined for both the hydrostatic and deviatoric stress conditions for all the three sets of tests, using the method of least squares applied to linear regression of log F(t) vs log(t) in which N was the slope of straight line and the intercept gave the value of log $(N.\eta_o)$. The value of η_o was thus computed. The values of N and η_o for all the sets of tests are presented in Table 2 for hydrostatic and deviatoric stress conditions respectively. A glance at Table 2 indicates that for the hydrostatic and deviatoric pressure ranges, the values of the constants N and η_o obtained do not vary significantly. A definite conclusion regarding the variation of the constants (N and η_o) with pressure can not be drawn in view of the limited tests conducted. However, in view of the absence of any significant variation in the values of N and η_o, their average values can be adopted for the analysis.

Table 2. Computations of N and η_o

Hydrostatic stress condition			Deviatoric stress condition		
Stress (KN/m^2)	N	η_o	Stress (KN/m^2)	N	η_o
180	0.499	23.110	40	0.314	27.107
225	0.422	27.080	50	0.274	30.617
270	0.495	24.600	60	0.336	25.601

APPLICATION IN FINITE ANALYSIS

The incorporation of rheological constants into constitutive relationships for three dimensional and plane strain finite element analyses is illustrated in Eqn. 19. In visco-elasticity, the strain versus time response is obtained at a constant applied stress. Therefore, in visco-elastic finite element analysis, the total load is considered as acting on the structure and an iterative (variable stiffness) method is followed. In the explicit time integration scheme, the solution for the first time step is started with the assumption of zero initial strain. In any ith time step, the rheological constants, namely, R'', R' and η'', η' are computed using respectively, Eqn. 20 as a function of the strain level and the current strain rate obtained at the end of the i-Ith time step. The stiffness matrix of finite elements in clay medium can be updated for every time step using current values of R'',R' and η'', η'. The convergence of the solution would naturally occur when the strain rate becomes almost equal to zero i.e. at a time when the steady state condition is attained in the saturated clay medium.

CONCLUSIONS

(i) Procedure has been laid down for evaluating the visco-elastic material constants from two types of constant stress triaxial tests under hydrostatic and deviatoric stress conditions.

(ii) The study reveals that the visco-elastic constants vary nonlinearly and can be represented by a nonlinear Kelvin model.

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