Instability of sand under applied shear stresses

Instabilité d'un sable sous contraintes de cisaillement

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SYNOPSIS *For elastoplastic solids Drucker's quasi-thermodynamical postulate is often regarded as the necessary condition for material stability. The non-applicability of this postulate to granular materials is demonstrated in the light of an experimental investigation conducted using a torsional shear device. The tests revealed that for all the combinations of applied shear stresses, instabilities can develop well below the failure surface in a region where, due to non-associated flow, the second derivative of work is negative.*

INTRODUCTION

Concerns about instability have created the needs for establishing theoretically the necessary and sufficient conditions under which a material is stable. Within the theory of plasticity several postulates that would guarantee the mechanical stability of a soil element have been proposed based on energy principles (Bishop and Hill, 1951; Drucker, 1951). However, recent experimental studies (Lade et al., 1987 and 1988) revealed that the behavior of sand violates these postulates along certain triaxial stress paths.

STABILITY POSTULATES

Based on the hypothesis of a rigid plastic material Bishop and Hill (1951) showed that for an aggregate of infinitesimal volume the following condition guarantees stability:

$$(\sigma_{ij} - \sigma_{ij}^{o}) \cdot d\varepsilon_{ij} \geq 0 \qquad (1)$$

Where σ designates the actual state of stress, σ^{o} is a stress state located anywhere inside the elastic domain, i.e. not violating the yield condition, and $d\varepsilon$ is the increment in strain. The above inequality is known as the Maximum Work Principle and its application to a small stress increment $d\sigma$ requires the second derivative of work to be positive:

$$d^2W = d\sigma_{ij} \cdot d\varepsilon_{ij} \geq 0 \qquad (2)$$

Since in their derivation Bishop and Hill considered the material to be inelastic, the strains in inequalities (1) and (2) are entirely plastic:

$$(\sigma_{ij} - \sigma_{ij}^{o}) \cdot d\varepsilon_{ij}^{p} \geq 0 \qquad (3)$$

$$d^2W^p = d\sigma_{ij} \cdot d\varepsilon_{ij}^{p} \geq 0 \qquad (4)$$

Using a completely different approach based on thermodynamical considerations Drucker (1951) proposed inequalities (3) and (4) as the necessary condition for material stability. Drucker's quasi-thermodynamical postulate is based on the hypothesis that energy is dissipated during a cycle of application and removal of a small load increment. Hence, energy must be supplied to the material in order for the plastic deformations to take place.

The stability condition contained in (3) and (4) was analyzed and discussed by Mandel (1964). He showed that Drucker's postulate is a sufficient but not a necessary condition for a material to be stable. Besides, one of the hypotheses behind the Maximum Work Principle is that sliding between grains obeys Schmid's law, which requires a constant sliding resistance independent of the intergranular forces such as found in metals (Schmid, 1924). The frictional nature of sliding between soil particles does not obey this law. Therefore discrepancies between inequalities (3) and (4) and the actual behavior of granular materials should be expected.

Based on the assumption that a stable material is able to propagate a small perturbation in the form of waves, Mandel (1964) proposed a necessary condition for stability. He showed that a wave can propagate in a material with an elastoplastic matrix **A**, along the direction α, if and only if all the eigenvalues λ of the matrix **B** are positive, where:

$$d\varepsilon_{ij} = A_{ijkl} \cdot d\sigma_{kl} \qquad (5)$$

$$B_{ik} = A_{ijkl} \cdot \alpha_j \cdot \alpha_l \qquad (6)$$

$$k = 1,2,3 \qquad \lambda_k > 0 \qquad (7)$$

If one of the eigenvalues λ is negative or null one of the components of the perturbation can not propagate. This implies instability, and the possible appearance of strain localization along a certain direction. This phenomena is known as bifurcation and condition (7) may be used to predict the appearance of shear bands.

ASSOCIATED AND NONASSOCIATED FLOW

If the yield surface, $f=0$, is smooth, Drucker's postulate requires the constitutive law to be associated. In geometrical terms it implies normality between the plastic strain increment and the yield surface (Drucker, 1951). Associated laws are mathematically attractive because theorems regarding the existence and uniqueness of a solution and minimum principles can be derived (Koiter, 1960). The use of associated flow laws is widespread in constitutive modeling and several authors have used them to explain and predict the behavior of soils and soil structures.

Because nonassociated laws reject the normality condition, the existence and uniqueness of a solution can not be guaranteed (Mroz, 1963), and these laws are therefore mathematically less attractive. Nevertheless, there is strong experimental evidence that granular materials do not obey Drucker's postulate:

(i) In series of drained conventional triaxial experiments with decreasing stresses and increasing stress ratio, stable behavior was observed although inequality (2) was violated, $d^2W<0$ (Lade et al., 1987).

(ii) Undrained strain controlled triaxial tests on loose sands often exhibit a drop of the deviator stress in a region well below failure. This behavior also implies a negative second derivative of work, (Moroto, 1985).

(iii) Series of tests performed by Lade et al. (1988) showed that switching from drained to undrained conditions can be essential to the stability of granular materials. Furthermore, on its way to failure inequality (2) is violated and the material is unstable.

Within the framework of plasticity the above observations can only be explained with nonassociated models. It also requires the stress paths to be directed within the wedge where plastic loading occurs with $d^2W^P<0$ (fig.1).

EXPERIMENTAL CONDITIONS AND PROCEDURES

Hollow cylindrical specimens of loose Antelope Valley sand (e=1.221, D_r=34%) were used in this study. The specimens had nominal dimensions of an inner radius of 90 mm, an outer radius of 110 mm and a height of 390 mm. The use of a torsional shear device allowed for individual control of vertical, torsional and radial stresses (fig.2). Under these conditions the increment of strain energy dW can be calculated using equation (8).

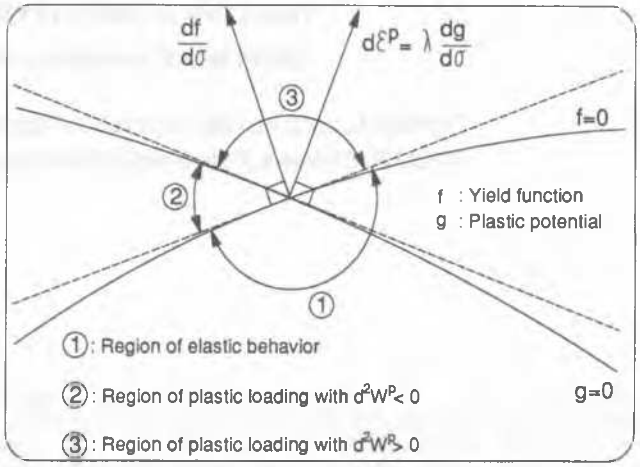

Fig.1 Different regions of nonassociated flow.

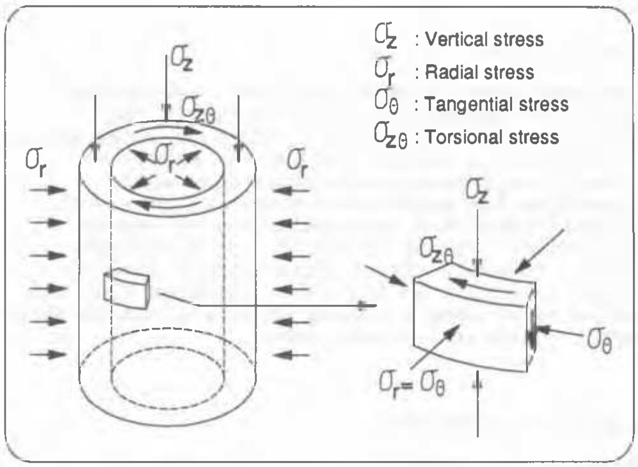

Fig.2 Definition of the stress components.

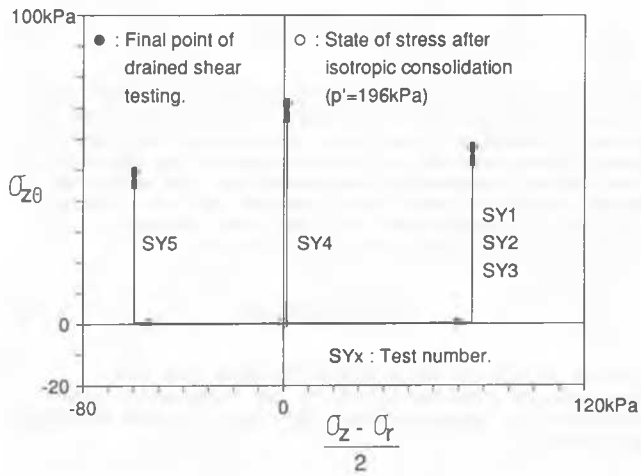

Fig.3 Direction of the stress paths.

$$dW = \sigma_z' \cdot d\varepsilon_z + \sigma_r' \cdot d\varepsilon_r + \sigma_\theta' \cdot d\varepsilon_\theta + 2\sigma_{z\theta} \cdot d\varepsilon_{z\theta} \tag{8}$$

Since equilibrium requires the tangential and radial stresses to be equal, equation (8) can be rewritten as:

$$dW = p' \cdot d\varepsilon_v + \Delta\sigma \cdot d\Delta\varepsilon + \sigma_{z\theta} \cdot d\gamma_{z\theta} \tag{9}$$

in which:

$$p' = \frac{1}{3} \cdot (\sigma'_z + \sigma'_r + \sigma'_\theta) \tag{10}$$

$$\varepsilon_v = \varepsilon_z + \varepsilon_r + \varepsilon_\theta \tag{11}$$

$$\Delta\sigma = \sigma_z - \sigma_\theta \tag{12}$$

$$\Delta\varepsilon = \frac{2}{3}\left(\varepsilon_z - \frac{\varepsilon_r + \varepsilon_\theta}{2}\right) = \varepsilon_z - \frac{1}{3} \cdot \varepsilon_v \tag{13}$$

and:

$$\gamma_{z\theta} = 2 \cdot \varepsilon_{z\theta} \tag{14}$$

In equation (9) the first right-hand term gives the volumetric contribution to energy, while the second and third terms define the contributions from shear distortion. This equation reflects that two very distinct shearing modes are possible in torsional shear. One is the triaxial mode created by the stress difference $\Delta\sigma$ and the other is the torsional mode in which shear stresses are applied to the surface of the specimen.

After isotropic consolidation, the specimens were triaxially loaded to the target values of the stress difference $\Delta\sigma$ and the effective mean normal stress p'=196 kPa. Then the torsional stress was applied under drained conditions while keeping $\Delta\sigma$ constant (fig.3). The shear stresses were applied very slowly, and each phase of the experiments lasted 4 hours or more. Hence, all the kinetic energy terms can be neglected and dW in equations (8) and (9) represents the increment in total energy.

If undrained conditions prevail the volumetric contribution to energy in (9) is null. Then the increment in energy dW and the second derivative of work become respectively:

$$dW = \Delta\sigma \cdot d\Delta\varepsilon + \sigma_{z\theta} \cdot d\gamma_{z\theta} \tag{15}$$

$$d^2W = d\Delta\sigma \cdot d\Delta\varepsilon + d\sigma_{z\theta} \cdot d\gamma_{z\theta} \tag{16}$$

INSTABILITY OF SAND IN TORSIONAL SHEAR

After reaching the desired torsional shear stress under drained stress controlled conditions, the drainage valve was closed. This caused a rapid increase in pore pressures (fig.4) while strains steadily accumulated (figs.5 and 6). During this phase of the experiments the torque and vertical force acting on top of the specimen could not be maintained.

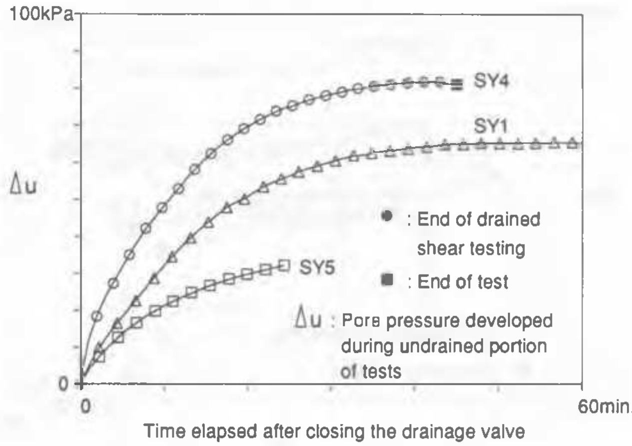

Fig.4 Pore pressure development in instability tests.

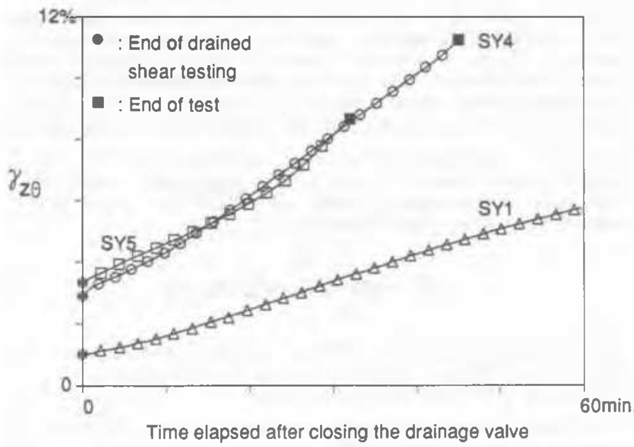

Fig.5 Torsional strains in instability tests.

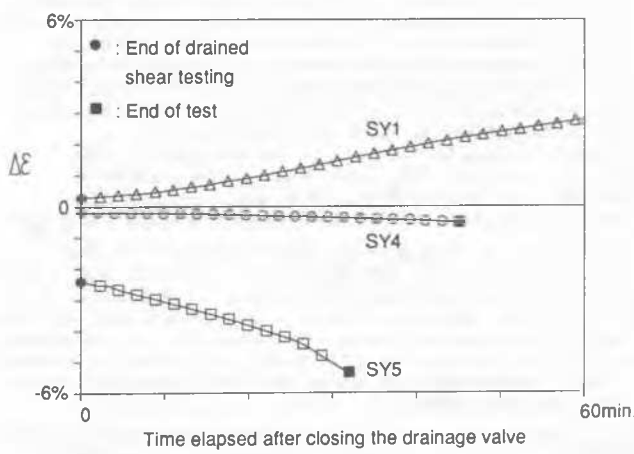

Fig.6 Strain difference $\Delta\varepsilon$ in instability tests.

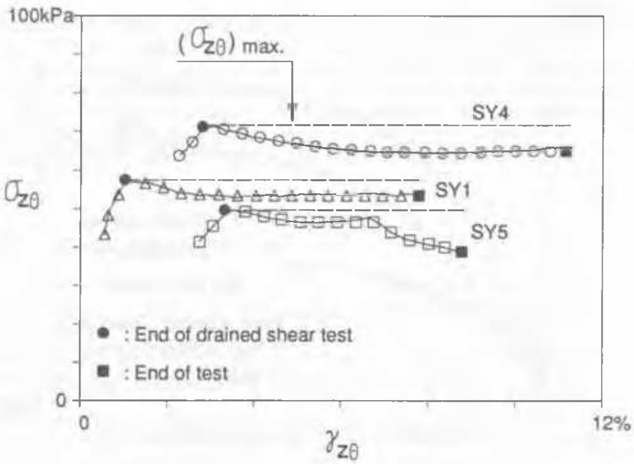

Fig.7 Torsional mode stress-strain behavior.

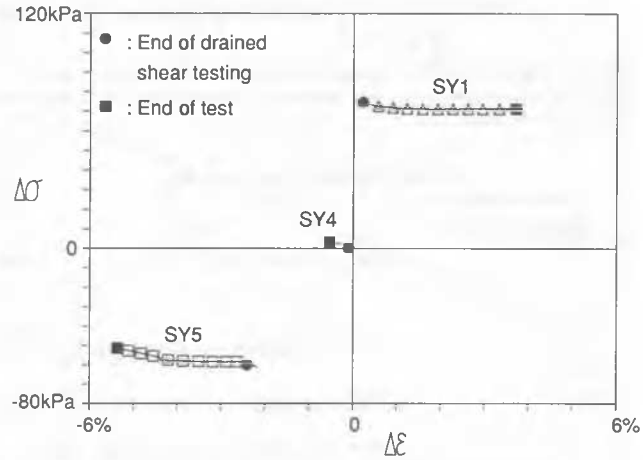

Fig.8 Triaxial mode stress-strain behavior.

Consequently a decrease in shear stresses (figs.7 and 8) was observed during the undrained portion of the stress paths. The stress-strain curves (figs.7 and 8) clearly demonstrates that from the moment the drainage valve was closed the specimens were unable to sustain the given load and were therefore by definition unstable.

The drop in shear stresses combined with increasing shear strains implies that each of the two right hand terms in equation (16) are negative, and therefore:

$$d^2W = d\Delta\sigma \cdot d\Delta\varepsilon + d\sigma_{z\theta} \cdot d\gamma_{z\theta} < 0 \qquad (17)$$

The elastic part of d^2W is always positive (Koiter, 1960), hence the above inequality requires d^2W^p to be negative.

This behavior is also observed in the softening range of a conventional drained triaxial compression test (fig.9) were the material is unstable. Hence, different experiments indicate the close relation between $d^2W<0$ and unstable behavior. However, a direct comparison between the type of instability observed in conventional triaxial tests and the one from the torsional shear tests presented here is impossible. In a triaxial test, d^2W becomes negative after the maximum stress ratio has been reached (at the failure surface), and the specimen is unconditionally unstable. In contrast, the hollow cylindrical specimens were unstable inside the failure surface well before the maximum stress ratios had been reached (fig.10).

Note that during the drained portion of the stress paths the specimens were stable and exhibited a positive d^2W. Hence inequality (17) is not the cause but rather a requirement of the unstable material behavior. This is in agreement with the theoretical findings obtained by Mandel (1964) and with the experimental observations by Lade et al. (1987).

The specimens were examined visually during the entire process of pore pressure build up and after the experiments were completed.

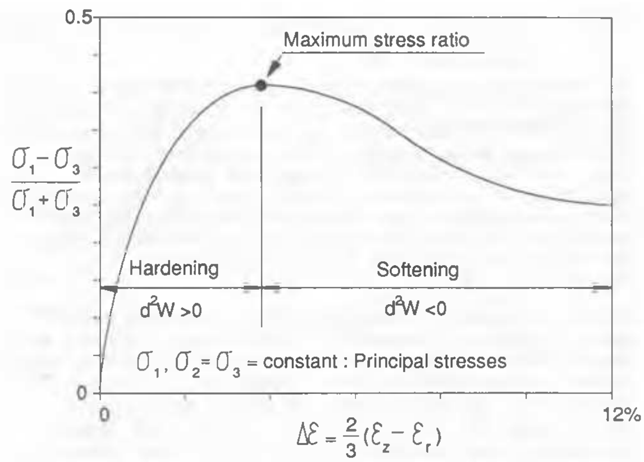

Fig.9 Stress-strain behavior in a triaxial test.

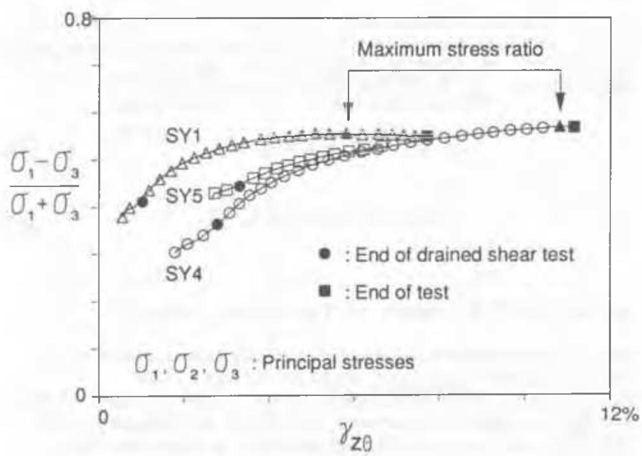

Fig.10 Stress ratio in instability tests.

No evidence of shear banding was detected in any of the specimens, which suggests that Mandel's stability condition (7) was satisfied along all the stress paths. Hence, neither the Maximum Work principle, nor Drucker's postulate, nor Mandel's wave propagation condition, seem to be able to explain the type of instability observed here.

A more plausible explanation to this type of unstable behavior can be derived from nonassociated flow. When the yield surface opens up in the outward direction of the hydrostatic axis, yielding can occur for a decreasing mean effective stress p′. In a material that tends to compress during plastic flow, undrained conditions lead to an increase in pore water pressures. Hence, loading is free to occur under constant or even dropping shear stresses, provided the yield surface f and the plastic potential g satisfy the following conditions:

$$\frac{\partial f}{\partial p'} \leq 0 \tag{18}$$

$$d\varepsilon_v^p = \lambda \cdot \frac{\partial g}{\partial p'} > 0 \tag{19}$$

Hence, an element of soil under undrained conditions and satisfying the above conditions is potentially unstable, and a small perturbation is sufficient to trigger the instabilities observed in the present study, for any combination of shear stresses.

YIELDING AND FLOW IN TORSIONAL SHEAR

In order to investigate the validity of this explanation, embodied in equations (18) and (19), the yield and plastic flow characteristics of the sand were studied prior to closing of the drainage valve. The material behavior along certain stress paths allows differentiation between the regions where the behavior of the material is elastic and the regions where yielding occurs. Consequently sections of the yield surface, $f=0$, can be obtained experimentally using the proper loading scheme. Stress paths involving a small cycle of loading and unloading, followed by reloading in a different direction, can be used for that purpose. Using this technique several investigations have been conducted using conventional triaxial (Poorooshasb et al., 1967; Tatsuoka and Ishihara, 1974), cubical (Yamada and Ishihara, 1982) and hollow cylindrical specimens (Pradel, 1988).

In the present study, the stress paths used for determination of yield surfaces involved changes in the mean effective stress p′ and torsional stress $\sigma_{z\theta}$, while keeping the stress difference $\Delta\sigma$ constant (figs.3 and 11). The yield point, i.e. the state of stress at which plastic strains reappear, was obtained from the stress-strain curve as exemplified in fig.12. In fig. 13 the experimental yield points are plotted together. Obviously, the directions of the yield surfaces drawn through these points

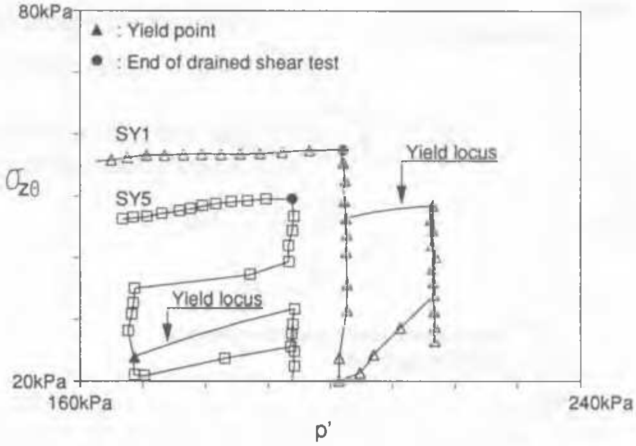

Fig.11 Stress paths in instability tests.

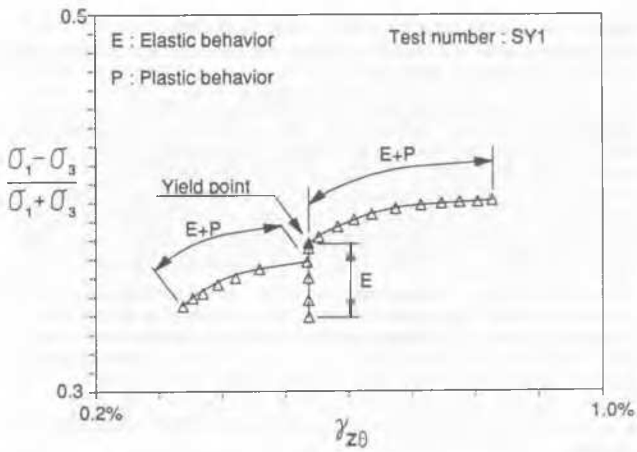

Fig.12 Plastic flow in instability test SY1.

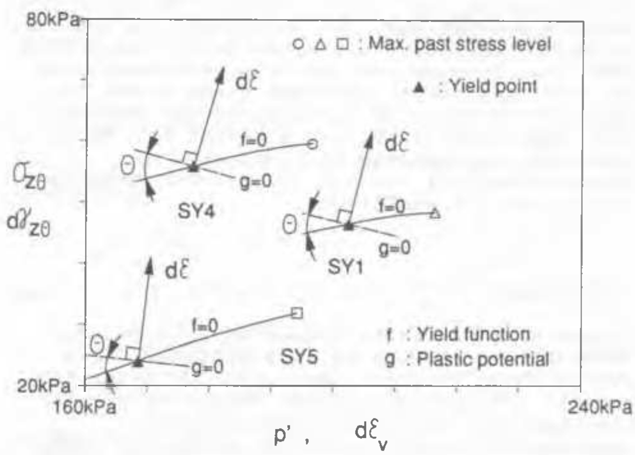

Fig.13 Yielding flow in instability tests.

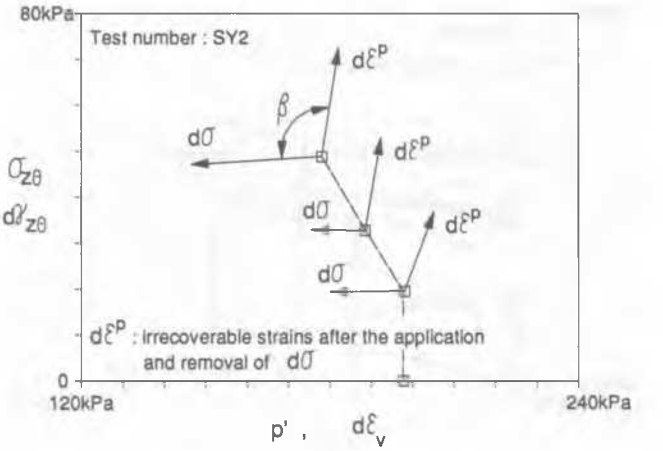

Fig.14 Plastic flow in test SY2.

satisfy inequality (18), for all the combinations of applied shear stresses employed in the present study.

In those portions of the tests in which torsional shear stresses were applied while the mean normal stress was kept constant, the measured volumetric strains were entirely plastic. The compressive behavior of all the specimens is reflected by the direction of the plastic strain increments shown in fig. 13, and implies that inequality (19) is also fulfilled.

The unloading branches of the stress-strain curves (fig.12) show that the elastic strains are very small compared to the plastic strains. The strain increments in fig.13 are then good approximations to the directions of plastic flow. These directions give evidence of the existence of nonassociated flow, $\theta \neq 0$, and of a wedge in which $d^2W^p<0$ (fig.1).

Further evidence of this wedge was obtained from a drained test in which the specimen was subjected to application and removal of small stress probes $d\sigma$. Fig.14 shows that the directions of the irrecoverable strains measured at the end of each cycle indicate values of the angle β greater than 90°. Since $\Delta\sigma=\sigma_z-\sigma_\theta$ was constant throughout the experiment, then $d^2W^p<0$. Note that because the test was performed under drained conditions, pore pressures could not rise and instability was not free to develop. This test shows stability although Drucker's postulate is violated and implies that inequalities (1) through (4) are not necessary conditions for stability.

CONCLUSIONS

Torsional shear tests conducted with various combinations of applied shear stresses show overwhelming evidence that granular materials exhibit nonassociated flow. They also reveal that unstable behavior, with all the consequences it may have, can occur well below the failure surface. This is in contradiction with the stability postulates by Drucker and Hill which demand associated plastic flow, and require that instability without shear banding occurs only when the failure surface is reached.

A possible explanation to the instability of a soil element suddenly under undrained conditions is provided from nonassociated flow. It requires the yield surface to open up in the outward direction of the hydrostatic axis and the material to be compressive. Both requirements were found to be in agreement with the observed yield and flow characteristics of the sand, which exhibited unstable behavior.

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