Mechanisms of fabric evolution in granular media

Evolution de la structure des milieux pulvérulents

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SYNOPSIS: Micromechanical mechanisms of fabric evolution in granular materials are examined with the objective to provide quantitative description of the elements of fabric relevant to macroscopic stress-strain response of granular materials. Physical ideas presented in the paper generalize numerous observations on numerically simulated plane granular assemblies. Theoretical results related to development of induced anisotropy, mechanisms of dilatancy and energy dissipation lead to stress-strain relationships for plane granular assemblies. Conventional macroscopic parameters such as angle of friction at constant volume and peak dilatancy rate are explicitly expressed in terms of microstructural characteristics. Basic theoretical relationships are verified on the basis of numerical experiments with plane granular assemblies.

1 INTRODUCTION

The difficulties in understanding the physics of granular materials are related to the fact that the fabric of sands (i.e. the arrangement of particles) continuously changes under applied loads. Such changes are traceable in experiments on photo-elastic disks, numerical simulations of plane granular assemblies and physical tests on sand in which the system of interparticle contacts is carefully monitored (e.g. Cundall and Strack 1979, Oda 1972). The objective of this paper is to introduce quantitative aspects of fabric description that are necessary for derivation of constitutive relationships for sands from physical considerations. Although the microscopic mechanisms discussed in the paper are believed to be of a general nature, the mathematical aspects of fabric description are simplified and directed to explain the behaviour of plane assemblies in experiments without stress rotations.

2 DESCRIPTION OF FABRIC

Perhaps the easiest way of gaining a physical insight into the nature of fabric changes is to monitor the total number of interparticle contacts in a simulated "biaxial" test on an assembly of disks (Bathurst 1985). The behaviour of the assembly of particles in this numerical experiment features attributes of sand response such as softening and dilatancy (Figure 1). The dilatant behaviour of the assembly leads to disintegration of fabric manifested by the loss of contacts (Figure 2a). The loss of contacts is orientationally non-uniform in the sense that contacts oriented along the direction of tensile strain disintegrate most rapidly while some contacts are

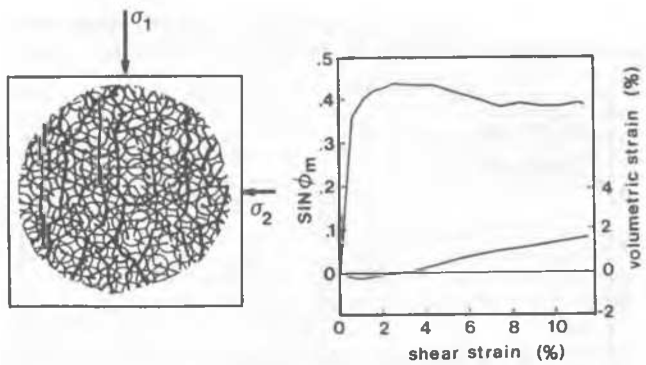

Figure 1 Biaxial test setup and stress-strain-dilatancy curves. Thickness of lines crossing the assembly is proportional to magnitudes of contact forces.

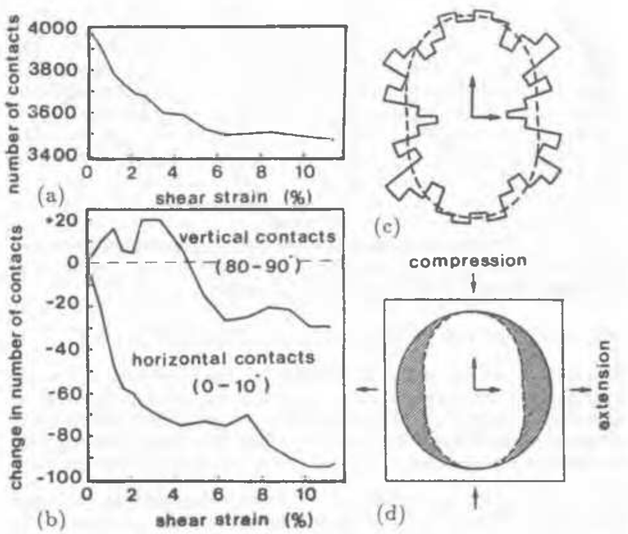

Figure 2 (a) Variation in the total number of contacts during simulated test. (b) Changes in the number of vertical and horizontal contacts. (c) Contact orientation distribution at large strain and analytical approximation. (d) Comparison of initial and limiting contact orientation distributions. The shaded area between the distributions represents disintegrated contacts.

created in the direction of compressive strain (Figure 2b). As a result, the polar distribution of contact orientations takes a characteristic "peanut" shape (Figure 2c). The normalized contact orientation distribution $S(\theta)$ can be adequately expressed as follows:

$$S(\theta) = \frac{1}{2\pi}\left[1 + a\cos 2(\theta - \theta_o)\right] \qquad (1)$$

where θ defines angular orientation with respect to horizontal direction; a is the parameter defining the anisotropy in contact orientations and represents the eccentricity of the "peanut" in Figure 2c; θ_o is the direction of anisotropy (vertical for the test in Figure 1).

3 DESCRIPTION OF CONTACT FORCES

Along with the changes in the total number of contacts, intergranular forces evolve due to changes in external loads and internal geometry. Although the variation of forces from one contact to another is highly irregular, it is quite clear that intergranular forces are generally higher on contacts oriented toward the major principal stress direction (Figure 1). Regular trends in the orientational distribution of contact forces can be observed by averaging forces over groups of contacts with similar orientations. Figure 3a illustrates the orientational histogram of normal components of contact forces averaged over contacts with normal vectors falling within 10 degree orientational intervals. The distribution of tangential components of contact forces is shown in Figure 3b.

Analytical expressions for average normal and tangential forces that adequately describe histograms in Figure 3a,b are as follows:

$$\overline{f}_n(\theta) = \overline{f}_o\left[1 + a_n \cos 2(\theta - \theta_f)\right]$$
$$\overline{f}_t(\theta) = -\overline{f}_o\left[a_t \sin 2(\theta - \theta_f)\right] \qquad (2)$$

where $\overline{f}_o$ is the average force over all contacts; a_n, a_t are non-dimensional coefficients defining the orientational variation of average contact forces and θ_f is the preferred direction of forces. In the described biaxial test, θ_f is coincident with the major principal stress direction. It should be noted that in tests that involve principal stress rotation the direction of maximum force does not necessarily coincide with the major principal stress direction.

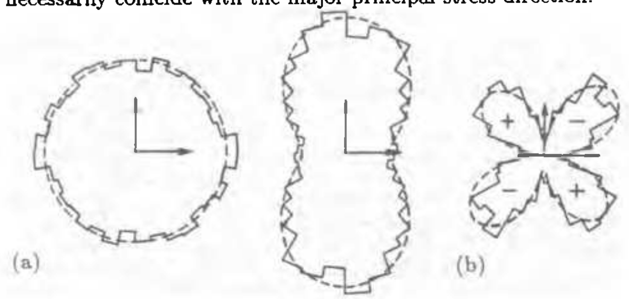

Figure 3 (a) Orientational distributions of average normal forces at initial state and at peak strength. (b) Orientational distribution of average tangential forces at peak strength.

4 EVOLUTION OF FABRIC AND CONTACT FORCES

Preferential loss of contacts oriented in the direction of tensile strain leads to an increase in the degree of anisotropy that can be expressed in terms of the relationship between the parameter of anisotropy a and shear strain (Figure 4a). The monotonic growth of a reflects a progressive destruction of the system of contacts.

The variation of the coefficients of force anisotropy a_n, a_t with shear strain (Figure 4b) reveal trends that can be interpreted in physical terms. For example, the initial increase in tangential forces (increase in a_t) continues only up to a point when deformations cease to be purely elastic. From there on a_t drops off due to particle rotations that tend to release tangential forces.

Normal forces, on the other hand, tend to grow steadily in the direction of the major principal stress and well into the range of plastic deformations. This corresponds to an increase in a_n (Figure 4b). Normal forces grow along stiff load paths (visible in Figure 1a) where the density of the assembly is greater than average. When the density of the assembly drops sufficiently, due to dilatancy, average normal forces on contacts oriented in the major principal direction diminish. This corresponds to macroscopic softening and reduction in a_n.

5 STRESS-FABRIC-CONTACT FORCES RELATIONSHIP

The evolution of microstructural characteristics with shear strain (Figure 4a,b) bears a qualitative resemblance to familiar stress strain curves for granular materials. This correspondence is not coincidental as it can be shown (Rothenburg and Selvadurai 1981) that the introduced microscopic parameters are related to the measure of deviatoric load as follows:

$$\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} = \frac{1}{2}(a + a_n + a_t) \qquad (3)$$

Figure 5a illustrates verification of the above theoretical relationship based on independent calculation of microstructural parameters a, a_n, a_t and comparison of their half sum with $(\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2)$ determined from boundary stresses. It is visually apparent that relationship (3) accurately relates the macroscopic measure of shear stress to characteristics of microstructure. It must be noted that the above relationship is only valid when the major principal direction of stress is coincident with the direction of anisotropy. This is the case for loading paths that do not involve principal stress rotations.

Relationship (3) is a simplified form of the following general expression for stress tensor for assemblies of spherical particles and their 2D analog (Rothenburg and Selvadurai 1981):

$$\sigma_{ij} = \overline{d} m_v \int \overline{f}_i(\mathbf{n}) n_j(\mathbf{n}) S(\mathbf{n}) d\,\mathbf{n} \qquad (4)$$

where $\overline{d}$ is the average diameter of particles (spheres or disks); m_v is the number of contacts per unit volume (area for plane systems); $\overline{f}_i(\mathbf{n})$ is the average force acting on contacts with normal vector $\mathbf{n}$ and $S(\mathbf{n})$ is the contact orientation distribution introduced previously for plane systems. Integration above is with respect to a full range of contact orientations. For plane systems, integration is performed in polar coordinates using $\mathbf{n} = \{\cos\theta, \sin\theta\}$, $d\,\mathbf{n} = d\,\theta$.

Relationship (3) follows immediately if expressions (1), (2) are substituted into (4) and only linear terms with respect to parameters of anisotropy are retained during integration. The neglected terms introduce little error, as comparison in Figure 5a suggests. If all terms are retained, the two curves in Figure 5a become indistinguishable.

From a physical point of view relationship (3) implies that the ability of a granular assembly to carry deviatoric loads is related to its ability to develop an anisotropic distribution of contact orientations ($a \neq 0$) or to sustain highly direction-dependent contact forces ($a_n, a_t \neq 0$). Since parameters a, a_n, a_t each make an additive contribution to the measure of strength according to $\sin\phi_m = (\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2)$, these parameters are essentially components of strength.

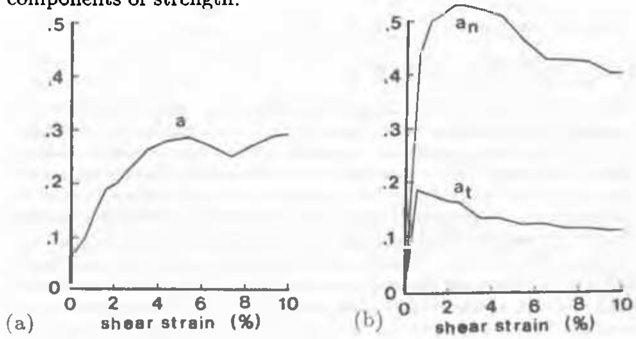

Figure 4 (a) Evolution of anisotropy in contact orientations. (b) Evolution of parameters of contact force anisotropy.

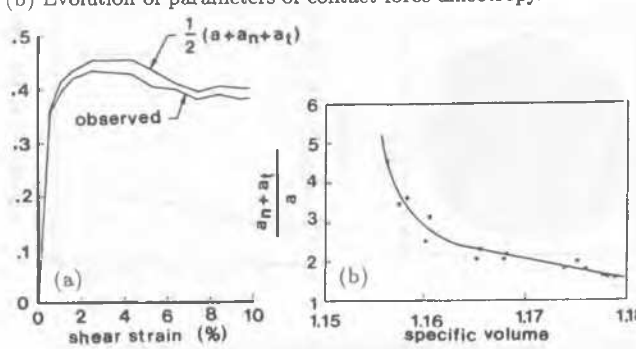

Figure 5 (a) Verification of stress-force-fabric relationship. (b) Ratio of parameters of anisotropy vs specific volume.

6 MECHANISMS OF ANISOTROPY DEVELOPMENT

The degree of anisotropy in a granular assembly is a function of the geometrical particle arrangement which varies with strain. If a granular assembly is initially under hydrostatic load, friction in contacts is not mobilized and the assembly can deform only elastically. When deviatoric load is applied, it preferentially increases forces on contacts oriented in the major principal direction. At the same time, forces on contacts oriented in the direction of minor principal stress are reduced. On some of these contacts the force is reduced to zero (i.e. some contacts disintegrate). In locations where the number of contacts is reduced some particles become mobile and deformation occurs when the overall number of mobile particles becomes sufficiently large. The macroscopic motion of the assembly can be stopped when some of the mobile particles form contacts that prevent their further motion.

Formation of contacts in the direction of compressive strain and contact disintegration in the direction of tensile strain results in development of oriented conglomerates of interlocked particles that move as rigid blocks. The location of these regions in the assembly coincides with stiff load paths in Figure 1. As deviatoric load increases and contacts disintegrate, the entire material becomes partitioned into subregions that remain interlocked and move as rigid entities. The kinematics of these regions was described by Drescher and De Josselin De Jong 1972 based on experiments with photo-elastic disks.

Although the parameter of anisotropy a reflects the difference in the number of vertical and horizontal contacts, it indirectly reflects the number and the size of blocks that move as rigid units. In the process of unidirectional deformations the assembly finally achieves a sufficient freedom for deformations to proceed unrestrained. In essence, at a limiting state the material has lost all redundancy eliminating contacts that restrain deformations. As there is a limit on the number of contacts that can be lost, there must be a limiting value of the parameter a. The limiting anisotropy $a = a_\infty$ is clearly identifiable in Figure 4a and development of anisotropy with shear strain recorded in the simulated test can be approximated as follows:

$$a(\gamma) = a_\infty \frac{\alpha\gamma}{1+\alpha\gamma} \tag{5}$$

where $\gamma = \epsilon_1 - \epsilon_2$ is shear strain and α is some constant. Specific reasons for choosing the above hyperbolic relationship can be explained by noting that it can be rewritten in an incremental form that conveys physical meaning:

$$da = \frac{\alpha}{a_\infty}(a_\infty - a)(1 - \frac{a}{a_\infty})d\gamma \tag{6}$$

The proportionality of the increment of a to the increment of shear strain $d\gamma = d\epsilon_1 - d\epsilon_2$ implies that da is proportional to the difference between the number of contacts created due to compressive strain $d\epsilon_1$ and the number of contacts disintegrated due to the tensile increment $d\epsilon_2$.

The origin of the coefficient of proportionality above is somewhat more complex. Its composition implies that the increment of a is proportional to the number of contacts $(a_\infty - a)$ that can still be disintegrated at a given state of fabric and to the probability $P(a) = (1 - a/a_\infty)$ that the system can remain stable after disintegration of a contact. The choice of this probability is somewhat arbitrary but $P(0) = 1$ reflects the notion that the assembly is most stable in an isotropic state and unstable in the state of limiting anisotropy when $P(a_\infty) = 0$.

It should be noted that the incremental equation (6) for anisotropy development may not be valid under conditions of principal stress rotation when internal stability appears to be more complex. Integration of (6) for monotonic paths with increasing γ and initially isotropic state at $\gamma = 0$ leads to (5).

7 MECHANISM OF ENERGY DISSIPATION

With the degree of anisotropy a being defined in terms of shear strain according to (5), further theoretical developments leading to stress-strain relationships must concentrate on terms a_n, a_t in the strength equation (3).

It is quite clear that the magnitude of the directional variation of contact forces reflected in a_n is limited by conditions of stable sliding at a given state of fabric. Although separate determination of a_n and a_t is difficult without complex statistical theory of friction mobilization, the sum $a_n + a_t$ required in expression (3) can be determined indirectly through equation of energy dissipation $\dot{E} = \sigma_1\dot{\epsilon}_1 + \sigma_2\dot{\epsilon}_2$. The latter can be conveniently rewritten as follows:

$$\frac{q}{p} + \frac{\dot{\epsilon}_v}{\dot{\gamma}} = \frac{\dot{E}}{p\dot{\gamma}} \tag{7}$$

where $\dot{\epsilon}_v = \dot{\epsilon}_1 + \dot{\epsilon}_2$, $q = (\sigma_1 - \sigma_2)/2$ and $p = (\sigma_1 + \sigma_2)/2$, while the ratio q/p is given by (3).

The assessment of the rate of energy dissipation requires knowledge of the number of sliding contacts at a given state of fabric. Observations on simulated systems suggest that the number of contacts where slip occurs is rather small and slip invariably occurs at contacts that are end points of elongated blocks of particles that slide as rigid elements. Since the number of such blocks is indirectly reflected in the parameter of anisotropy a, it is reasonable to postulate the following expression for the rate of energy dissipation:

$$\dot{E} = \mu a p \dot{\gamma} \tag{8}$$

where μ is some constant.

Further utilization of the energy dissipation equation (7) requires specification of the dilation rate $\dot{\epsilon}_v/\dot{\gamma}$. In this respect it should be noted that dilation of granular materials is associated with macroscopic movement of elongated conglomerates of particles that act like wedges disrupting the material. From this point of view it is clear that the macroscopic rate of volume change should be proportional to the size of interlocked conglomerates and their total number. Both are proportional to a. More detailed statistico-geometrical analysis leading to this conclusion is described elsewhere (Rothenburg and Selvadurai 1985).

It can, therefore, be expected that $\epsilon_v/\gamma \approx a$ with the coefficient of proportionality being a function of a parameter reflecting the density of packing, i.e.:

$$\dot{\epsilon}_v/\dot{\gamma} = -aD(v) \tag{9}$$

where $D(v)$ is a function of specific volume v. Since dilation rate of granular assemblies must be zero at "critical" specific volume v_c, $D(v)$ must be such that $D(v_c) = 0$. If $D(v)$ is decomposed into Taylor series and only linear terms retained, this function can be taken as follows:

$$D(v) = \delta(1 - v/v_c) \tag{10}$$

where δ is some constant controlling the rate of dilation (or construction when $v > v_c$).

Substitution of (8) and (9) into (7) gives the following expression for the mobilized angle of friction for a given state of fabric:

$$\sin\phi_m = a(\mu + D) \tag{11}$$

where D is given by (10). The sum of the coefficients of force anisotropy can be found from (3) as follows:

$$a_n + a_t = a(2D + 2\mu - 1) \tag{12}$$

It should be noted that the combination of the postulate on energy dissipation and specification of the dilation rate are sufficient to obtain the expression for the mobilized angle of friction without recourse to the strength equation (3). In that sense relationship (12) may appear as somewhat redundant as far as stress-strain relationships for plane systems are concerned. The value of the relationship (12) in the present context is that it suggests that the ratio $(a_n + a_t)/a$ depends on specific volume (linearly with the choice of $D(v)$ according to (10)). Figure 5b presents a plot of the ratio $(a_n + a_t)/a$ vs specific volume of the assembly during the simulated biaxial test. Although the scatter of data points is significant, the correlation of the $(a_n + a_t)/a$ ratio with specific volume is unquestionable. The relationship in Figure 5b is only linear in the vicinity of the critical specific volume. This is expected as $D(v)$ in the form (10) is only the linear term in the Taylor expansion of $D(v)$ in the vicinity of the critical specific volume.

8 STRESS-STRAIN RELATIONSHIPS

Stress-strain relationship for plane assemblies can be derived analytically for the linear choice of the dilation function $D(v)$ according to (10). This involves integration of the dilatancy equation (9) with $a(\gamma)$ according to (5). The final form of the volumetric curve can represented as follows:

$$\frac{\epsilon_v}{\epsilon_{max}} = 1 - \Gamma(\bar{\gamma}) \quad \text{where}: \quad \Gamma(\bar{\gamma}) = (1 + \bar{\gamma}/\nu)^{\nu^2} \exp^{-\nu\bar{\gamma}} \qquad (13)$$

where ϵ_{max} is the maximum volumetric strain from initial to critical state (positive if the material dilates); $\bar{\gamma}$ is shear strain normalized in such a way that $\bar{\gamma} = 1$ at a point where the rate of dilation is maximum (i.e. $\bar{\gamma} = \gamma/\gamma_{dil}$) where the point of peak dilation rate is related to microstructural parameters according to $\gamma_{dil} = 1/\sqrt{\alpha\delta a_\infty}$; ν is a non-dimensional parameter controlling the magnitude of peak dilation rate and is related to microstructural parameters ($\nu = \sqrt{a_\infty \delta/\alpha}$). It should be noted that the above relationship excludes small terms of the order of the square of maximum volumetric strain.

With the volumetric strain curve known, specific volume during the test can be computed and $D(v)$ evaluated according to (10). The relationship for the mobilized angle of friction can be determined immediately from (11) to obtain:

$$\frac{\sin\phi_m}{\sin\phi_{cv}} = \left[1 + \frac{\nu}{\sin\phi_{cv}} \frac{\epsilon_v^{max}}{\gamma_{dil}} \Gamma(\bar{\gamma})\right] \frac{\bar{\gamma}}{\nu + \bar{\gamma}} \qquad (14)$$

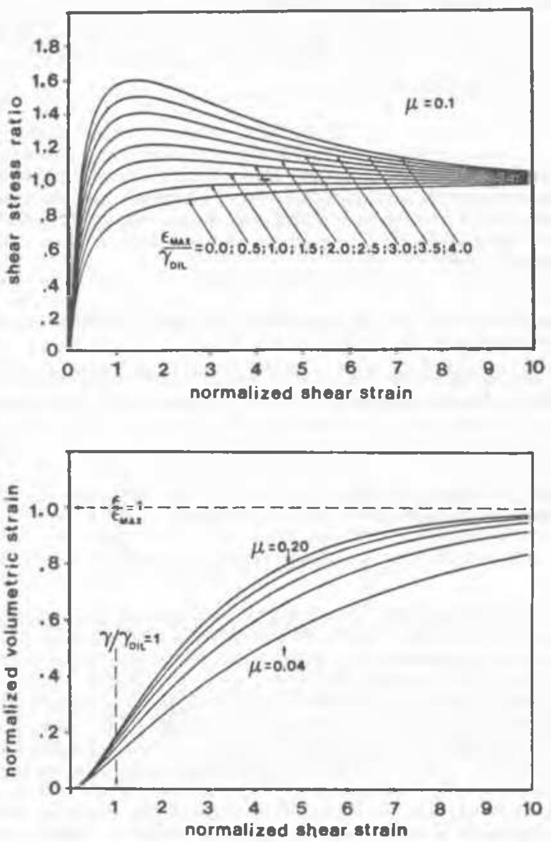

Figure 6 Stress-strain relationship for constant mean pressure test.

where $\sin\phi_{cv}$ is the mobilized angle of friction at constant volume and is related to microstructural parameters by $\sin\phi_{cv} = \mu a_\infty$.

The above two equations define stress-strain curves in terms of conventional parameters such as $\sin\phi_{cv}$, γ_{dil}, ϵ_{max} and another parameter ν related to the peak dilation rate. The shape of stress-strain dilation curves is shown in Figure 6. It should be noted that the volumetric curve determined above excludes the elastic component of compression and lacks a characteristic initial "dip" typical of tests with increasing hydrostatic confinement.

Although the developed relationships are for plane systems, the resulting analytical expressions are very versatile in approximating results of conventional triaxial tests when elastic compression is included. The key reason for this feature is that analytical curves are given in terms of easily measurable parameters. Also, the hyperbolic nature of the fabric mobilization equation leads to an essentially hyperbolic shape for stress-strain relationships at stresses below peak strength.

9 CONCLUDING REMARKS

The results of the present study unequivocally established that the major characteristic of microstructure that controls macroscopic response of granular materials is the contact orientation distribution. A parameter of this distribution defining the degree of microstructural anisotropy was uniquely related to the measure of deviatoric load and density of the plane granular assembly. This result essentially gives a quantitative meaning to the term "stress induced anisotropy".

The presented theory did not address the question of elastic deformations and the important topic of pressure-sensitivity of granular materials. The latter feature is of utmost importance to granular materials as it is well known that even a very dense sand can behave as a seemingly loose material when sheared under high confining pressures. In this respect it should be noted that recent experimental studies by Been and Jefferies (1985) convincingly demonstrated that major features of sand behaviour are controlled not so much by pressure and density but by a potential for volume change under given ambient stress conditions. The present study has confirmed this result indirectly by demonstrating that stress-strain relationships for plane systems depend only on volumetric strain developed from the initial to critical state. Given the empirical results of Been and Jefferies (1985), pressure sensitivity can be easily introduced into the presented stress-strain relationships by considering volumetric strain from initial to critical state to be dependent on mean intergranular stress.

REFERENCES

Bathurst, R.J. (1985). A Study of Stress and Anisotropy in Idealized Granular Assemblies. Ph.D Dissertation, Queen's University at Kingston, Canada

Been, K. and Jeffries, M.G. (1985). A State Parameter for Sands. Géotechnique, Vol. 35, No. 2, pp. 99-112

Cundall, P.A. and Strack, O.D.L. (1979). A Discrete Numerical Model for Granular Assemblies. Géotechnique, Vol. 29, No. 1, pp. 47-65

Drescher, A. and De Josselin De Jong, G. (1972). Photoelastic Verification of a Mechanical Model for the Flow of a Granular Material. Jour. Mech. Phys. Solids, Vol. 20, pp. 337-351

Oda, M. (1972). The Mechanism of Fabric Changes During Compressional Deformation of Sand. Jap. Soc. Soil Found. Eng., Vol. 12, No. 2, pp. 1-18

Rothenburg, L. and Selvadurai, A.P.S. (1981) A Micromechanical Definition of the Cauchy Stress Tensor for Particulate Media Proc. Int. Symp. on the Mech. Behaviour of Structured Media, Selvadurai (ed), Ottawa

Rothenburg, L. and Selvadurai, A.P.S. (1985). Anisotropic Fabric of Plane Granular Assemblies and Elements of their Mechanical Response. Plastic Behaviour of Anisotropic Solids, J.B. Boehler (ed). Edition du Centre National de la Recherche Scientifique Anisotropic Solids, CNRS, J.P. Boeler (ed)