Problems of three-dimensional slope stability

Problèmes de la stabilité de talus à trois dimensions

M. DENNHARDT, Research Assistant, Bergakademie Freiberg, GDR
W. FÖRSTER, Professor of Soil Mechanics, Bergakademie Freiberg, GDR

SYNOPSIS In this paper is represented the problem of estimation of the factor of safety for slopes with spatial failure surfaces taking into account all possible limit equilibrium equations. As with two-dimensional (plane) methods we have a statical indeterminate problem due to not considering of stress-strain relations. So assumptions must be made regarding various unknown functions. One of four different methods is represented in detail in which the equilibrium conditions are considered for the failure body as a whole. An assumption concerning the distribution of normal stress along the failure surface is used. A comparison with plane calculation shows the advantage of spatial calculation. Consideration of all equilibrium equations and of a smooth failure surface increases the reliability of results of spatial calculation in comparison with other known three-dimensional methods.

INTRODUCTION

One of the main problems in soil mechanics is the estimation of the stability of slopes. Since the seventies the commonly used two-dimensional limit equilibrium methods are followed by the three-dimensional methods (Baligh and Azzouz, 1975/77; Hovland, 1977). The reason for this development is that without consideration of spatial effects the factor of safety is underestimated when slopes are fixed on both sides or loaded with high surcharges. The above mentioned methods, however, do not use all possible equilibrium conditions, but a composed failure surface.
We know that it is not useful to increase the number of methods which can be observed in the case of the two-dimensional calculation. In this paper we represent a way taking into account all possible equilibrium conditions of the failure body with a justifiable mathematical and numerical expense. For this purpose two vectorial equilibrium equations concerning forces and moments are the base of consideration of four possible types of methods for calculation of the stability of a symmetric but otherwise arbitrary shaped failure body. One of these is discussed in detail. Numerical results are indicated.

THE GEOMETRY OF THE FAILURE BODY

A cartesian x_1-x_2-x_3-co-ordinate system is given. Failure surface and surcharge are symmetrical to x_1-x_3-plane but otherwise arbitrary shaped. The slope satisfies this condition too.
The known local vectors of the failure surface are expressed by:

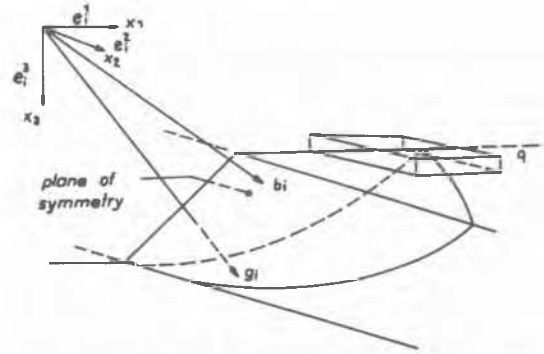

Fig. 1 Co-ordinate System, One Symmetric Half Of The Failure Body

$$g_i = (g_1, g_2, g_3) = (x_1, x_2, g(x_1, x_2)). \quad (1)$$

The vectors of the tangents to the failure surface are:

$$c_k^{\alpha} = \frac{\partial g_k}{\partial x_{\alpha}} . \quad (2)$$

Therefore we have for these vectors:

$$c_k^{1} = \frac{\partial g_k}{\partial x_1} = (1, 0, \frac{\partial g_3}{\partial x_1}) , \quad (3)$$

$$c_k^2 = \frac{\partial g_k}{\partial x_2} = \left(1,\ 0,\ \frac{\partial g_3}{\partial x_2} \right) . \qquad (4)$$

With

$$a_{(\alpha)} = 1 + \left(\frac{\partial g_3}{\partial x_\alpha} \right)^2 \qquad (5)$$

the tangential vectors of unit length can be written as:

$$\overset{\circ}{c}{}_k^{\alpha} = \frac{1}{\sqrt{a_{(\alpha)}}}\, c_k^{\alpha} \quad (\alpha=1;2) \qquad (6)$$

The normal vector to the failure surface is:

$$n_k = \varepsilon_{ijk} \frac{\partial g_i}{\partial x_1} \frac{\partial g_j}{\partial x_2} \quad , \qquad (7)$$

Where ε_{ijk} is the symbol of co-ordinates of the E-tensor.
With

$$a = 1 + \left(\frac{\partial g_3}{\partial x_1}\right)^2 + \left(\frac{\partial g_3}{\partial x_2}\right)^2 \qquad (8)$$

we obtain the unit normal vector:

$$\overset{\circ}{n}_k = \frac{1}{\sqrt{a}} n_k = \frac{1}{\sqrt{a}} \left(-\frac{\partial g_3}{\partial x_1}, -\frac{\partial g_3}{\partial x_2},\ 1 \right). \qquad (9)$$

An infinitesimal element of area can be expressed as:

$$dA = \sqrt{a}\, dx_1 dx_2 \ . \qquad (10)$$

Further the partial derivatives of the co-ordinate $()_i$, (i=1,2,3) of a vector with respect to the independent variable x_α, (α=1;2) is written as:

$$\frac{\partial ()_i}{\partial x_\alpha} = ()_{i,\alpha} \qquad . \qquad (11)$$

POSSIBLE METHODS FOR ESTIMATING STABILITY

The following varieties are possible:

(i) Whole failure-body method
This variant will be represented in detail. It is comparably easy concerning the mathematic and computation expense. The equations of equilibrium are integrals of acting stresses over the whole failure body.

(ii) Method of slices parallel to x_1-x_3-plane
The governing equations are derived from the equations of method (i) by partial integration over x_2 from x_2=0 to x_2=t and differentiation with respect to x_2.

(iii) Method of slices parallel to x_2-x_3-plane
In analogy to method (ii) the governing equations originate by partial integration and differentiation with respect to x_1.

(iv) Method of soil columns
This method devides the failure body into infinitesimal vertical oriented soil columns. The obtained differential equations are similar in their structure to the equations of the plane problem, but they are partial ones.

With all four methods we can investigate inhomogeneous slopes. The methods (ii) and (iii) are, indeed, "hybrid" methods because they contain also elements of the whole-failurebody method. Necessary in this context assumptions are very complex. The method (iv) needs a high mathematical and numerical expense and a high number of assumptions which are physically difficult to explain. Therefore we represent only method (i). Methods (ii)- (iv) will be reported elsewhere in detail.

THREE-DIMENSIONAL WHOLE-FAILURE-BODY METHOD

Failure criterion , definition of factor of safety, soil mechanical and surcharge parameter

The Mohr-Coulomb failure criterion is introduced into equations as:

$$\tau = \frac{1}{F} (\sigma \tan \phi + c) \qquad (12)$$

where τ = mobilised shear strength on the failure surface,
σ = normal stress on the failure surface,
ϕ = angle of internal friction,
c = cohesion,
F = factor of safety.

In this eqution and in the following considerations stresses can be effective or total ones. The same applies to the parameters of shear strength. It is assumed that the mobilised shear stresses always act in direction of the tangent vector c_k^1 . The vertical surcharge q on the upper bench as well as the soil mechanical parameters ϕ , c, and γ (unit weight) may be taken as functions of spatial co-ordinates:

$$q = q(x_1,\ x_2) \quad , \ \gamma = \gamma(x_1,\ x_2,\ x_3) \ ,$$
$$\phi = \phi(x_1,\ x_2,\ x_3), \quad c = c(x_1,\ x_1,\ x_3) \ . \qquad (13)$$

The equilibrium equations for the failure body

In the whole-failure-body method the equilibrium equations are integrals taking into account all acting forces. The integration regions are (Fig.2) :
G - projection of the failure body into the x_1- x_2-plane,
$\mathcal{G}$ - surface of the real failure body and
V - volume of the failure body.

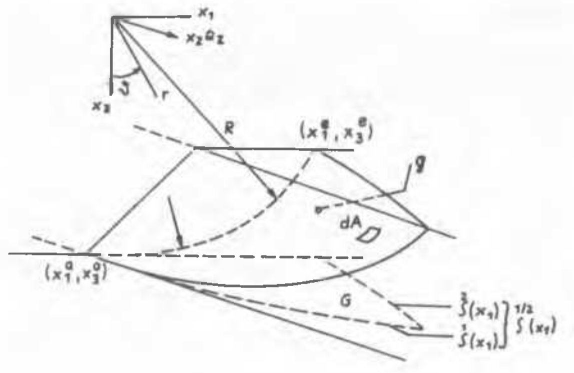

fig. 2 Failure Body, Cylindric Co-ordinates, Areas Of Integration

So first we obtain for translative motion

$$-\iint_g \sigma \mathring{n}_i \, dA + \iint_g \tau \mathring{c}^1_i \, dA + \iiint_V \gamma e^3_i \, dV + \iint_G q e^3_i \, dx_1 dx_2 = 0 . \qquad (14)$$

And for rotation about the origin of the co-ordinate system we have:

$$M_k = -\iint_g \sigma \varepsilon_{ijk} \mathring{n}_i g_j \, dA + \iint_g \tau \varepsilon_{ijk} \mathring{c}^1_i g_j \, dA + \iiint_V \gamma \varepsilon_{ijk} e^3_i x_j \, dV + \iint_G q \varepsilon_{ijk} e^3_i b_j \, dx_1 dx_2 = 0 . \qquad (15)$$

Introducing the failure criterion and the definition of safety from equation (12) we derive as components of these vectorial equations six equations. The equilibrium of forces for the x_2-direction and the equilibrium of moments about x_1- and x_3-axis are already fulfilled due to presumed symmetry. The following three equations remain:

$$\iint_g \sigma \left(g_{3,1} \frac{1}{\sqrt{a}} + \frac{\tan\phi}{F} \frac{1}{\sqrt{a_1}}\right) dA + \iint_g \frac{c}{F} \frac{1}{\sqrt{a_1}} \, dA = 0 , \qquad (16)$$

$$\iint_g \sigma \left(\frac{\tan\phi}{F} \frac{1}{\sqrt{a_1}} g_{3,1} - \frac{1}{\sqrt{a}}\right) dA + \iint_g \frac{c}{F} g_{3,1} \frac{1}{\sqrt{a_1}} \, dA + \iiint_V \gamma \, dV + \iint_G q \, dx_1 dx_2 = 0 , \qquad (17)$$

$$\iint_g \sigma \left((g_{3,1} g_3 + x_1) \frac{1}{\sqrt{a}} + \frac{\tan\phi}{F} \frac{1}{\sqrt{a_1}} (g_{3,1} x_1 - g_3)\right) dA + \iint_g \frac{c}{F} (g_{3,1} x_1 - g_3) \frac{1}{\sqrt{a_1}} \, dA + \iiint_V \gamma x_1 \, dV + \iint_G q x_1 \, dx_1 dx_2 = 0 . \qquad (18)$$

Now we suppose that the failure surface originates by rotation of a curve described by $r = r(x_2) = r(-x_2)$

Then the equations for the differential geometrical relations of the failure surface follow:

$$g_3 = \sqrt{r^2 - x_1^2} ,$$
$$g_{3,1} = -\frac{x_1}{g_3} ,$$
$$a = \frac{r}{g_3} \sqrt{1 + (r')^2} ,$$
$$a_1 = \frac{r}{g_3} . \qquad (19a\text{-}e)$$

Equations (16), (17), and (18) reduce to:

$$\iint_g \sigma \left(\frac{\tan\phi}{F} \frac{g_3}{r} - \frac{x_1}{\sqrt{1+(r')^2}}\right) dA + \iint_g \frac{c}{F} \frac{g_3}{r} \, dA = 0 \qquad (20)$$

$$\iint_g \sigma \left(\frac{\tan\phi}{F} \frac{x_1}{r} - \frac{g_3}{r} \frac{1}{\sqrt{1+(r)^2}}\right) dA - \iint_g \frac{c}{F} \frac{x_1}{r} dA + \iiint_V \gamma \, dV + \iint_G q \, dx_1 dx_2 = 0 , \qquad (21)$$

$$\iint_g \sigma \frac{\tan\phi}{F} \cdot r \, dA + \iint_g \frac{c}{F} r \, dA + \iiint_V \gamma x_1 \, dV + \iint_G q x_1 \, dx_1 dx_2 = 0 . \qquad (22)$$

These three equations contain as unknowns the functions σ and the parameter F . The task is to determine F as well as σ so that the equations (20), (21), and (22) are satisfied. Since σ occurs in terms of determined integrals a unique solution does not exist without consideration of auxiliary conditions. We have a statical undetermined problem in which either the stress- strain relations must be considered or assumptions must be made concerning the distribution of normal stresses. Because there are three equations with one free parameter (F), the shape of the distribution of normal stresses is described by a two-parametric function. In two-dimensional cases a sickle-shaped distribution of normal stresses or concentration of stresses in a single point are assumed. We introduce a trigonometric function for distribution of normal stresses. Magnitude and position of the maximum of σ in the plane of symmetry are controlled by two free parameters. The distribution in x_2-direction is cosinus-shaped. The function equals zero at the boundaries of the failure surface. We introduce cylindric co-ordinates (Fig. 2).
The relations of transformation to the cartesian co-ordinates are:

$$x_1 = r \sin\vartheta , \quad x_2 = z , \quad x_3 = r \cos\vartheta . \qquad (23)$$

Then we receive for the distribution function with the parameters χ and f :

$$\sigma(\vartheta, z) = \chi \sin\left[\frac{(\vartheta - \vartheta_a)^f}{(\vartheta_e - \vartheta_a)^f}\pi\right] \cos\left[\frac{z}{\overset{1/2}{s}_2(x_1)} \frac{\pi}{2}\right] = \chi F(\vartheta, z, f) = \chi F(x_1, x_2, f) , \qquad (24)$$

where $\vartheta = \arcsin\frac{x_1}{R}$,

$$\vartheta_a = \arcsin\frac{x_1}{R}\,,\quad x_1 = x_1^a\,,$$
$$\vartheta_e = \arcsin\frac{x_1}{R}\,,\quad x_1 = x_1^e\,. \qquad (25a\text{-}c)$$

This enables to return to cartesian co-ordinates. The distribution of normal stresses has a shape shown in fig.3 as an example.
The maximum σ_{max} of the normal stress belongs to $\vartheta = \vartheta_{max}$:

$$\vartheta_{max} = 2^{-\frac{1}{\xi}}(\vartheta_e - \vartheta_a) + \vartheta_a\,. \qquad (26)$$

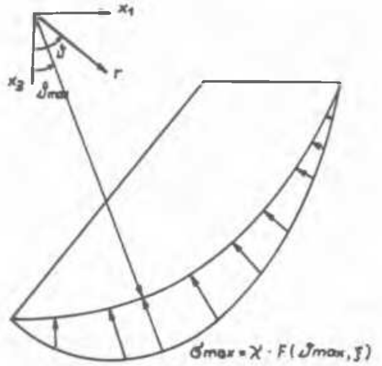

Fig. 3 Shape Of The Distribution Of Normal Stress

Introducing assumption (24) leads to a non-linear system of equations for the unknowns χ, ξ , and F.

Using three new unknowns u_1, u_2, u_3 according to:

$$u_1 = \frac{\chi}{F}\,,\quad u_2 = \chi\,,\quad u_3 = \frac{1}{F} \qquad (27a\text{-}c)$$

we obtain a system of linear equations which contains the unknown ξ implicitely. The coefficients a_{ij} and b_j can be obtained from equations (20) - (22) using (24).
Now for the system of equations we write:

$$a_{ij}\, u_i = b_j \quad (i,j=1;2;3) \qquad (29)$$

Wecan find the solution for ξ by an iterative process with the aid of the equation

$$u_1 = u_2 u_3\,. \qquad (30)$$

It is to be satisfied by the solution of the equation system (21). The factor of safety results from equation (25c). Probably a parameter ξ_o exists which fulfils equation (29) and (30) because the used distribution σ probably well corresponds to the physical reality. If more than one value ξ_o fulfils the equations it is necessary to decide which is the physically meaningful solution.

Example

To demonstrate the three-dimensional whole-failure-body method we use an example with a very simple geometry (see Fig.4).

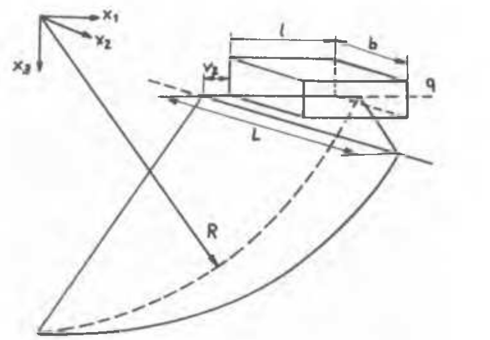

Fig. 4 Geometry Of The Slope Of The Example

The stability of a slope of 9m height, consisting of boulder-marl, with a slope angle of β=55°, a surcharge of q=55kN/m² on an area of b×l=4m×5m in a distance of v_L=1m from the upper edge is to estimate. The soil parameters are:

$$\phi' = 32{,}7^\circ \quad , \quad c' = 20{,}2\ \mathrm{kN/m^2},\ \gamma = 22{,}1\ \mathrm{kN/m^3}$$

The failure surface is assumed to be a part of the surface of a rotary ellipsoid with the equation:

$$1 = \frac{x_1^2}{R^2} + \frac{x_2^2}{c^2} + \frac{x_3^2}{R^2} \quad , \qquad (31)$$

where R = radius of the rotary body in the plane of symmetry,
c = semimajor axis of the ellipsoid .

The radius R, the distance v and the length L (see Fig.4) are to determine under the condition $F = F_{min}$.
The values of $x_1^a, x_1^e, x_3^a, x_3^e$, e ,and c (Fig.4) as well as the equations of the boundaries $\overset{1}{s}(x_1)$ and $\overset{2}{s}(x_1)$ of the failure surface in a horizontal projection are easily to obtain by geometrical considerations. The radius $r=r(x_2)$ is:

$$r(x_2) = R\sqrt{1 - (\frac{x_2}{c})^2}, \qquad (32)$$

the derivative $r'(x_2)$ is:

$$r'(x_2) = \frac{R^2}{c^2}\frac{x_2}{r}\ . \qquad (33)$$

For the numerical calculation of the double integrals an equation similar to the Newton-Cotes quadrature formulas was used.
The projection of the failure body to the x_1- x_2 - plane was devided into rectangular areas.

The remaining small triangular areas at the boundary were neglected. We used ν (j=1...ν) slices in x_1- direction and divided the j-th slice into ν-j+1 (i=1...ν-j+1) subareas (Fig.5). The total number of areas becomes:

$$\bar{\nu} = \frac{\nu}{2}(\nu + 1) \qquad (34)$$

In each of the generated areas (i,j) a function $f(x_1, x_2)$ is supported at 9 definite points (x_1, x_2). The integral over the area G is then:

$$\iint_G f(x_1, x_2)dx_1dx_2 = \sum_{j=1}^{\nu}\sum_{i=1}^{\nu-j}(\overset{ij}{f}_1+\overset{ij}{f}_3+\overset{ij}{f}_7+\overset{ij}{f}_9+4(\overset{ij}{f}_2+\overset{ij}{f}_4+\overset{ij}{f}_6+\overset{ij}{f}_8)+16\overset{ij}{f}_5)\frac{\overset{ij}{\Delta x}_1\overset{ij}{\Delta x}_2}{36} \qquad (35)$$

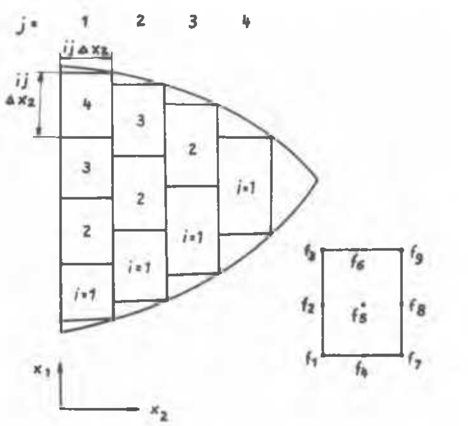

Fig. 5 Division Of The Failure Body Projection Into Rectangular Areas

For computation a table computer was used. The programm allowed the observation of the iteration process for determination of ρ. Though of course only these values of F, u_i,(i=1,2,3) are of physical significance for which ρ_o satisfies equation (30) their dependence on ρ is of interest. Near $\rho = \rho_o$ the function F(ρ) has a pole. The factor of safety is very sensible to inaccuracies in the determination of the parameter ρ_o.Certainly the real value of the factor of safety depends on the width of our net for calculation of double integrals. In case of our example it revealed an accuracy of 0.01 for the chosen scanning (ν=6, $\bar{\nu}$=21). Increasing ν leads to increasing F. Estimations on the safe side are possible with a small value ν. The minimal factor of safety and the critical failure surface (rotary ellipsoid) were determined for the given slope. It turns out that the safety is substantially larger than for a two-dimensional calculation using the methods of Morgenstern/Price (1965) or Fröhlich (Tab. I) .
The geometry of the sliding body is quite plausible with a length of the body of 2L=8.0m. The parameter ρ_o =0.87 shows that, as expected, the maximum of the normal stress distribution is situated in the lower part of the failure surface.

TABLE I

Comparison Between Results Of Plane Methods And The Whole-Failure-Body Method.

	Fröhlich	Morgenstern Price	Whole-Failure Body Method
F	1.22	1.29	1.54
R/m	14	19	20
v/m	3	4	4
L/m	-	-	4
ρ_o	-	-	0.8735

CONCLUSIONS

A three-dimensional general method for estimation stability of slopes has been developed. Special properties are:

(i) All equilibrium equations are fulfilled.
(ii) Layered structures can be considered.
(iii) Pore water pressures can be considered.
(iv) Necessary assumptions regard the shape of the distribution of normal stresses on the sliding surface. A chosen function has two determinable parameters, the maximum normal stress σ_{max} and its location.
(v) Necessary integrations are performed by utilising numerical methods.

Assumptions about the distribution of normal stresses seem as reliable as assumptions concerning interslice forces.
An example shows the possibility of application of the method and the influence of three-dimensional consideration in comparison to two-dimensional ones. Obviously under special conditions a three-dimensional calculation leads to economical advantages. The presented method seems to be more reliable than all other today known methods for three-dimensional cases.
In further studies it is necessary to check:
- another shape of distribution for the normal stresses , and
- another general form of the sliding surface.

REFERENCES

Baligh, M. M., Azzouz, A. S.(1975). End Effects on Stability of cohesive Slopes. ASCE J. Geot. Div. (101), 11, 1105-1117.
Baligh, M. M., Azzoz, A. S.(1977). Line Loads on cohesive Slopes. Proc. 9th ICSMFE, 2, 13-16,Tokyo.
Hovland, H. J.(1977). Three-Dimensional Slope Stability Analysis Method. ASCE J. Geot. Div. (103), 9, 971-986.
Morgenstern, N. R., Price, V. E.(1965). The Analysis of the Stability of general Slip Surfaces. Geotechnique (15), 15, 79-93.