Constitutive relations – Some conclusions from a workshop

Lois de comportement – Résultats d'un atelier

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SYNOPSIS Element tests yield an appropriate framework for checking and developing constitutive laws. The ability of constitutive laws to describe such tests is discussed on the basis of the results of a Workshop held 1982 in Grenoble where the participants were given some experimental data in order to calibrate their equations and predict the results of further experiments. In addition, the limitations for test evaluation imposed by bifurcation are discussed.

1 INTRODUCTION

Constitutive relations for soils are intended to describe the mechanical properties of the material in a realistic and feasible manner. They are needed and are often crucial in soil mechanics investigations; for example in the planning and in evaluation of physical element and model tests, numerical and closed-form calculations. Considering the wide spectrum of soils and their behaviour one can easily see that constitutive relations become more complex the more realistic they are (it is hard to believe that a universal formulation will ever be at hand). Thus research and publications in this field are growing more and more numerous, intricate, and controversial. Even specialists can scarcely read and judge the papers - let alone be familiar with all aspects of the field.

An international workshop hold in September 1982 in Grenoble was one effort - and not the only one - to improve this rather unsatisfactory situation. Data from some triaxial, biaxial, and truly triaxial tests carried out in Karlsruhe with a sand and a clay had been distributed 8 months earlier to so-called Predictors. The latter had been selected so that the main groups of approaches would be represented. They were invited to predict the results of some other tests made immediatedly prior to the workshop. The discussions in Grenoble dealt with the experimental basis, the different predictions and the theories behind them, and some general aspects.

Predictors and discussion chairmen were invited afterwards to submit revised evaluations and comments. The collection has now been published (Gudehus et al. 1984). It may serve as a kind of handbook for those who work on or with constitutive relations. The present article is based on this book and the work behind it. The points outlined here are divided into the following categories: testing of soil elements, asymptotic behaviour of the material, bifurcation in samples. In order to avoid lengthy mathematical formulism and symbol conventions the representation is mainly verbal and graphical. A more detailed evaluation is published elsewhere (Gudehus 1984). A comprehensive treatise on constitutive relations is not yet at hand. So this paper cannot be more than a flashlight photo of the present scene. Some numerical aspects are treated in another publication (Gudehus & Kolymbas 1985).

2 SOIL ELEMENT TESTS

The expression 'element test' implies a straining of a sample with uniform stresses and strains. If the material is 'simple' in the sense of Truesdell & Noll (1965) such tests suffice to reveal all the material's mechanical properties. In the case of a fully saturated soil sample an element test cannot strictly be drained. We assume, however, that slow drained tests can come sufficiently close to the condition of uniform stress and strain. The effective stress principle is applied. Diffuse or localized bifurcation is excluded here although this will not always be possible as outlined in Sec. 4.

Most of the standard laboratory tests of soil mechanics do not impose uniform stress and strain and thus cannot directly reflect material properties. Results from such tests evaluated assuming uniformity can lead to properties being observed which are not in the material, e.g. softening in case of localized pore pressure production.

So-called rectilinear extension tests, i.e. cuboidal tests, can be carried out with thin rubber moulds, fluid pressure and/or rigid lubricated boundary plates. To this group of tests belong

1) isotropic compression tests with arbitrarily shaped rubber moulds (difficult because of grain penetration and drainage);
2) oedometric compression and extension tests (require wall lubrication and irrotational top plate);
3) cylindrical compression and extension tests (with lubricated end plates and fluid confining pressure);
4) biaxial compression or extension tests (with 4 lubricated plates and fluid pressure, Vardoulakis et al. 1978, or 6 plates, Hambly 1969);
5) general cuboidal ('true triaxial') tests (with 6 lubricated plates, Goldscheider &

Gudehus 1973).

In these tests the principal axes of stress and strain remain the same. Soil element tests with relative axis rotation - i.e. shear or torsion tests - cannot yet be performed. Note that the so-called simple shear test and the hollow cylinder torsion test cannot produce uniform stress and strain. For this reason it was decided to restrict the tests for the workshop to those of the rectilinear extension type. Two materials were chosen, viz.

- a dry pure quartz sand of 0.1 to 0.5 mm grain size,
- a saturated kaolin clay.

The samples were either pluviated or pumped into the rubber mould. Minor imperfections were wiped out by subsequent isotropic compression.

Although the penetration of sand grains into the rubber mould was compensated for in the cuboidal tests, a scatter could not be totally avoided even with perfect servo-control. This point was vividly discussed in Grenoble. These tests were repeated using a cuboidal apparatus in Grenoble but without compensation for sand grain penetration and were essentially confirmed. Another sand test series was made with cylindrical samples of 80 cm diameter and 28 cm height and exceptionally precise strain measurements (Hettler & Vardoulakis 1984).

The clay was deformed in triaxial and biaxial compression and extension tests. Back pressure was not applied in order to avoid non-uniformities due to seepage. Therefore, the pore fluid was more compressible, which was criticized and even allowed for by one Predictor. A scatter of initial water content could not be avoided. Some tests were made with up to 4 cycles of axial strain.

Some results were debated as they indicated material properties being at variance with some widely used constitutive assumptions. Mobilized friction and dilatancy of the sand were found to be independent of the confining pressure (Fig.1).

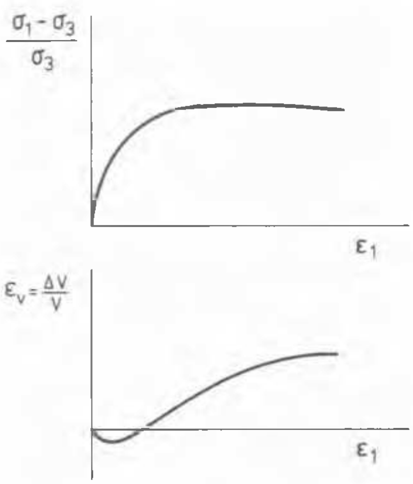

Fig.1

This finding is not always valid: other sands with softer grains tested in the same apparatus showed the usual pressure influence (Hettler & Vardoulakis 1984). Non-hydrostatic stress paths starting from isotropic stress in the triaxial cell produce strain paths which are always uniaxial initially ($\dot{\varepsilon}_2 = 0$, i.e. $\dot{\varepsilon}_v/\dot{\varepsilon}_1 = 1$ in Fig.1). This property, which was observed with two independent testing devices, supports Winkler's assumption of separable subgrade reaction (Gudehus et al. 1985). The response to stress cycles measured with precision in the large triaxial cell was the one known from soil dynamics; thus a distinction into static and dynamic moduli is no longer justified.

The clay test results showed some significant deviations from the well-known Cam Clay assumptions. The plots of void ratio vs. log stress (Fig. 2) can be approximated by straight lines

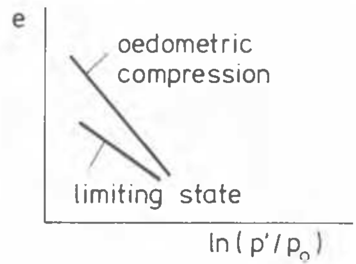

Fig. 2

which are not parallel. Therefore shear strength is not generally proportional to consolidation stress. Furthermore, undrained stress paths reached a constant shear stress while the pore pressure continued to increase (Fig. 3). Thus the limiting state (i.e. $d\tau/d\varepsilon = 0$) of unpreloaded clay is not generally the so-called critical one. In such cases the determination of

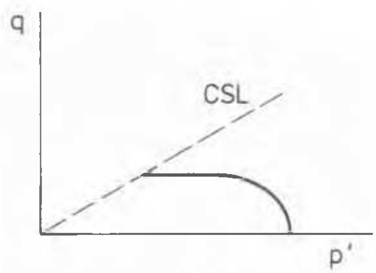

Fig. 3

the friction angle ϕ' is ambiguous. Critical limiting states were not reached prior to the formation of slip bands in cylindrical compression tests and biaxial deformation tests for $\varepsilon_1 = 15 \div 25$ %, and could not be reached in cylindrical extension tests due to unavoidable bifurcation. These facts are of importance for stability analysis (Sec. 4).

3 ASYMPTOTIC PROPERTIES

The constitutive relation of a soil can be represented by a system of ordinary differential equations, which can be directly integrated in the special case of an element test. Neglecting viscosity, these equations are invariant with respect to transformations of the time scale. As the equations are generally non-linear it is very difficult to determine their mathematical properties. Some insight can be gained by considering asymptotically monotonic or cyclic solutions. Consider first constant strain rates - i.e. proportional strain paths - imposed on arbitrary initial states. The most important example is a constant-volume deformation; this is supposed to lead to a stationary stress state (somewhat misleadingly called the critical state in soil mechanics). This property is implied by those elasto-plastic relations which start from Cam Clay. In other constitutive relations it imposes a restriction on the material parameters, which was not observed in many of the Grenoble papers.

Consider now proportional strain paths with 'small' positive or negative volume changes (Fig. 4a). For a certain range of initial void ratios and initial hydrostatic pressures the corresponding stress paths exhibit temporarily stationary stress states (peak), which are frequently called limiting. Fig. 4b, based upon tests by Goldscheider, represents sand behaviour at low mean stresses so that the grains are not crushed. If the material is initially dense a dilatant deformation (1) leads to a stationary stress ratio and a stress increase whereas a compressive deformation (2) produces a peak stress ratio and a subsequent stress decrease. With initially loose material (plotted for the case of extension) a dilatant deformation (3) can produce stress decrease without reaching a peak ratio, whereas a compressive deformation (4) is not essentially different from the above case 2.

These behaviours are not fully covered by the constitutive relations presented in Grenoble, most of which involve parameters which are functions only of the initial and not the instantaneous void ratio. In Grenoble Krawietz and Rettig proposed evolution equations in which the critical void ratio plays a central role.

For a clay the influence of mean stress is somewhat more marked and complicated; Fig. 4c is partly based on tests by Kuntsche and partly deduced from various related tests (proportional strain tests with clay are rare). If the material

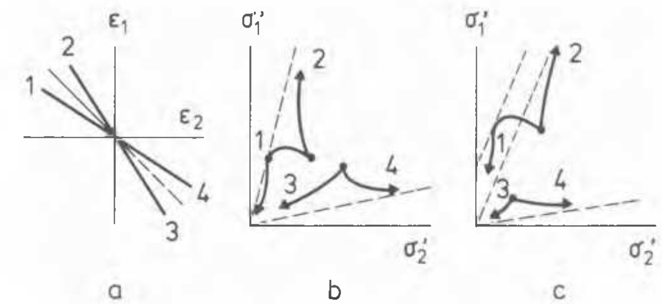

Fig. 4

is initially denser than critical, a dilatant deformation (1) produces a peak stress ratio and subsequent decrease; the envelope of peaks associated with the same void ratio has a cohesion intercept. With a compressive volumetric strain (2) the stress path approaches to a ray from the origin. In the case of a void ratio above the critical one (often called unpreloaded) a dilatant deformation (3) leads to a decay, whereas a contractant deformation (4) produces stress increase with an asymptotically constant stress ratio.

For lack of numerical investigations it cannot by fully judged how far the constitutive relations represented in Grenoble describe these properties. In order to describe the behaviour stated above several constitutive relations including further state variables were proposed. It is recommended that move numerical analyses should be carried out to detect the asymptotic behaviour (cf. Gudehus & Kolymbas 1985).

The situation is simpler for proportional deformations with 'large' relative volume decreases, especially under uniaxial and isotropic compression (Fig. 5a). With sand in cuboidal tests Goldscheider has found that proportional stress paths are always reached (Fig. 5b). For example a uniaxial compression tests results in stresses approaching the stress path $\sigma_2 = \sigma_3 = K_\sigma \sigma_1$ (1a) or else remaining on this stress path and an isotropic compression test results in the stress path $\sigma_1 = \sigma_2 = \sigma_3$. According to Kuntsche's (1982) biaxial tests (Fig. 5c), the same behaviour holds for clay; the deformations required to reach proportional stress paths with clay are bigger than with sand. Such asymptotic states are called swept-out-memory or SOM states (Gudehus et al. 1977) as any remnant of history except stress and density is erased.

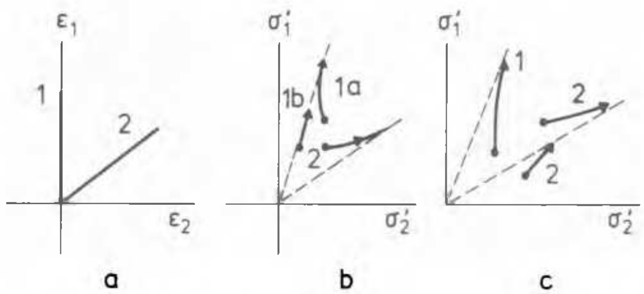

Fig. 5

This property is covered by some of the rate type relations presented in Grenoble (but not the 'endochronic' one). Chen (1984) has shown that a simple ('cap' type) elasto-plastic model also describes this behaviour. Computer experiments are recommended in order to check other constitutive relations. The numerical tractability of some of the Grenoble papers has been checked (Gudehus 1984), but unfortunately this is not the case in many publications.

We now turn to the response to stress or strain cycles. Consider first stress cycles (Fig. 6a). It appears that both sand and clay develop strain cycles then (Fig. 6b). There is always some hysteresis which is larger if limiting stress states are approached. It is not yet clear if this so-called shakedown is accompanied by a strain increase with the logarithm of the number of cycles (Goldscheider 1978). It may well be that minor imperfections prevent samples from reaching a perfectly elastic cyclic response.

Elasto-plastic relations seem to be particularly suited for describing this cyclic behaviour. The rate type relations using only stress as the state variable produce unlimited strain accumu-

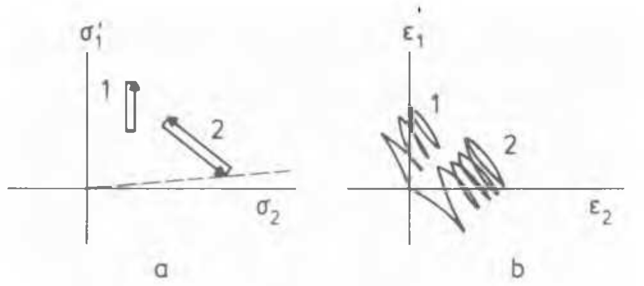

Fig. 6

lation and are not suitable at present. Improved soil models involving additional state variables produce, for the present, a reasonable fit for some isolated experiments.

Consider now strain cycles. Fig. 7a represents constant volume cycles with small (1) and large (2) amplitudes. The response (Fig. 7b) is asymptotically cyclic for sufficiently small amplitudes (1), whereas successive stress decrease occurs with large amplitudes (2). (The response to other strain cycles is not treated here for lack of experimental data). Elasto-plastic relations can easily cover case 1. To model case 2 (the so-called cyclic degradation)

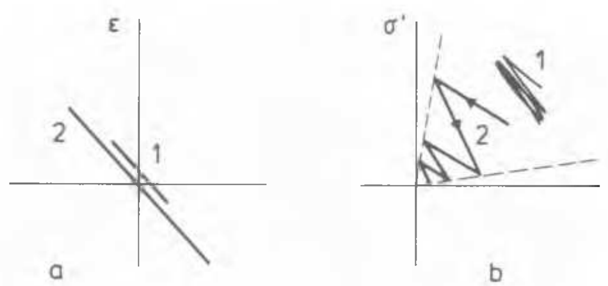

Fig. 7

additional hidden variables have to be introduced with consequences which are difficult to foresee. In fact, the corresponding Grenoble predictions appear to be partly realistic and partly inconsistent. The rate type relations without hidden variables predict case 2 directly and so one could apply them to analyses of incremental collapse. However, case 1 requires internal variables, and the proposals made in Grenoble are not more than preliminary guesswork.

Again it is recommended that computer models should be used to clarify these important aspects of asymptotic behaviour. It appears that some of the cyclic response predictions for Grenoble contain numerical errors. This observation indicates that the well-posedness of such problems is not easily attained.

4 ONSET OF INSTABILITY

Stability is a very general notion. It is restricted here to some consequences of constitutive relations as far as they reflect instabilities observed with soil samples. Numerical instabilities due to inadequate discretization, error accumulation, cageing etc. are not considered. The discussion concentrates upon the onset of instability. Stability of the asymptotic solutions is partly treated in the previous Section 3. Non-asymptotic processes following bifurcation are still beyond the current state of soil mechanics knowledge. Even so, a detailed inspection of the onset of instability helps to avoid the sort of confusion which arose again in Grenoble.

Consider first samples of dry sand in biaxial tests. If two faces are loaded by fluid pressure, a plane narrow slip zone indicates the loss of stability (Fig. 8a). As this occurs very close to the peak of the stress-strain curve (Fig. 8b) the classical analysis works: limit equilibrium provides the stress condition, and Coulomb's extremum principle yields the inclination of the slip band. The same case can also be treated as a bifurcation problem (Vardoulakis et al. 1978): It turns out that the slip band develops a little bit prior to the peak and can be inclined somewhat more or less than the Coulomb angle.

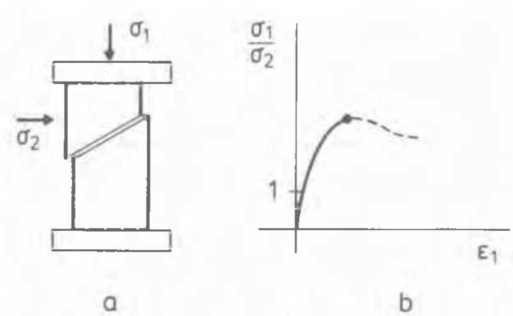

Fig. 8

What is the benefit of the bifurcation analysis then? Related practical problems - such as active earth pressure upon a translated wall - are satisfactorily covered by the classical analysis. However, the bifurcation analysis is powerful in evaluating soil tests and helps in the selection of constitutive relations. It turns out that elasto-plastic relations with smooth yield surfaces are not realistic: they imply too high an incremental shear stiffness at the onset of bifurcation (Vardoulakis 1980). However, elasto-plastic relations with vertices or without yield surfaces, and some rate type relations (but not the endochronic ones) are more suitable. Furthermore, bifurcation analysis can support classical stability analyses. Note that the static limit condition is, in general, neither necessary nor sufficient for a collapse, and that Coulomb's extremum principle is not based on principles of general mechanics.

Another instructive case of bifurcation in dry sand occurs in the so-called triaxial compression test (Fig. 9). Vardoulakis(1979) has shown that barrelling (B), i.e. diffuse bifurcation, occurs in a sample with ideally lubricated end plates just prior to the limiting state of the material. Bulging can be suppressed by keeping the ratio h/d below about 0.5, whereas necking under triaxial extension occurs with any h/d. The stress ratios associated with bifurcation decrease with increasing pressure even if friction and dilatancy are assumed to be pressure

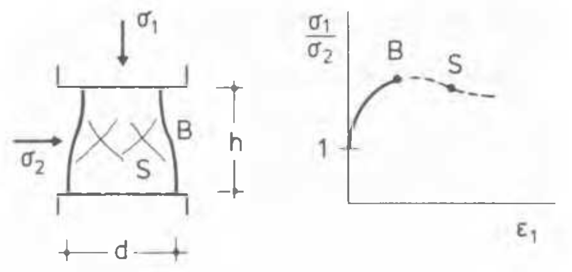

Fig. 9

independent. Shear bands (S), i.e. localized bifurcations inevitably occur beyond the peak; their inclinations are far off the Coulomb value. All these theoretical results are corroborated by experiments with quartz sand.

What is the lesson for constitutive relations? First, one should be aware that soil element tests (cf. Sec. 2) are not generally possible even with ideally lubricated end plates. From the first bifurcation onwards(dashed in Figs.8 and 9) it would be misleading to evaluate tests as if they were element tests. If they are evaluated as element tests then observed material properties, such as strong softening and strong pressure dependence of friction, are not correct. Second, comparing calculated and observed modes of bifurcation reveals material properties which are not detectable by element tests. For example, diffuse or localized bifurcations imply the onset of well-defined shearing (which cannot be uniform in any test). Incremental shear moduli turn out to be overestimated by most of the elasto-plastic relations.

Coming to water saturated sand or clay, bifurcation analyses appear to be promising (Vardoulakis 1985). The interaction of solid and fluid constituents is described by Biot's (1956) version of Darcy's law and by Terzaghi's principle of effective stress. At the onset of bifurcation, sudden changes of pore pressure with subsequent dissipation occur. A decisive property is the volume change tendency of the background solid skeleton. If this is contractant, as in the case of an anisotropically preconsolidated sample, a catastrophic pore pressure increase is to be expected for the case of dead external loading. The onset of such bifurcations need not coincide with limit states of maximum shear stress or stress ratio or, especially, so-called critical states. The associated modes (Fig.10) are a single shear band, or shear bands not reaching rigid boundaries ('internal buckling'), or diffuse bulging, each time without overall volume change.

This theory is helpful for understanding some experimental findings from undrained tests (which have also been discussed in Grenoble). The failure of an unpreloaded saturated undrained sample must be interpreted with caution. Failure does not generally imply maximum shear stress or stress ratio, nor a critical state. The observation of strongly non-elliptical undrained stress paths (Fig. 3), and of markedly different peak stress ratios (expressed as ϕ' values) for cylindrical compression and extension, is easily covered by the bifurcation theory. Therefore such phenomena as spontaneous (non-cyclic) liquefaction and sensitivity are very dependant on sample preparation and the loading device. An evaluation of test results assuming uniform stress and strain must be misleading. This statement implies that pore pressures measured only at one point of the sample are insufficient in case of diffuse bifurcation, and that pore pressures during localized bifurcation can scarcely be measured.

Wrong constitutive modelling with respect to contractancy can have dramatic consequences for boundary value problems. Consider a young clay deposit loaded by a dam or a tower. A c_u-analysis with c_u from strain-controlled tests may indicate a safety factor just above one. If the soil is not prone to contraction, a very slow ductile flow in the ground can be expected. Otherwise, a rapid collapse due to pore pressure production is likely to occur. This danger is not revealed by the conventional c_u-analysis, nor by finite element calculations with a conventional elasto-plastic relation.

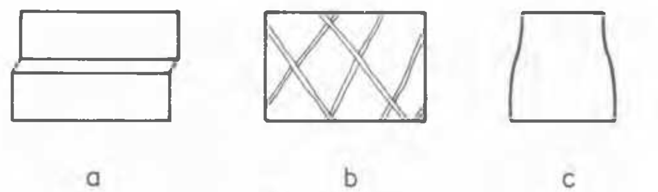

Fig. 10

5 CONCLUDING REMARKS

The present report has focussed on some important aspects of constitutive relations. The behaviour of samples can be better understood with the aid of the concepts of element tests (Sec. 2) and bifurcation (Sec. 4). The asymptotic response to monotonic or cyclic loading is very instructive (Sec.3).

Further aspects can only be mentioned here:
- The determination of constitutive parameters for a chosen relation from experimental data is a difficult calibration problem;
- soils can be self-similar so that centrically similar stress paths are associated with identical strain paths, which renders 1g model tests feasible;
- the response described by constitutive relations should be smooth in order to obtain numerical convergence.

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