Soil stresses under a polygonal area uniformly loaded

Efforts du sol sous un polygon uniformément chargé

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SYNOPSIS Boussinesq's, Westergaard's and Fröhlich's equations for normal stress σ_z are integrated over a triangle. The integration is generalized for any polygonal area.

INTRODUCTION

In Soil Mechanics it is very useful to obtain the soil stresses produced by superficial loading. This is accomplished using the formulas for the normal vertical stress σ_z at a point, caused by a vertical concentrated load applied at the surface. The most widely used formulas are given below.

(i) Boussinesq's equation

$$\sigma_z = \frac{3Q}{2\pi z^2}\left[1 + (r/z)^2\right]^{-5/2} \qquad (1)$$

(ii) Westergaard's equation

$$\sigma_z = \frac{KQ}{2\pi z^2}\left[K^2 + (r/z)^2\right]^{-3/2} \qquad (2)$$

(iii) Fröhlich's equation

$$\sigma_z = \frac{\chi Q}{2\pi z^2}\left[1 + (r/z)^2\right]^{-(\chi+2)/2} \qquad (3)$$

where:
Q = vertical concentrated load
r = horizontal projection of the distance between the load Q and the point P where the stress is evaluated
z = depth of point P

$$K = \sqrt{\frac{1-2\nu}{2(1-\nu)}}$$

ν = Poisson's ratio of the soil
χ = parameter with values 2, 3 or 4

Observe that when $\chi = 3$, in Fröhlich's equation, one obtains the Boussinesq's equation.

In order to determine the normal stress σ_z caused by a uniform vertical load, it is necessary to integrate the expression given by the formulas (1), (2) or (3) over the area where the load is distributed.

An exact solution, is known for the stress under the corner of a rectangular area uniformly loaded (Fadum) and some approximate solutions are known for the case of an area of any shape, such as those proposed by H. G. Poulos in 1974 who used the so called sector method, as well as the popular chart by N. M. Newmark (1942).

There is no known solution for an area of any polygonal shape. This paper presents a method conducive to an exact solution.

INTEGRATION OF THE STRESS UNDER A VERTEX OF A UNIFORMLY LOADED TRIANGULAR AREA

The stress σ_z is obtained by integration under the center P of a circular sector (Fig. 1) with radius R and central angle Θ, loaded with a uniformly distributed vertical load q. The various solutions are

(i) Boussinesq's equation

$$\sigma_z = \frac{q\Theta}{2\pi}\left\{1 - \left[1 + (R/z)^2\right]^{-3/2}\right\} \qquad (4)$$

(ii) Westergaard's equation

$$\sigma_z = \frac{q\Theta}{2\pi}\left\{1 - K\left[K^2 + (R/z)^2\right]^{-1/2}\right\} \qquad (5)$$

(iii) Fröhlich's equation

$$\sigma_z = \frac{q\Theta}{2\pi}\left\{1 - \left[1 + (R/z)^2\right]^{-\chi/2}\right\} \qquad (6)$$

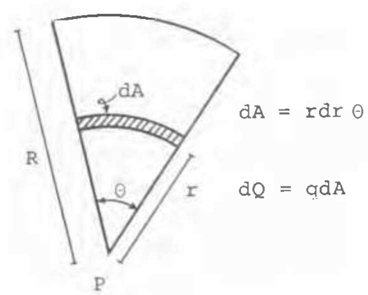

Fig. 1 Circular Sector

Using the results given in (4), (5) or (6) it is possible to obtain σ_z under a vertex P of any triangle. In Fig. 2 a differential circular sector whose central angle is $d\Theta$ is considered. The distance R is a function of Θ as shown in the figure:

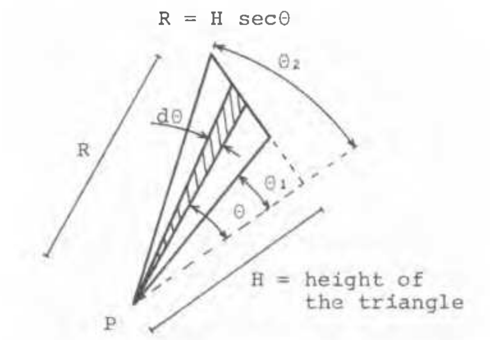

Fig. 2 Triangular Area

The following solutions are obtained

(i) Boussinesq's equation

$$\sigma_z = \frac{q}{2\pi}\int_{\Theta_1}^{\Theta_2}\left\{1 - a^3\left[a^2 + \sec^2\Theta\right]^{-3/2}\right\}d\Theta$$

integrating we obtain

$$\sigma_z = \frac{q}{2\pi}\left\{\Theta_2 - \Theta_1 - \tan^{-1}\left[\frac{a\tan\Theta_2}{\sqrt{a^2+\sec^2\Theta_2}}\right] + \tan^{-1}\left[\frac{a\tan\Theta_1}{\sqrt{a^2+\sec^2\Theta_1}}\right] + \right.$$

$$\left. + \frac{a}{1+a^2}\left[\frac{\tan\Theta_2}{\sqrt{a^2+\sec^2\Theta_2}} - \frac{\tan\Theta_1}{\sqrt{a^2+\sec^2\Theta_1}}\right]\right\} \qquad (7)$$

where a is the ratio z/H and H is the height of the triangle

(ii) Westergaard's equation

$$\sigma_z = \frac{q}{2\pi}\int_{\Theta^1}^{\Theta_2}\left\{1 - b\left[b^2 + \sec^2\Theta\right]^{-1/2}\right\}d\Theta$$

integrating we obtain

$$\sigma_z = \frac{q}{2\pi}\left\{\Theta_2 - \Theta_1 - \tan^{-1}\left[\frac{b\tan\Theta_2}{\sqrt{b^2+\sec^2\Theta_2}}\right] + \right.$$

$$\left. + \tan^{-1}\left[\frac{b\tan\Theta_1}{\sqrt{b^2+\sec^2\Theta_1}}\right]\right\} \qquad (8)$$

where b is the ratio z/H multiplied by the constant K.

(iii) Fröhlich's equation

$$\sigma_z = \frac{q}{2\pi}\int_{\Theta_1}^{\Theta_2}\left\{1 - a^{\chi}\left[a^2 + \sec^2\Theta\right]^{-\chi/2}\right\}d\Theta$$

Integrating with $\chi = 2$, we obtain

$$\sigma_z = \frac{q}{2\pi\sqrt{1+a^2}}\left\{\tan^{-1}\left[\frac{\tan\Theta_2}{\sqrt{1+a^2}}\right] - \tan^{-1}\left[\frac{\tan\Theta_1}{\sqrt{1+a^2}}\right]\right\} \qquad (9)$$

Integrating with $\chi = 4$, we obtain

$$\sigma_z = \frac{q}{4\pi(1+a^2)}\left\{\frac{3a^2+2}{\sqrt{1+a^2}}\left[\tan^{-1}\left(\frac{\tan\Theta_2}{\sqrt{1+a^2}}\right) - \tan^{-1}\left(\frac{\tan\Theta_1}{\sqrt{1+a^2}}\right)\right] + \right.$$

$$\left. + a^2\left[\frac{\tan\Theta_2}{a^2+\sec^2\Theta_2} - \frac{\tan\Theta_1}{a^2+\sec^2\Theta_1}\right]\right\} \qquad (10)$$

The process of integration was, in general, simple. However in the case of the Boussinesq's equation the integration was rather complicated.

INTEGRATION FOR AN AREA OF ANY POLYGONAL SHAPE

Since a polygon can be subdivided in triangular areas (Fig. 3), it is possible to apply the integration formulas obtained above in sequential form to all the resulting triangles.

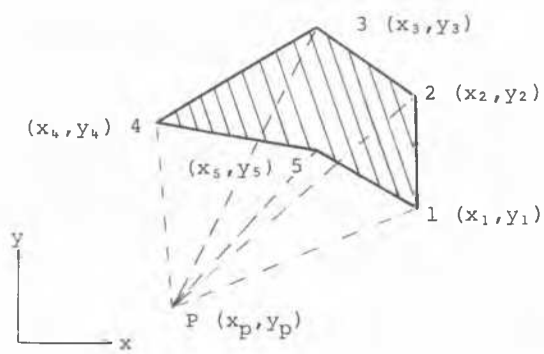

Fig. 3 Polygonal Uniformly Loaded

In Fig. 3, the triangles P12, P23 and P34 give a positive contribution to the σ_z-values under P while the triangles P45 and P51 contribute negatively. Note that the numbering of the vertices is counterclockwise.

The formulas for a polygon of n vertices will be obtained applying reiteratedly the equations (7), (8), (9) or (10) at the n-triangles formed by the point P and two consecutive vertices i and i+1.

(i) Boussinesq's equation

$$\sigma_z = \frac{q}{2\pi}\sum_{i=1}^{n}\left\{\Theta_{2i}-\Theta_{1i}-\tan^{-1}\left[B_{2i}\right]+\tan^{-1}\left[B_{1i}\right]+\right.$$
$$\left.+\left[B_{2i}-B_{1i}\right]/\left[a_i^2 + 1\right]\right\} \qquad (11)$$

(ii) Westergaard's equation

$$\sigma_z = \frac{q}{2\pi}\sum_{i=1}^{n}\left\{\Theta_{2i}-\Theta_{1i}-\tan^{-1}\left[W_{2i}\right]+\tan^{-1}\left[W_{1i}\right]\right\} \qquad (12)$$

(iii) Fröhlich's equation

With $\chi = 2$

$$\sigma_z = \frac{q}{2\pi}\sum_{i=1}^{n}\frac{1}{\sqrt{1+a_i^2}}\left\{\tan^{-1}\left[J_{2i}\right]-\tan^{-1}\left[J_{1i}\right]\right\} \qquad (13)$$

With $\chi = 4$

$$\sigma_z = \frac{q}{4\pi}\sum_{i=1}^{n}\frac{1}{1+a_i^2}\left\{\frac{3a_i^2+2}{\sqrt{1+a_i^2}}\left[\tan^{-1}\left(J_{2i}\right)-\tan^{-1}\left(J_{1i}\right)\right]+\right.$$
$$\left.+ N_{2i} - N_{1i}\right\} \qquad (14)$$

where:

$\Theta_{1i} = \tan^{-1} C_{1i}$

$\Theta_{2i} = \tan^{-1} C_{2i}$

$C_{1i} = \left[x'_i\,(x'_{i+1} - x'_i) + y'_i(y'_{i+1} - y'_i)\right]/F_i$

$C_{2i} = \left[x'_{i+1}(x'_{i+1} - x'_i) + y'_{i+1}(y'_{i+1} - y'_i)\right]/F_i$

$x'_i = x_i - x_p$

$x'_{i+1} = x_{i+1} - x_p$

$y'_i = y_i - y_p$

$y'_{i+1} = y_{i+1} - y_p$

$x_p,\ y_p$ = coordinates of point P

$x_i,\ y_i$ = coordinates of vertex i

$x_{i+1},\ y_{i+1}$ = coordinates of vertex i+1

$F_i = x'_i y'_{i+1} - x'_{i+1} y'_i$

$a_i = \left|zL_i/F_i\right|$

$L_i = \sqrt{(x'_{i+1} - x'_i)^2 + (y'_{i+1} - y'_i)^2}$

$$B_{ki} = \frac{a_i C_{ki}}{\sqrt{1 + a_i^2 + C_{ki}^2}} \qquad (k=1,2)$$

$$W_{ki} = \frac{Ka_i C_{ki}}{\sqrt{1 + K^2 a_i^2 + C_{ki}^2}} \qquad (k=1,2)$$

$$J_{ki} = \frac{C_{ki}}{\sqrt{1 + a_i^2}} \qquad (k=1,2)$$

$$N_{ki} = \frac{a_i^2 C_{ki}}{1 + a_i^2 + C_{ki}^2} \qquad (k=1,2)$$

CONCLUSIONS

With the proposed formulas (11), (12), (13) or (14) will be easy to calculate the normal stress σ_z at any point P within a soil loaded by a uniform vertical load distributed over a polygonal area. In fact one of the author (C. Casales G.) has written a computer-program for the determination of σ_z in a HP-41CV computer.

FINAL COMMENT AND ACKNOWLEDGMENT

Both authors obtained the same results presented in this paper working independently.

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