An exact finite element for beams on elastic foundation

Un élément fini exact pour les poutres sur fondation élastique

D. Z. YANKELEVSKY, Faculty of Civil Engineering, Technion – Israel Institute of Technology, Haifa, Israel
M. EISENBERGER, Faculty of Civil Engineering, Technion – Israel Institute of Technology, Haifa, Israel

SYNOPSIS An exact stiffness matrix for a beam on elastic foundation has recently been developed by the authors, leading to exact solutions with only a small number of elements. A general formulation is presented which considers transverse as well as axial loading. The elastic foundation is also generalized to include linear and rotational spring and shear interaction between springs. Stiffness terms are derived from the exact solution of the governing differential equation. The exact stiffness approach requires nodal points only at points of abrupt changes in loading, stiffness or local supports. The efficiency of the present formulation is demonstrated through numerical examples.

INTRODUCTION

The problem of a beam on elastic foundation has been treated by numerous authors [Hetenyi (1964)]. Finite Difference solutions have been applied [Beaufait and Hoadley (1980)] to yield approximate solution which converges to the exact solution when the finite difference mesh is refined. Finite element approximate solution of the problem may roughly be obtained by using beam elements to which discrete springs are connected to the models [Cook (1981)]. A more complicated model in which the foundation springs are smeared is presented by Cook (1981) and Mohr (1980) however the shape function of a free beam is used in these foundations thus yielding approximate solutions to the problem.

Recently an exact finite element stiffness matrix for a beam on a Winkler foundation has been developed by the authors [Eisenberger and Yankelevsky (1984)]. In a later work a general foundation for the exact stiffness matrix of beam columns on elastic foundation is presented [Ynakelevsky and Eisenberger, (1984)]. Recent work by Zhaohua and Cook (1983) generalized the elastic foundation. In the present paper a general exact finite element formulation for a beam subjected to transverse and axial force resting on a generalized elastic foundation is derived and numerical examples are shown.

BASIC EQUATIONS

A beam column on a generalized elastic foundation as shown hereunder, is considered. The beam is subjected to transverse loading and a pair of tensile axial forces N, acting at the end cross sections. The generalized foundation is composed of three components:

(a) Winkler type springs [Winkler (1867)], defined by the Winkler foundation modulus k which apply a reaction $k \cdot y(x)$ being normal to the undeformed beam axis and directly proportional to the beam deflection $y(x)$.

(b) Pasternak [Selvadurai (1979)] shear interaction between springs which results by a reaction component normal to the beam. The shear property is defined by the parameter k_s yields a reaction $k_s \cdot y''(x)$.

(c) Rotational springs which apply reaction moments being proportional the the angle of rotation through the parameter k_θ. The distributed moment may be replaced by equivalent shear $k_\theta \cdot y''(x)$.

Force equilibrium in y direction of an infinitesimal element yields:

$$\frac{ds(x)}{dx} = p(x) - q(x) \tag{1}$$

where: $s(x)$ is the shear force
$q(x)$ is the applied transverse loading
$p(x)$ is the foundation reaction:

$$p(x) = k\, y(x) - k_s \cdot y''(x) - k_\theta y''(x) \tag{2}$$

Moment equilibrium yields:

$$\frac{dM(x)}{dx} = N \frac{dy(x)}{dx} - S(x) \tag{3}$$

and the moment curvature equation is:

$$EI \frac{d^2 y(x)}{dx^2} = M(x) \tag{4}$$

From eqs. 1-4 we obtain the governing differential equation:

$$EI \frac{d^4 y(x)}{dx^4} - D \frac{d^2 y(x)}{dx^2} + ky(x) = q(x) \tag{5}$$

in which:

$$D = N + k_s + k_\theta \tag{5a}$$

The homogeneous solution of eq. 5 becomes:

$$y(x) = C_1 \cos\beta x \cosh\alpha x + C_2 \cos\beta x \sinh\alpha x + C_3 \sin\beta x \cosh\alpha x + C_4 \sin\beta x \sinh\alpha x \tag{6}$$

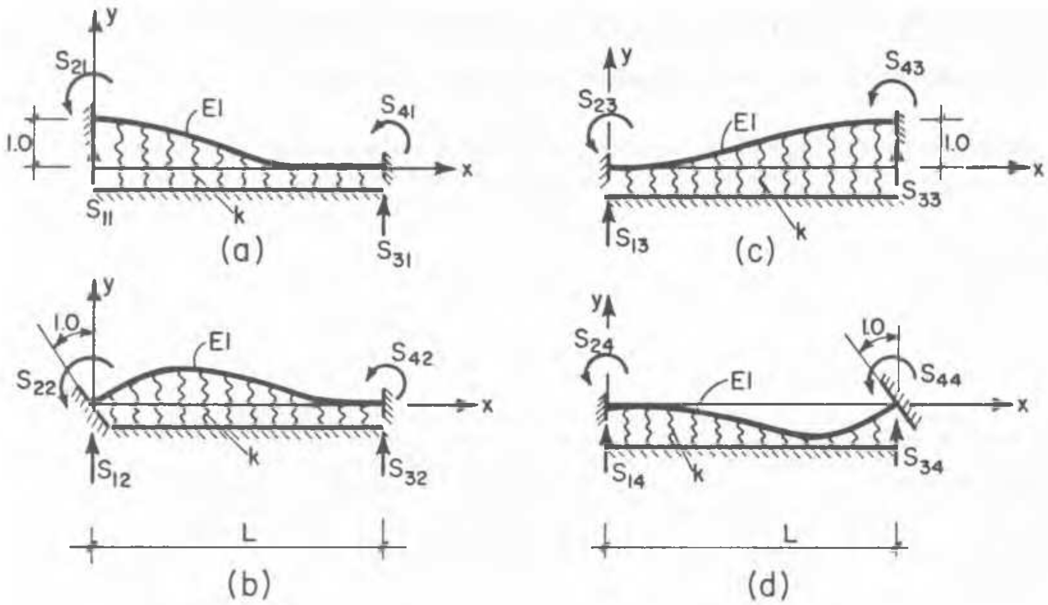

Fig. 1 Definition of Stiffness Terms

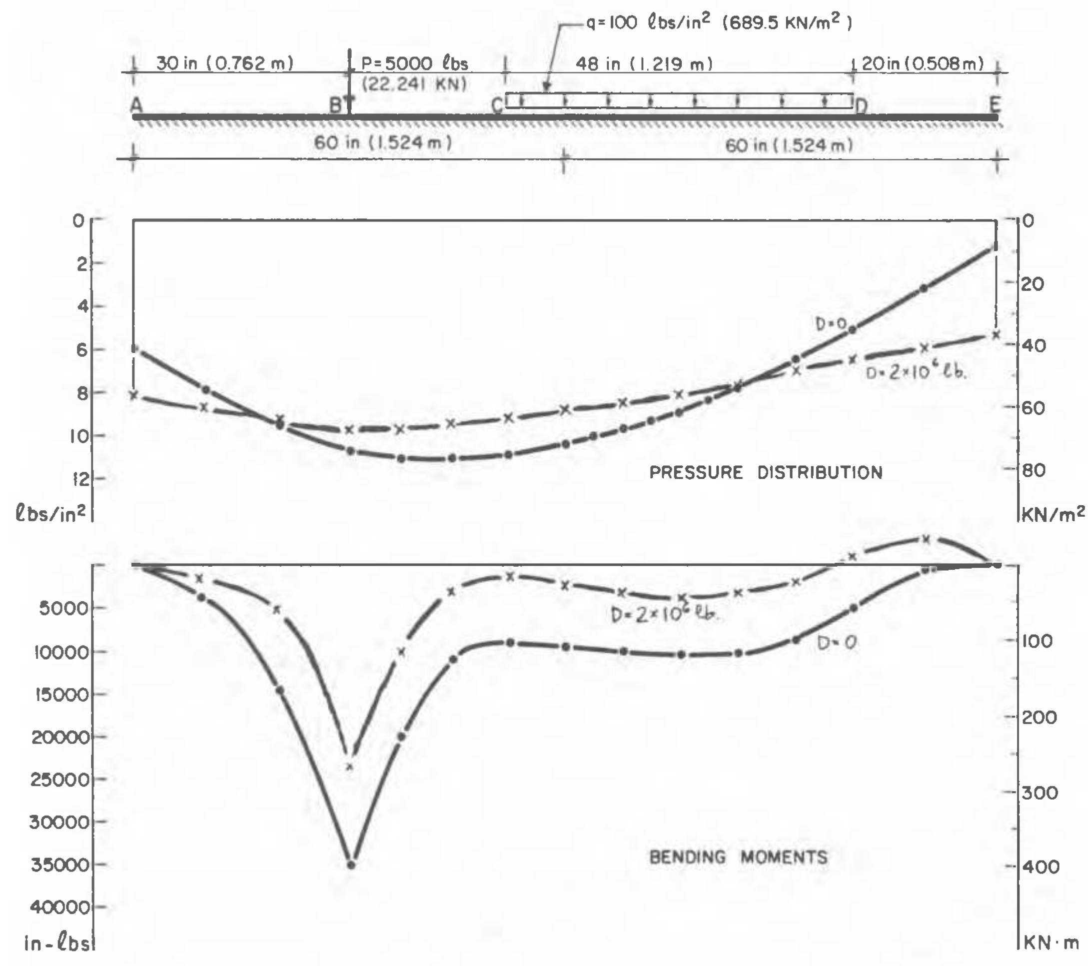

Fig. 2. Example I - Beam Geometry , Loading and Computed Results

in which:

$$\alpha = \sqrt{\lambda^2 + \delta} \quad (6a)$$

$$\beta = \sqrt{\lambda^2 - \delta} \quad (6b)$$

$$\lambda = \sqrt[4]{\frac{k}{4EI}} \quad (6c)$$

$$\delta = \frac{D}{4EI} \quad (6d)$$

STIFFNESS MATRIX

Consider a finite element of a beam on a generalized elastic foundation. The basic definition for the stiffness terms of that element are shown in Fig. 1. These are the terms of the symmetric 4x4 stiffness matrix of which the values may easily be found by substituting the corresponding end conditions into eq. 6. As a result the following expressions for the stiffness terms are found [Yankelevsky and Eisenberger (1984)]:

$$S_{13} = -\frac{4\alpha\beta\lambda^2 EI}{M}\left[\alpha\sin\beta\ell\,\cosh\alpha\ell + \beta\cos\beta\ell\,\sinh\alpha\ell\right] = S_{31} \quad (7)$$

$$S_{23} = -\frac{4\alpha\beta\lambda^2 EI}{M}\sin\beta\ell\,\sinh\alpha\ell = S_{32} = -S_{14} = -S_{41} \quad (8)$$

$$S_{43} = -\frac{2\lambda^2 EI}{M}\left[\alpha^2\sin^2\beta\ell + \beta^2\sinh^2\alpha\ell\right] = S_{34} = -S_{12} = -S_{21} \quad (9)$$

$$S_{33} = \frac{4\alpha\beta\lambda^2 EI}{M}\left[\beta\sinh\alpha\ell\,\cosh\alpha\ell + \alpha\sin\beta\ell\,\cos\beta\ell\right] = S_{11} \quad (10)$$

$$S_{22} = S_{44} = \frac{2\alpha\beta EI}{M}\left[\beta\sinh\alpha\ell\,\cosh\alpha\ell - \alpha\sin\beta\ell\,\cos\beta\ell\right] \quad (11)$$

$$S_{24} = S_{42} = -\frac{2\alpha\beta EI}{M}\left[\beta\sinh\alpha\ell\,\cos\beta\ell - \alpha\sin\beta\ell\,\cosh\alpha\ell\right] \quad (12)$$

These are exact expressions for the stiffness as they are derived from the exact solution. As a result an exact displacement field is obtained and therefore when continuity conditions exists along a certain part of the beam, a single finite element is sufficient to yield an exact solution. Nodes are therefore placed at points of abrupt changes in loading, foundation or beam stiffnesses and local supports. The stiffness matrix has been incorporated into a beam program, and its solution is very efficient because only a small number of elements is required to yield an exact solution.

NUMERICAL EXAMPLES

Example 1: The beam analysed by Hetenyi (1964) was solved using the computer program (Fig. 2). For this loading system, 4 elements are required to obtain the exact solution (elements AB, BC, CD, and DE). The resulting ground pressure distribution and bending moment diagram are shown in the Figure, and these are identical with those given by Hetenyi.

For the case where the generalized foundation is non zero (i.e. pasternak model, rotational springs model, axial force, or combination of the above, $D \neq 0$) the figure shows different ground pressure distribution and bending moment diagram. As can be seen these are significant changes.

Example 2: A simply supported beam on elastic foundation (Fig. 3) was analysed by Mohr (1980) using a contact stiffness matrix.

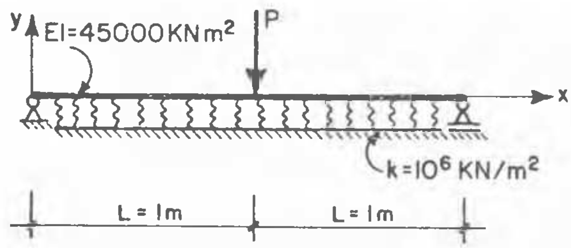

Fig. 3 Example 2 - Geometry and Loading

The same problem is solved by the proposed stiffness matrix formulation using one segment, for different values of the generalized foundation modulus D. The results are given in Table 1. Results are given for the following quantities: rotation angle at the support, deflection under the applied load, reaction, and bending moment under the applied load. All the values are linear function of the applied force, F.

TABLE I

D KN	ϕ $\times 10^{-5}$ Rad	R $\times 10^{-2}$ KN	δ $\times 10^{-4}$ cm	M KN-m
-15000.	1.098	-0.602	0.8607	0.1832
-10000.	1.086	-0.098	0.8525	0.1820
-5000.	1.075	0.396	0.8444	0.1809
0.	1.064	0.880	0.8366	0.1798
5000.	1.053	1.355	0.8288	0.1787
10000.	1.042	1.821	0.8212	0.1777
15000.	1.032	2.279	0.8137	0.1766

CONCLUSIONS

The paper presents the exact stiffness matrix of a beam subjected to transverse and axial load resing on a generalized elastic foundation. The foundation is composed of Winkler springs including shear transfer and rotational springs. An exact solution may be achieved by using a small number of elements. Nodes are placed at points of abrupt change in loading, stiffness or at the location of local supports. The stiffness matrix is easily assembled into a beam program and efficient and exact solutions are obtained.

REFERENCES

Beaufait, F.W., Hoadley, P.W. (1980). Analysis of elastic beams on nonlinear foundations. Computers and Structures, Vol. 12, pp.669-696.

Cook, R.D. (1981). Concepts and applications of finite element analysis. John Wiley & Sons.

Eisenberger, M., Yankelevsky, D.Z. (1984). Exact stiffness matrix for beams on elastic foundation. To be published in Computers and Structures.

Hetenyi, M. (1964). Beams on elastic foundation. Univ. of Michigan press, Ann-Arbor, Mich.

Mohr, G.A. (1980). A contact stiffness matrix for finite element problems involving external elastic restraint. Computers and Structures, Vol. 12, pp. 189-191.

Selvadurai, A.P.S. (1979). Elastic analysis of soil foundation interaction. Elsevier Scientific Publishing Company, Amsterdam, Holland.

Winkler, E. (1867). Die lehre von der elastizität und festigkeit. Prague, Dominicus.

Yankelevsky, D.Z., Eisenberger, M. (1984). Exact F.E. solution for a beam column on elastic foundation. Submitted for publication.

Zhaohura, F., Cook, R.D. (1983). Beam elements on two parameter elastic foundations. Jour. of Eng. Mech., Div., ASCE, Vol. 109, No. 6, Dec.