Probabilistic analysis of stability of earth slopes

Analyse probabiliste de la stabilité des talus en terre

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SYNOPSIS A probabilistic method for the analysis of stability of slopes is presented. Determination of the probability of failure anywhere along the slope axis involves two steps. The first step concerns determination of the probability of occurrence of a potentially unstable zone, i.e. an area where the conventional factor of safety is less than unity. It is demonstrated that a real failure can only occur within such area. The second step involves evaluation of the probability that a real failure occurs, assuming the presence of a potentially unstable zone. In this step the effects of the finite width of a real failure mode must be taken into account. A procedure is suggested for updating the estimate of the probability of failure, if it is observed or may be assumed that certain states of loading actually have occurred and did not involve failure.

INTRODUCTION

There has been a continuous development of probability based methods for computational analysis of reliability of earth slopes during the past decade. In the earliest publications attention was restricted to a probabilistic treatment of the conventional analysis of stability in a cross sectional plane of the slope (Wu & Kraft 1970, Cornell 1972, Alonso 1975, Tang Yucemen & Ang 1976). Key feature of these analyses is that natural variability of shearing strength of soil and uncertainty, originating from limited sampling and testing defects, leads to uncertainty about the actual value of the factor of safety. The probability of failure of the slope was considered to equal the probability that the factor of safety is less than unity. Either only the most critical slip circle or a more or less representative set of potential slip circles was considered, taking into account decay of mutual correlation (Morla Catalan & Cornell 1976).
A basically three dimensional failure mode has been examined in a probabilistic context by Vanmarcke (1977). He considered a rigid cylinder of finite width in the along slope direction. His analysis accounts for "end section" contributions to the failure resisting moment. Combination of the effects of these contributions, on the one hand, and the effect of "averaging" of shearing strength variations along the slip surface, on the other hand, yields a so called critical width of the failure mode for which the corresponding probability of occurrence takes a maximum. The analysis, according to Vanmarcke, thus predicts the probability of failure as well as the most probable width of the failure area.

The analysis presented here adopts Vanmarcke's fundamental concept of modeling spatial variations of the soil's shearing strength as a random process, as well as his concept of a "finite width" failure mode. It differs from his analysis in the sense that this width is not preassigned to some critical value. Instead, it is taken to be equal to the expected width of the zone where the conventional factor of safety is less than unity, the so called potentially unstable zone. It is demonstrated that a real failure mode, if it occurs, necessarily coincides exactly with such a zone. Whether or not a real failure actually occurs, depends on the "end section" contributions to the failure resisting moment. The analysis yields estimates of the probability that a failure occurs and the expected width of the failure mode. The adopted analytical description enabled a further development, which would otherwise have been cumbersome. A kind of Bayesian procedure could be designed, by which it is possible to update estimates of the probability of failure based on observation of the history of survival of the slope.

DESCRIPTION OF THE FAILURE MODE

The failure mode adopted here consists of a cylindrical failure surface, which extends over a finite width ℓ in the along slope direction (figure 1). Analysis of equilibrium of failure generating (overturning) moment and failure resisting moments predicts failure if

$$\int_{\ell} M_r + 2.M_e < \int_{\ell} M_o \tag{1}$$

Here M_r denotes local failure resisting moment due to mobilized friction along the (potential) failure surface, M_e the "end section" contribution to the failure resisting moment and M_o the local overturning moment.

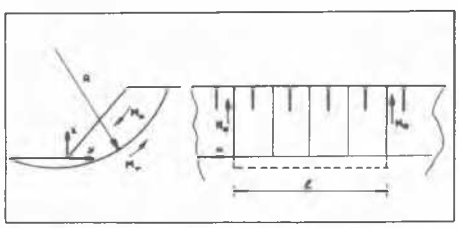

Figure 1. Definition of failure mode

From equation 1 it is found that existence of a real failure mode necessarily implies existence of a potentially unstable zone. Figure 2 indicates a potentially unstable zone of width ℓ. Now consider the possibility that a real failure occurs, and its width is less than ℓ (figure 2a). The part of the potentially unstable zone complementary to the failure mode would be excluded from failure. However, comparison of failure causing and failure resisting moments, acting on this part of the zone indicate its instability, since

$$\int_{\ell_1} M_r + M_e' < \int_{\ell_1} M_o + M_e \qquad (2)$$

where it is assumed that the partly mobilized end section resistance M_e' is less than M_e. Next, consider the possibility that the width of a real failure mode exceeds ℓ (figure 2b). Clearly the part of the failure mode outside ℓ is generated by internal interaction of shearing forces M_e' (with $M_e'<M_e$). Comparison of the moments acting on this part of the failure mode yields a condition for its instability

$$M_e' > \int_{\ell_2} M_r - \int_{\ell_2} M_o + M_e \qquad (3)$$

which is obviously not satisfied since $M_e'<M_e$ and $M_r<M_o$. So it appears that a real failure mode can only occur within a potentially unstable zone, and its width equals the width of such zone. It is believed that this statement holds approximately true when a less idealized description of a failure mode is adopted.

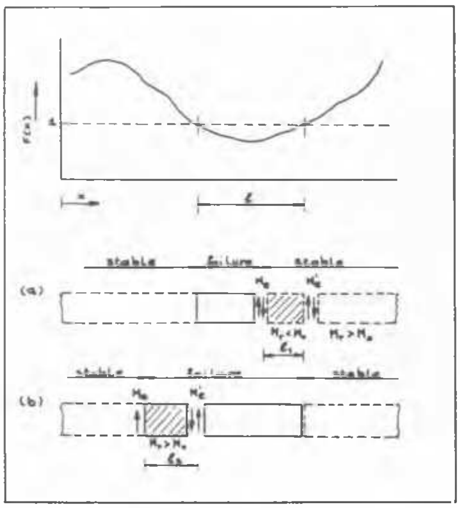

Figure 2. A potentially unstable zone and hypothetical failure modes

The factor of safety of a failure mode, including the effects of end section contributions, reads

$$F^{\ell} = F + \frac{2M_e}{\ell M_o} \qquad (4)$$

where F is the conventional factor of safety. The probability that a failure occurs is expressed as

$$\begin{aligned} P(f) &= P(\, F^{\ell}<1 \,) = P(\, F^{\ell}<1 \wedge F<1 \,) \\ &= P(\, F^{\ell}<1 \mid F<1 \,)\, P(\, F<1 \,) \end{aligned} \qquad (5)$$

Here, P( ) denotes probability of occurrence of an event, f the event "failure", F<1 the event "the (conventional) factor of safety is less than unity somewhere along the slope axis", and $F^{\ell}<1$ the event "the factor of safety of a failure mode of width ℓ is less than unity". In the following F(x)<1 will be used to indicate the event "the factor of safety at the location x in the along slope direction is less than unity", and F↓1 is used to indicate a down crossing event somewhere along the slope axis. In equation 5, $\wedge$ denotes intersection of events and | denotes "conditional to".

PROBABILITY OF OCCURRENCE OF A POTENTIALLY UNSTABLE ZONE

It has been well established that most, if not all, of the properties attributed to a soil in a natural deposit may exhibit significant spatial variations (Lumb 1966, Schultze 1971). In a slope stability analysis, variations of soil density and shearing strength properties yield the failure causing and the failure resisting moments and consequently the factor of safety to be varying quantities in the along slope direction. It has been demonstrated by Alonso (1975) that the major part of variations of the factor of safety is due to variations of the failure resisting moment. These variations originate from variations of shearing strength properties, if variations of the pore pressure distribution is left out of consideration. So, only variations of shearing strength properties will be considered here. Moreover, in order to avoid complexity of notation, the analysis here is restricted to a cohesive soil in a one layer stratum. This imposes no restriction to applicability of the methodology in the case of soils with internal friction angle, applying an effective stress analysis, or in the case of multilayered strata.

Statistical techniques have been introduced as an adequate tool for analytical description of the pattern of variations. In this paper, a random field model, more or less similar to the model introduced by Vanmarcke (1977), has been adopted. The erratic pattern of cohesion c is conceived as a normally (gaussian) distributed random variable at each location within the soil layer. Parameters of the distribution, i.e. expected mean value μ_c and standard deviation σ_c, may be estimated from a relatively limited number of borehole samples. Furthermore, the random field model requires the assessment of a "decay of correlation" model, which more or less indicates the "average wavelength" in the pattern of variations. In our analysis, the following decay-of-correlation model has been assumed

$$\rho_c = \exp\{-\frac{\Delta x^2+\Delta y^2}{d_h^2} - \frac{\Delta z^2}{d_v^2}\} \qquad (6)$$

where $\rho_c=\rho_c(\Delta x,\Delta y,\Delta z)$ denotes correlation among deviations from the mean value of cohesion in two points of the soil layer, separated by a distance Δx and Δy in horizontal and Δz in vertical direction. Parameters d_h and d_v will be referred to as the autocorrelation parameters or distances of correlation.

In order to reveal the statistics of variations of the factor of safety F in the along slope direction, consider the potential slip circle C, indicated in figure 3. The failure resisting moment can be obtained from integration of cohesion along the slipcircle:

$$M_r(x) = R \int_C c(x,y_C,z_C)\, dC \qquad (7)$$

where C symbolically denotes the slip circle arc with radius R and $c(x,y_C,z_C)$ the local value of cohesion. The expected mean value of the failure resisting moment reads:

$$\mu_{M_r} = \mu_c\, |C|\, R \qquad (8)$$

where $|C|$ denotes the length of the failure arc. Note that μ_{M_r} is independent of x since μ_c is so. The expression for $M_r(x+\Delta)$ is similar to equation 7 if x is replaced by x+Δx. The autocovariance of both resisting moments reads:

$$\begin{aligned} \mathrm{cov}(M_r(x),M_r(x+\Delta x)) &= E(M_r(x)\, M_r(x+\Delta x)) - \mu_{M_r}^2 \\ &= R^2\, \sigma_c^2 \int_{C_1}\int_{C_2} \rho_c(\Delta x, y_{C_1}-y_{C_2}, z_{C_1}-z_{C_2})\, dC_1\, dC_2 \end{aligned} \qquad (9)$$

where E( ) denotes mathematical expectation and C_1 and C_2 denote identical slip circles at x and x+Δx respectively. The variance of the resisting moment equals the covariance for Δx=0 :

$$\sigma^2_{M_r} = \mathrm{cov}(M_r(x), M_r(x)) \qquad (10)$$

and the coefficient of correlation among failure resisting moments at x and x+Δx equals by definition:

$$\rho_{M_r}(\Delta x) = \mathrm{cov}(M_r(x), M_r(x+\Delta x))/\sigma^2_{M_r} \qquad (11)$$

Correlations among factors of safety are identical to correlations among failure resisting moments, since the overturning moment is considered as a deterministic constant. So, for simplicity of notation, the indices M_r or F will be dropped if we refer to correlation. For the present case it turns out that:

$$\rho(\Delta x) = \exp(-(\Delta x/d_h)^2) \qquad (12)$$

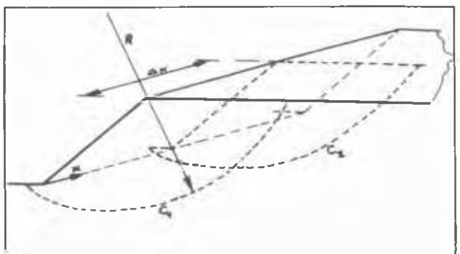

Figure 3. Definition sketch

Cross covariances and cross correlations among failure resisting moments and factors of safety, associated to different slip circle arcs, may be evaluated according to equation 9, if C_1 and C_2 refer to different slip circles.

Failure resisting moments and factors of safety are normally distributed, since a normal distribution has been assumed for the cohesion. The probability that the factor of safety $F(x)=M_r(x)/M_o$ is less than unity somewhere along a slope, which extends from x=o to x=L, equals:

$$P(\,F<1\,) \approx P(\,F(o)<1\,) + P(\,F(o)>1\,)\,P(\,F\downarrow 1\,) \qquad (13)$$

if it is assumed that $L \gg d_h$. Introducing the so called index of reliability:

$$\beta = (\mu_{M_r}-M_o)/\sigma_{M_r} \quad \text{or} \quad \beta = (\mu_F-1)/\sigma_F \qquad (14)$$

it is found, for small target probabilities:

$$P(\,F<1\,) \approx \Phi(-\beta) + \Phi(\beta)\,\frac{L}{2\pi}\,\exp(-\tfrac{1}{2}\beta^2)\,\sqrt{-\rho''(o)} \qquad (15)$$

where Φ( ) the standardized Gaussian probability function, and double prime denotes second derivative w.r.t. x. The formula for the level crossing probability $P(\,F\downarrow 1\,)$ can be found in any standard textbook on random processes (e.g. Papoulis 1965). It may easily be verified that the expected width of a potentially unstable zone can be expressed as:

$$\ell = 2\pi\,\Phi(-\beta)\,\exp(\tfrac{1}{2}\beta^2)\,/\,\sqrt{-\rho''(o)} \qquad (16)$$

In figure 4 the ratio $\ell\sqrt{-\rho''(o)}$ is given as a function of the index of reliability, according to equation 16. Note that $1/\sqrt{-\rho''(o)}=0.7d_h$ in the present case.

PROBABILITY OF OCCURRENCE OF A FAILURE

Recalling equations 4 and 5 and applying the results of the previous section, the probability of failure can be evaluated as:

$$P(f) = \frac{\Phi\left(-\beta - \frac{2M_o}{\ell\sigma_{M_r}}\right)}{\Phi(-\beta)}\,P(\,F<1\,) \qquad (17)$$

Note that P(f) approaches P(F<1) as the expected width ℓ increases largely, and P(f) tends to zero if ℓ decreases to zero.

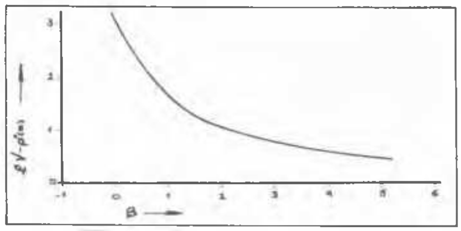

Figure 4. Standardized width $\ell\sqrt{-\rho''(o)}$ versus β

A PROCEDURE FOR UPDATING ESTIMATES OF THE PROBABILITY OF FAILURE

Until now, estimates of the probability of failure of a slope have been based purely on statistical information concerning shear strength properties. For any state of loading, and pore pressure distibution if an effective stress analysis is considered, a corresponding estimate of the probability of failure can be determined. These estimates will be referred to as prior estimates. Suppose that one of the analyzed states of loading is effectuated, and it is observed that no real failure occurs. If the fundamental properties of soil do not alter in the course of time, then there will occur no failure either, if the same state of loading is effectuated a next time. In other words, the probability of failure in this state of loading reduces to zero. This new estimate will be referred to as the posterior estimate. In this example, the reduction of the probability of failure is trivial. More interestingly is the question how the probability of failure under extreme conditions of loading is reduced, if it is observed that other conditions have been survived. Procedures, which refer to this problem, applicable to specific situations, have been reported by Matsuo (1983) and by the author (1983).

In order to obtain a general procedure to reduce estimates of the probability of failure, consider two different states of loading of a slope, referred to as states 1 and 2. The corresponding factors of safety F_1 and F_2 may refer to different critical failure arcs. Variances, auto and cross correlations of these factors of safety and the associated prior estimates of the probabilities of failure can be obtained from numerical evaluation, in accordance with the previous sections. Now suppose that state 1 is actually effectuated, and no failure is observed. The posterior estimate of the probability of failure in state 2, applying the total probability theorem, is expressed as:

$$P(f_2|\bar{f}_1) = (P(f_2) - P(f_1 \wedge f_2))/(1-P(f_1)) \qquad (18)$$

An overbarred event denotes negation of the event. $P(f_1)$ and $P(f_2)$ are prior estimates of the probabilities of failure. In equation 18 only the probability of intersection of events $P(f_1 \wedge f_2)$ is yet unknown. Recalling equation 5 this probability equals:

$$P(f_1 \wedge f_2) = P(\,F_1^{\ell}<1 \wedge F_2^{\ell}<1 \wedge F_1<1 \wedge F_2<1\,) \qquad (19)$$

If potentially unstable zones occur in both states of loading, then these zones either coincide or they are separated by a distance, sufficiently large to assume weak cross correlation among the local factors of safety within these zones. If the two zones coincide, then the correlation among F_1 and F_2, and thus among F_1^{ℓ} and F_2^{ℓ} can be evaluated as the cross correlation at zero lag. The r.h.s.

probability of equation 19 can thus be decomposed into two pobabilities, each one conditional to one of the two mutual exclusive events. A tedious and lengthy derivation, which will not be given here, yields the formula:

$$P(f_1 \wedge f_2) \approx P(F_j<1)\{S(\rho)\, Q_{ij} + S(o)(1-Q_{ij})\, P(F_i \downarrow 1)\} \qquad \ldots\ldots(20)$$

where $i=2, j=1$ if $P(F_2<1) > P(F_1<1)$, and $i=1, j=2$ otherwise, and ρ is the coefficient of cross correlation at zero lag among the factors of safety. In equation 20 is:

$$S(\rho) = \frac{\Psi\left(-\beta_1 - \dfrac{2M_{e_1}}{\ell_1 \sigma_{M_{r1}}}, -\beta_2 - \dfrac{2M_{e_2}}{\ell_2 \sigma_{M_{r2}}}, \rho\right)}{\Psi(-\beta_1, -\beta_2, \rho)} \qquad (21)$$

and analogously $S(o)$,

$$Q_{21} = \Psi(-\beta_1, -\beta_2, \rho) / \Psi(-\beta_1, \infty, \rho) \qquad (22)$$

$$Q_{12} = \Psi(-\beta_1, -\beta_2, \rho) / \Psi(\infty, -\beta_2, \rho) \qquad (23)$$

where $\Psi(\ ,\ ,\rho)$ is the standardized bivariate normal probability function. Note that $\Psi(a,b,o)=\Phi(a)\Phi(b)$. Equation 20 may easily be verified for extremal situations of cross correlation among F_1 and F_2. A computerized procedure is indicated for efficient numerical solution of the bivariate integrals involved in equations 21,22,23.

APPLICATION TO A FICTITIOUS SLOPE

Based on the analysis of the previous sections, the computer code PROSTAB has been developed. The programme is suitable for either (quasi) undrained or effective stress analysis of stability of a slope in homogeneous or stratified soils. It applies the (iterative) Bishop method of slices.

As a demonstration of the theory, the stability of a slope in a cohesive soil has been analyzed. Figure 5 shows a cross sectional view of the slope. Seven different levels of overburden loading have been considered. The corresponding conventional factors of safety range from 1.33 to 0.97, as the overburden increases from zero to 30 kN/m^2. The associated critical failure arcs are indicated in figure 5. The selected soil parameters, as well as the key results of the computations have been summarized in table I. This table also includes some results, applying Vanmarcke's theory (Vanmarcke 1977). The computed probabilities of failure, according to both theories, match reasonably well, except for the case of zero overburden, where they differ by a factor of 4. The estimated widths of failure areas are found to agree reasonably well, except for the case of high probability of failure.

TABLE I

Results of analysis of the slope in figure 5.

q	F	β	$P(F<1)$	$P(f)$	ℓ	$P(f_2 \mid \bar{f}_1)$ [2]	$P(f)$ [1]	ℓ [1]
0	1.33	2.03	0.48	0.009	33	-	0.002	21
5	1.26	1.77	0.70	0.02	37	0.018		
10	1.20	1.34	0.88	0.07	44	0.051	0.11	36
15	1.13	0.95	0.97	0.18	53	0.12		
20	1.08	0.55	0.99	0.34	66	0.20		
25	1.02	0.17	1.0	0.54	87	0.30	0.76	262
30	0.97	-.22	1.0	0.72	120	0.40		

[1]) according to Vanmarcke's theory

[2]) prob. of failure at q=10 if no failure at q=5 etc.

q in kN/m^2, ℓ in m.; $\mu_c=17.5$, $\sigma_c=2.62\ kN/m^2$; $d_h=45$ m., $d_v=3.0$ m. $L=1000$ m.

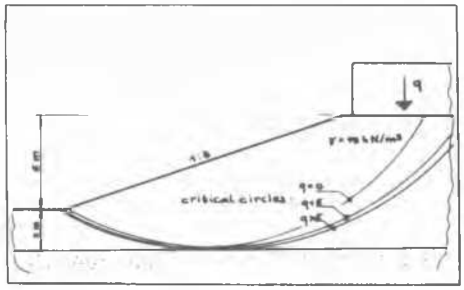

Figure 5. Cross sectional view of the analyzed slope

CONCLUSIONS

A probability based method for judgement of reliability of earth slopes has been presented. The method applies to long "uniform" slopes. Finite width failure modes are considered, and the effects of "end section" contributions to resistance against failure are taken into account. The method predicts the probability of occurrence of a failure somewhere along the slope axis, and the expected width of a failure mode. Furthermore, it is demonstrated that global information, obtained from observation of the slope's history of survival, can be formalized into a procedure for reduction of the probability of failure.

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