Probabilistic theory and kinematical element method

Théorie probabiliste et méthode des éléments kinématiques

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SYNOPSIS Probabilistic theory in combination with the Kinematical Element Method (KEM) seems to be an adequate method for a great number of different soil mechanics and foundation engineering problems. Applied to slope stability it can be shown that even with a highly complicated failure mechanism convergency can be achieved.

INTRODUCTION

For many slopes, dams or retaining structures and cut-off-walls the probabilistic analysis seems to become more and more an appropriate way of considering the problem rather than the deterministic one. The reason is that the mechanical behaviour of soil in terms of density, permeability, water content, deformation conditions and strength will be distributed in space and time within the earth structure always in a relatively random manner -inspite all site controls.

To quantify the reliability of a structure in a statistical way a mechanical model for different failure situations must be implemented into the probabilistic theory. This model should be flexible enough to represent variable geometry and strength behaviour and, on the other hand,simple enough to permit a great number of calculations. One of the objectives of this paper is to show that the Kinematical Element Method is such a model.

PROBABILISTIC THEORY

In recent years engineers tried to define the safety of a structure by means of a probabilistic concept, especially in the field of structural engineering.

The results of a probabilistic analysis are the probability of failure (p_f) of a system and a statement about the importance (α_i) of the involved parameters, the so-called basic variables ($\tilde{x}_i$) .The result reflects the safety including all possible combinations of material properties and loads. In traditional analysis this is only possible by means of a parameter variation. More recent approaches try to introduce the material behaviour and the behavour due the loads into the calculation by means of their distributions.

A measure for the factor of safety is -according to Hashofer/Lind (1974)- the shortest distance in the standardized numerical system from the origin to the generally curved failure surface. It is calculated by a suitable algorithm where the failure surface will be approximated by a tangent hyperplane in a common point.

This point of design (P*) has to be found by iteration.

The procedure is done as follows:

-determination of the basic variables and their distributions (($\tilde{x}_i$)-system),

-decoupling of the basic variables; non-Gauss-distributed variables have to be approximated by Gauss-distributions ((x_i)-system),

-transformation of the system into a standardized independent system ((u_i)-system),

-linearisation of the condition of failure $g(\mathbf{u})$ by a Taylor-expansion in the point of design (P*) = ($\mathbf{u}$*),

-interpretation of the expansion as a hyperplane; the absolute term of this tangential plane represents the shortest distance (β) which is denoted as safety index.

-iteration; the vector ($\mathbf{u}$) of the (i)th step is taken as the point of design for the (i+1)th step until the condition $\beta_{i+1} - \beta_i \leq \varepsilon$ is satisfied. Normally this method will converge rapidly (Formulas see appendix A1).

Into the probabilistic concept the KEM is introduced as a system state description. The object function (f) itself or the resulting force (RS) of the displaced element is taken as the system description. For RS > 0 the system is safe, for RS < 0 it is unsafe and if RS = 0, the critical state occurs Sturm/Mikota (1983).

KINEMATICAL ELEMENT METHOD

The KEM is an upper bound method in limit load theory; it consists of the following steps (Gussmann, 1982):

- design of an admissible and appropriate failure mechanism of rigid blocks;

- failure criteria (mostly Mohr-Coulomb) to be defined at the boundaries;

- mathematical description of the geometry and assignment of elements, boundaries and nodes (topology);

- kinematic analysis;

- static analysis;

- definition of the objective function to be minimized;

- variation of the geometry by means of optimization procedures.

For problems according to failure mechanisms like those of Fig. 1, the kinematics can be solved by a successive solution of two linear equations for two unknown displacements and the statics in a similar way for two unknown boundary forces for each element (see appendix A2).

To avoid internal iterations for a slope stability problem the objective function should be defined rather in terms of forces or energy than as a factor of safety.

The optimization procedure in finding the absolute minimum of the chosen function with due consideration of nonlinear inequality constraints (no tension forces, no overlapping of elements during the variation of geometry) is the main numerical problem of the method. This will be discussed later.

NUMERICAL TREATMENT

The combination of the probabilistic theory with minimizing the reliability-index related to the basic variables (i.e. shear parameters, weight, hydrostatic and static loads), and the KEM - with an internal minimization of the objective function (with respect to geometry)- would seem to be difficult to obtain convergency. But inspite of the coupled minimization procedures the difficulties are relevant only in the beginning; they reduce, however, notably within the neighbourhood of the minimum of β . The reasons are:

- the starting geometry for a set of fixed basic variables improves in the course of the mathematical procedure;

- the change of the basic variables is small close to the minimum of β.

These positive effects will only hold, if the starting geometry is really adequate to the first set of basic variables, this being the truly hard point of the method. It is felt by the authors that this problem should not be solved by means of highly sophisticated optimization procedures only, but should be supported by more physically based considerations (for details see Gussmann, 1984):

- adequate failure geometry according to experimental tests;

- successive refinement of the internal failure mechanism -for a set of fixed basic variables- starting with a very simple mechanism consisting a few elements only;

- grafically supported interactive optimization procedure;

- various optimization levels, starting with a robust and simple (but mostly slow) procedure without any use of derivatives, and ending with a highly sophisticated optimization procedure (Davidon/Nazareth, 1977).

The inequality constraints problem should also not be treated by mathematically means only (penalty function) but it should be searched for a starting geometry without active constraints. The authors found out, that in this case the constraints, which appear in the course of the optimization, will only be small and would normally disappear without further action.

It should also noted, that the combined problem can be solved even on a micro-computer within reasonable time.

APPLICATION TO SLOPE STABILITY

The applicability of the combined proceeding may be illustrated with an example of homogeneous slope (slope angle 33.4°;height 13.2 m; strength parameters φ and c).The unit weight γ = 20 kN/m³ is introduced as fixed datum. The slope is subdevided into 14 elements. Fig 1.a shows the initial geometry, Fig. 1.b the output geometry of the last iteration and Fig. 1.c the corresponding field of displacements. The results with 3 calculations for average mean values $\varphi = \bar{x}_1$; $c = \bar{x}_2$ and standard deviations σ_1 and σ_2 are as follows:

1. Example

Input: Gauss N- distribution for x_1, x_2

$\bar{x}_1 = 19.5°$; $\sigma_1 = 4.6°$

$\bar{x}_2 = 10.0$ kN/m², $\sigma_2 = 3.0$ kN/m²

$\gamma = 20.0$ kN/m³

Output: 1. Iteration: RS(1) = 4.5 kN/m

7. (= last) Iteration:

RS(7) = .000002 kN/m

$\alpha_1 = 0.856$; $\alpha_2 = 0.517$

$\beta = 0.05$; $p_f = 0.50$

($F_s = 1.0$ according to Taylor)

The result can be interpreted as follows:

An operative probability of failure of p_f =50% was finally obtained. This result may be compared to conventional slope stability analysis -see e.g. Taylor (1948)- which would yield a factor of safety F_s = 1 with the aforementioned mean values of φ and c.

The α-values indicate the probability of failure to be more influenced by deviations of φ than of c.

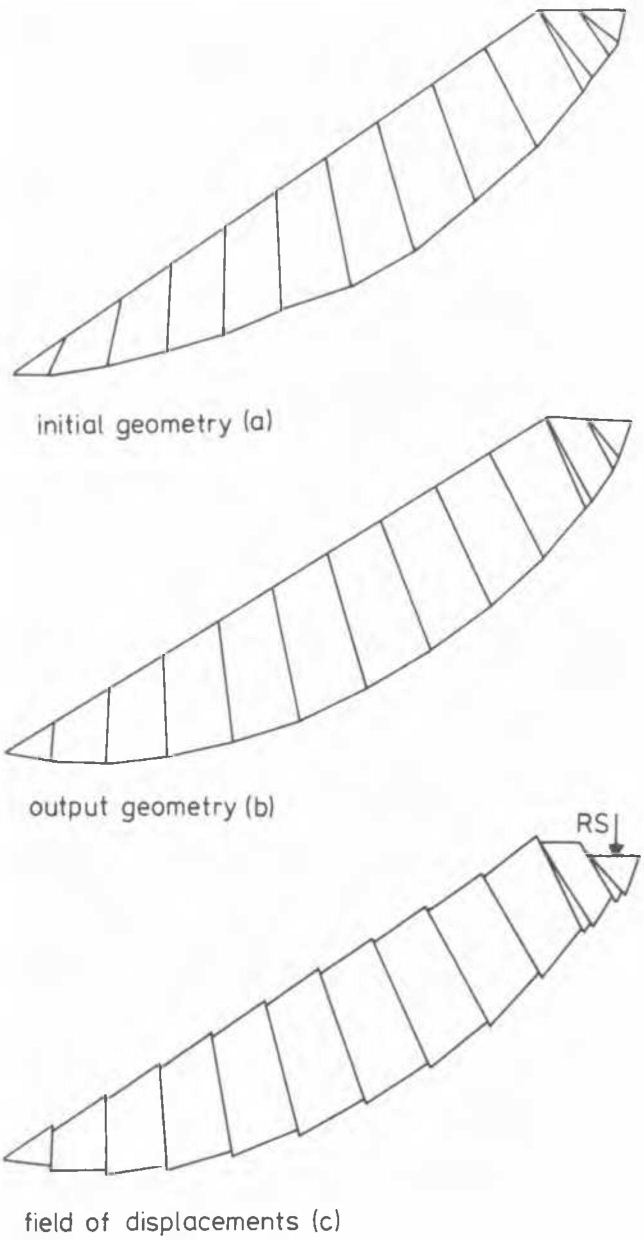

Figure 1

2. Example

Input: Gauss N- distribution for x_1, x_2

$\bar{x}_1 = 21.0°$; $\sigma_1 = 6.3°$

$\bar{x}_2 = 10.0$ kN/m², $\sigma_2 = 3.0$ kN/m²

$\gamma = 20.0$ kN/m³

Output: 1. Iteration: RS(1) = 309 kN/m

10. (= last) Iteration:

RS(10) = 0.00000007 kN/m

$\alpha_1 = 0.908$; $\alpha_2 = 0.418$

$\beta = 1.033$; $p_f = 0.15$

($F_s = 1.1$ according to Taylor)

3. Example

Input: Gauss log N- distribution for x_1, x_2

$\bar{x}_1 = 27.0°$; $\sigma_1 = 2.9°$

$\bar{x}_2 = 25.0$ kN/m², $\sigma_2 = 2.0$ kN/m²

$\gamma = 20.0$ kN/m³

Output: 1. Iteration: RS(1) = 1132 kN/m

7. (= last) Iteration:

RS(7) = 0.00012 kN/m

$\alpha_1 = 0.761$; $\alpha_2 = 0.648$

$\beta = 5.24$; $P_f = 10^{-7}$

($F_s = 1.8$ according to Taylor)

CONCLUSIONS

It could be shown for a particular slope stability problem -which is assumed to be typical for the statistically distributed strength-behaviour of soil structures- that the combination of two different theories with minimizing aspects can be solved by a staggered iterative procedure.The result is the probability of failure, which is a different but more appropriate measure of the reliability of a structure than the conventional factor of safety if a statistically sufficient amount of input data is available.

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APPENDIX A1

PROBABILISTIC concept

Independend basic variables (x_i)-system

$$(\mathbf{x}) = (x_1, x_2, \dots x_n)$$

Mean values of $(\mathbf{x})$

$$(\bar{\mathbf{x}}) = (\bar{x}_1, \bar{x}_2, \dots \bar{x}_n)$$

Standard deviations of $(\mathbf{x})$

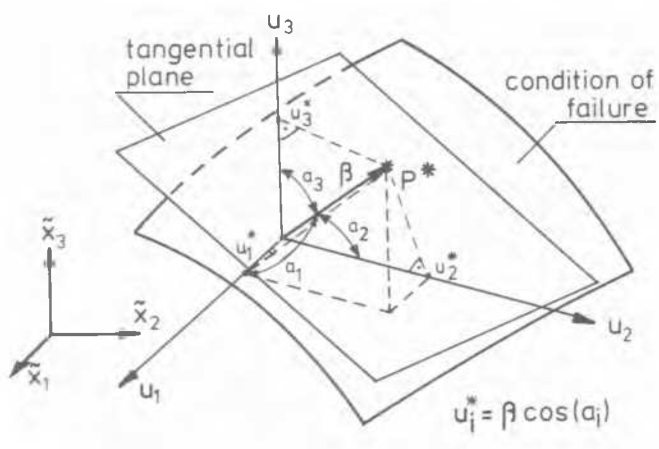
$$(\boldsymbol{\sigma}) = (\sigma_1, \sigma_2, \dots \sigma_n)$$

Transformation to (u_i)-system

$$u_i = \frac{x_i - \bar{x}_i}{\sigma_i} \longrightarrow (\mathbf{u})$$

Transformed state function

$$g(\mathbf{u}) = g(u_1, u_2, \dots u_n)$$



Taylor expansion in the point P*

$$g(\mathbf{u}) = g(\mathbf{u}^*) + \sum_{i=1}^{n} \left.\frac{\partial g(\mathbf{u})}{\partial(u_i)}\right|_{P^*} (u_i - u_i^*)$$

$$g(\mathbf{u}) = g(\mathbf{u}^*) + \sum_{i=1}^{n} \left.\frac{\partial g(\mathbf{u})}{\partial(u_i)}\right|_{P^*} (u_i) - \sum_{i=1}^{n} \left.\frac{\partial g(\mathbf{u})}{\partial(u_i)}\right|_{P^*} (u_i^*)$$

Hyperplane

$$\sum_{i=1}^{n} - (\alpha_i\, u_i) - \beta = 0$$

it is

$$\alpha_i = \frac{\left.\dfrac{\partial g(\mathbf{u})}{\partial(u_i)}\right|_{P^*}}{\left[\sum_{i=1}^{n} \left(\left.\dfrac{\partial g(\mathbf{u})}{\partial(u_i)}\right|_{P^*}\right)^2\right]^{1/2}}$$

$$\beta = \frac{\left[-\sum_{i=1}^{n} \left.\dfrac{\partial g(\mathbf{u})}{\partial(u_i)}\right|_{P^*} u_i^*\right] + g(\mathbf{u}^*)}{\left[\sum_{i=1}^{n} \left(\left.\dfrac{\partial g(\mathbf{u})}{\partial(u_i)}\right|_{P^*}\right)^2\right]^{1/2}}$$

$$u_i^* = -\alpha_i\, \beta$$

because of

$$\left.\frac{\partial g(\mathbf{u})}{\partial(u_i)}\right|_{P^*} = \left.\frac{\partial g(\mathbf{u})}{\partial(u_i)}\right|_{P^*} \sigma_i$$

the calculation is done in the (x_i)-system. At the end the probability of failure is evaluated by:

$$p_f = \Phi(-\beta)$$

APPENDIX A2

KINEMATICAL ELEMENT METHOD

Geometry

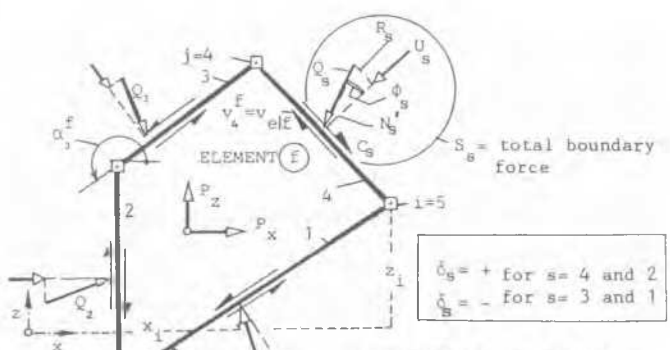
$$x_{j,i} = x_j - x_i = -x_{i,j} \; ; \; z_{j,i} = z_j - z_i = -z_{i,j}$$

$$l_{j,i} = \sqrt{(x_{j,i}^2 + z_{j,i}^2)} = l_{i,j}$$

$$\sin\alpha_{i,j} = z_{j,i}/l_{i,j}$$

$$\cos\alpha_{i,j} = x_{j,i}/l_{i,j}$$

$$\alpha_{i,j} = \alpha_s^f = \alpha_{e|f}$$



KINEMATICAL ELEMENT

Kinematics

$$v_{e|f,x} = v_{e|f}\cos\alpha_{e|f} = v_x^f - v_x^e = -v_{e|f,x}$$

$$v_{e|f,z} = v_{e|f}\sin\alpha_{e|f} = v_z^f - v_z^e = -v_{e|f,z}$$

$$v_{e|f} = v_s^f = v_s^e = v_{f|e}$$

$$\delta_s^f = \delta_s^e = \delta_s = \mathrm{sign}(v_s)$$

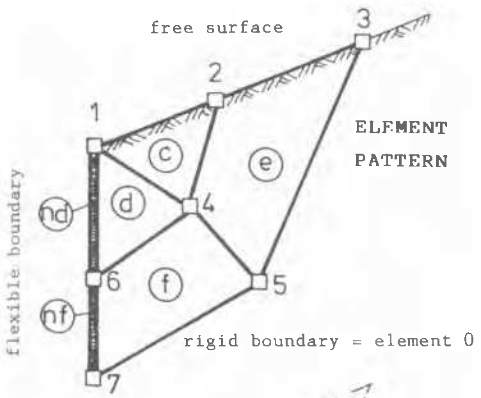
Successive linear equations

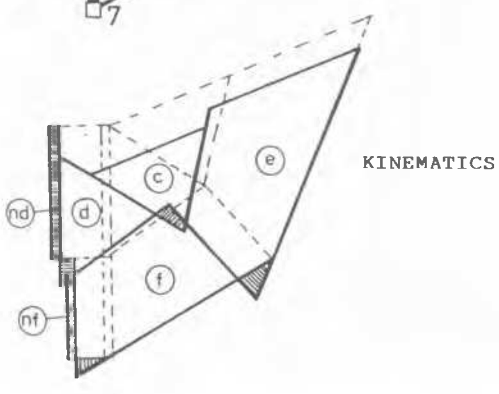
$$\mathbf{k}_v \; \mathbf{v} = \mathbf{b}$$

$$\mathbf{k}_v = \begin{bmatrix} k_{1,x} & k_{2,x} \\ k_{1,z} & k_{2,z} \end{bmatrix} \; ; \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \; ; \quad \mathbf{b} = \begin{bmatrix} b_x \\ b_z \end{bmatrix}$$

$$k_{1,x} = \cos\alpha_1 \; ; \; k_{2,x} = -\cos\alpha_2$$

$$k_{1,z} = \sin\alpha_1 \; ; \; k_{2,z} = -\sin\alpha_2$$





Statics

$$\varphi_s = \delta_s|\varphi_s| \; ; \; c_s = \delta_s|c_s|$$

$$\mathbf{k}\,\mathbf{q} + \mathbf{f} = \mathbf{0}$$

$$\mathbf{k} = \begin{bmatrix} k_{1,x} & k_{2,x} \\ k_{1,z} & k_{2,z} \end{bmatrix} \; ; \quad \mathbf{q} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} f_x = P_x + \gamma_x A^f - \sum_1^n s\,(c_s\cos\alpha_s + u_{m,s}\sin\alpha_s)\,l_s - \sum_1^n s\; k_{s,x} Q_s \\ f_z = P_z + \gamma_z A^f - \sum_1^n s\,(c_s\sin\alpha_s - u_{m,s}\cos\alpha_s)\,l_s - \sum_1^n s\; k_{s,z} Q_s \end{bmatrix}$$

$$k_{s,x} = -\sin(\alpha_s^f + \varphi_s)$$

$$k_{s,z} = \cos(\alpha_s^f + \varphi_s)$$

Objective function (alternative definitions)

$f = D - A$ (D = dissipative, A =kinetical work)

$f = \mathbf{v}^t\,\mathbf{S} = \Sigma\,(\hat{v}_x\bar{S}_x + \hat{v}_z\bar{S}_z)$; $\bar{S}$ = total boundary force on flexible boundary

$f = RS = \sqrt{(\bar{S}_x^2 + \bar{S}_z^2)}$

Optimization

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \lambda\,\mathbf{M}^k\,\mathbf{g}^k$$

$\mathbf{x}$ = vector of (kinematical) variables (x_i, z_i)

λ = parameter, $f(\mathbf{x}^{k+1}) < f(\mathbf{x}^k)$

$\mathbf{g}$ = gradient vevtor in $\mathbf{x}$

$\mathbf{M}$ = matrix, which approximates successively the inverse Hessian matrix of second derivatives. The authors proposal: $\mathbf{M}$ according to Davidon (1975).