Pore size distribution in filtration analyses

Répartition des vides dans les analyses de la filtration

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SYNOPSIS - With reference to the analysis of filter stability, a theoretical model based o information theory to obtain the pore size distribution of a particulate medium is presented in th paper. The maximum probability of occurrence is provided to the distribution so determined b imposing to the particulate system a condition of maximum entropy, without the introduction o fictitious or oversimplified hypotheses on soil packing. The theoretical pore size distributio curve depends on porosity and, then, it allows to investigate the development of clogging process accounting for the whole range of filter density, from the loosest to the densest state. The physica significance of the piping ratio is explored and the known empirical limit value justified.

INTRODUCTION

Two phenomena can usually reveal the loss of filters efficiency. The first, known as "clogging", refers to the action of retaining fine base particles up to complete filling of filter pores, thus impairing its drainage function to an unacceptable degree. On the other hand, filtering action fails if grains, eroded from the base material at the boundary with filter, are transported by seeping water, thus initiating backward erosion.
In both cases, the physical background of filter stability is to be looked for in the aptitude of filter voids to catch the smaller base particles within a certain length of penetration. The latter check, in turn, calls for the analysis of the diffusion process of particles of some assigned size through voids of aggregates of coarser particles.
Though unsolvable in a deterministic way, the problem is still approachable by considering the geometric hindrance of the movement of base soil grains through voids of the filter (Silveira, 1965).
The methods based on this principle aim mainly at determining the filter pore size distribution by assuming some ideal arrangement of particles and calculating the frequency of pores formed by specified groups of particles. The maximum length of the probable path within the filter of an assigned base particle can then be calculated for a given confidence level.
However, as pointed out by De Mello (1977) and later by Wittmann (1979), these methods suffer of limitations, some of which still unsolved. Among them, the distribution of pores accounting for the true packing of grains needs further research effort. As the maximum density is often assumed in computations, in fact, the existing methods reveal scarce flexibility in accounting for the actual density of filter materials.
In the paper, authors propose a theoretical model based on information theory arguments to find the pore size distribution of particulate systems, taking in due account the porosity.
The proposed approach offers the possibility of obtaining in a straightforward manner the void distribution curve in percentage of the number of pores, thus eliminating the fallacy hidden in the fact that pore size curves constructed on the basis of grain distributions by weight overlook the large number of small particles, which actually are available to build up denser arrangements (De Mello, 1977).
Finally, it is discussed whether and up to what extent the proposed computation model could be used to investigate the physical significance of empirical filter criteria that refer to some values of grain diameters, both of base and filter materials, corresponding to a few percentages by weight of granulometric fractions.

FUNDAMENTALS OF THE THEORETICAL MODEL

The volume of voids contained in a particulate material is usually expressed by the porosity or the void ratio. Even with a constant value of void ratio, however, different arrangements of particles do occur, corresponding to different pore size distributions.
Of course, a "local" description of soil structure, which takes into account location, average size, and shape of all voids is unfeasible; then, to face the problem, it is necessary to adopt some macroscopic variables, that do not

imply special hypotheses on soil particles packing but can be related to possible states or configurations of the discrete mass (Gudehus, 1968), (Mogami, 1969).
Owing to extremely high number of particles that are to be considered in the construction of a significant model of the porous medium, the study can be carried out by means of statistical mechanics methods, introducing the concept of "state probability".
In order to apply this line of thought, consider a volume frequency distribution of voids belonging to a particulate medium and let n, V_{min}, V_{max}, be respectively, the porosity of the system, and the minimum and maximum value of the volumes of pores.
The entire interval from V_{min} to V_{max} is divided into Q sub-intervals, each containing N_j pores of volume V_j and of associated frequency f_j defined as:

$$f_j = N_j / N_V \qquad (1)$$

where N_V is the overall number of pores.
It is worth observing that, if the population composing the system is sufficiently numerous, as it is the case, the number N_V of pores can be replaced by the number N_p of particles; moreover, the sum of all volumes $N_j \cdot V_j$ equals the whole volume of pores, the latter depending upon the porosity **n** of the medium.
The distribution of numbers N_j among the Q sub-intervals V_j must then obey the following "compatibility" conditions:

$$\sum_{j=1}^{Q} N_j = N_p \qquad (2')$$

$$\sum_{j=1}^{Q} N_j V_j = n V_{tot} \qquad (2'')$$

In (2"), V_{tot} represents the total volume (particles plus voids) of the system; while, in (2'), the term N_p can be expressed as a function of V_{tot}, if one observes that the following relation between the number N_i of particles of diameter D_i and the corresponding percentage by weight Δp_i holds:

$$N_i \frac{\pi D_i^3}{6} \gamma_s = (1-n) V_{tot} \gamma_s \Delta p_i \qquad (3)$$

in which γ_s is the specific gravity of grains.
From (3) one obtains:

$$N_i = \frac{6 V_{tot}}{\pi} (1-n) \frac{\Delta p_i}{D_i^3} \qquad (4)$$

And summing up over all the **N** diameters D_i:

$$N_p = \sum_{i=1}^{N} N_i = \frac{6 V_{tot}}{\pi} (1-n) \sum_{i=1}^{N} \frac{\Delta p_i}{D_i^3} \qquad (5)$$

Observe, now, that a given configuration (macro-state) of the system can be obtained in all the different ways (micro-states) in which the number N_j can be associated to volumes V_j. Recalling that there exist N_1 indistinguishable pores of volume V_1, N_2 of volume V_2, ... N_Q of volume V_Q, the number **W** of all possible micro-states can be obtained from combinatorial analysis as:

$$W = \frac{N_p!}{N_1!\, N_2! \ldots N_Q!} \qquad (6)$$

If the ways of forming a given configuration are all equally probable, as it can be supposed in absence of particular structural features of the porous medium, then the number **W** of micro-states leading to the considered configuration is proportional to the probability of occurrence of the configuration itself.
The value of such probability is not known in general; however, the degree of uncertainty about the knowledge concerning the state of the material depicted by the distribution $N_j(V_j)$ is expressed, according to the theory of information, as:

$$E = k \ln W \qquad (7)$$

where **K** is a constant.
The function **E** plays a role similar to the entropy in classical thermodynamics, hence it is called entropy also in the theory of information.
Looking for a stable configuration of the system, a distribution $N_j(V_j)$ which has the maximum probability of occurrence so as to supply the maximum uncertainty about the micro-states forming the configuration itself, has to be identified.
The latter requirement, in turn, is assured by imposing to the system the condition of maximum entropy.
One has then to find a distribution $N_j(V_j)$ which maximizes the function **E** expressed by (7) under the constraints dictated by (2') and (2").
This problem of conditional maximum has been solved by means of Lagrange's multipliers method. Without going further into mathematical details, the distribution of pores can be expressed as follows:

$$N_j = N_p \frac{e^{-\beta V_j}}{\sum_{j=1}^{Q} e^{-\beta V_j}} \qquad (8)$$

in which the well known Maxwell-Boltzmann

distribution law in ricognizable.
The frequency of voids can thus be cast in the form:

$$f_j = \frac{N_j}{N_p} = \frac{e^{-\beta V_j}}{\sum_{j=1}^{Q} e^{-\beta V_j}} \qquad (8')$$

In order to determine the parameter β, letting:

$$\delta(\beta) = \sum_{j=1}^{Q} e^{-\beta V_j} \qquad (9)$$

one can observe that:

$$-\sum_{j=1}^{Q} V_j e^{-\beta V_j} = \frac{\partial}{\partial \beta}\left[\sum_{j=1}^{Q} e^{-\beta V_j}\right] = \delta'(\beta) \qquad (10)$$

where $\delta'(\beta)$ indicates the derivative of the function $\delta(\beta)$ with respect to β.
On the other hand, replacing (8) in (2"), one also obtains:

$$\sum_{j=1}^{Q} V_j e^{-\beta V_j} = \frac{n V_{tot}}{N_p} \qquad (11)$$

Hence, equating (10) and (11), the following differential equation is obtained:

$$\frac{\delta'(\beta)}{\delta(\beta)} = -\frac{n V_{tot}}{N_p} \qquad (12)$$

The relation (12) holds true for whatever function $V_j(j)$;the latter, however, is conditioned by the necessity of describing the whole range of pore sizes.
After some trials, the following expression has been chosen:

$$V_j = V_{min} + \ln c^{(j-1)} \qquad (13)$$

where $c = \frac{1 + \Delta V}{\Delta V}$ and ΔV is a prescribed increment of pore volume.
It is easily verified that the adopted expression for V_j allows the calculation of $\delta(\beta)$ and $\delta'(\beta)$, so that replacing them in (12) the parameter β is finally obtained:

$$\beta = \frac{\ln \dfrac{V_{min} - \bar{V}}{V_{min} - \bar{V} + \ln c}}{\ln c} \qquad (14)$$

having indicated with $\bar{V}$ the term nV_{tot}/N_p.
One can observe that β is a function of porosity; moreover, it depends upon the grain size distribution of the medium through the number N_p of particles (see eq. (5)), and upon the range of pore volumes.

COMPARISON WITH EXPERIMENTAL DATA

Very few experimental results are available in technical literature on pore size distribution of granular media.
Some of them have been reported by Kezdi (1968) with reference to various materials, from natural pit gravel to quarried rock.
Test data are presented, in semi-logarithmic plot, as void volume distribution curves (fig. 1), and therefore they are particularly well suited for comparison with theoretical values predicted by the calculation model presented in the preceding section. Experimental results are

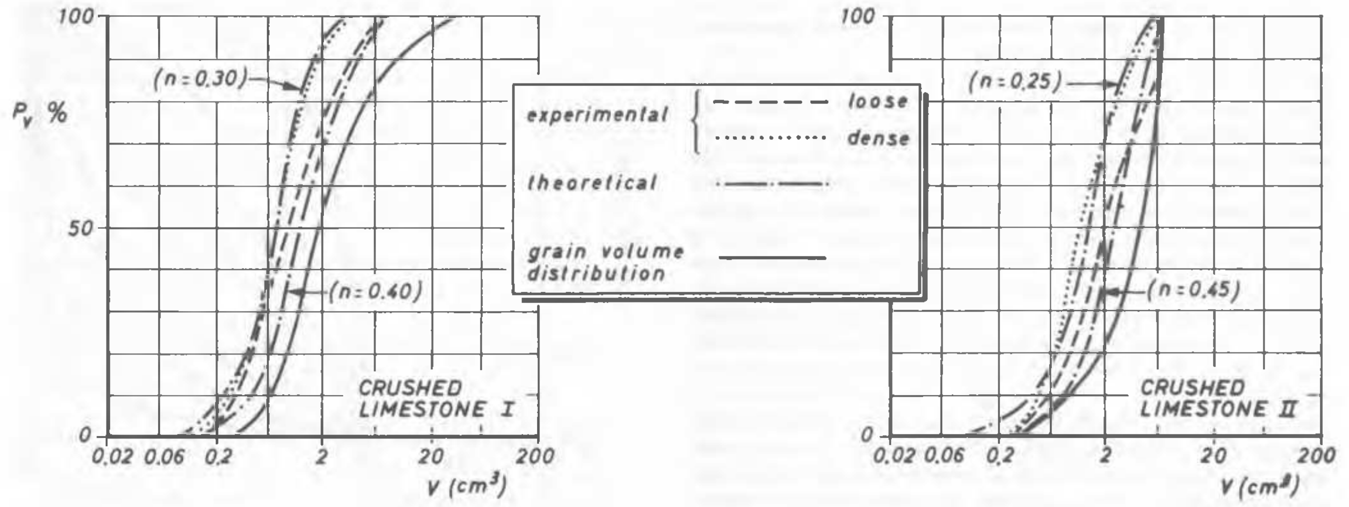

Fig. 1 - Theoretical and experimental volume distribution curves; solid particles and pores.

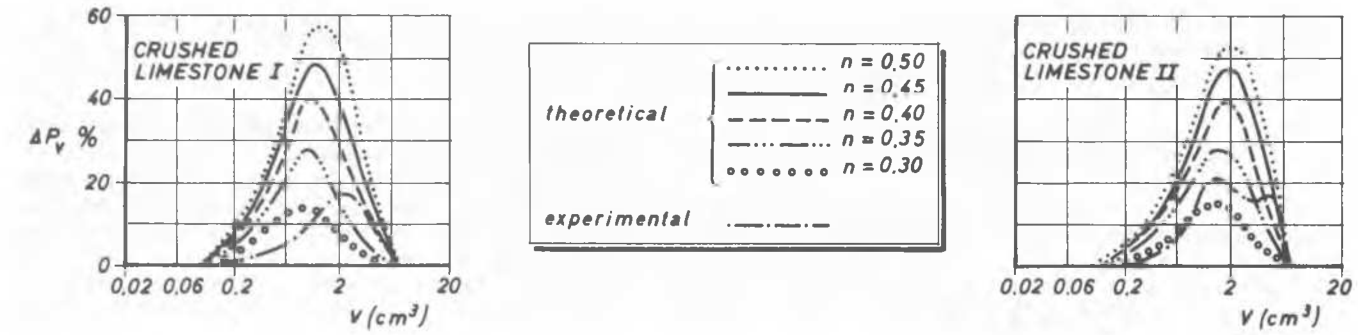

Fig. 2 - Difference among percentages of pore volumes at various porosities.

furnished for the dense and for the loose state as well, although nothing is reported in the original paper by Kezdi about the corresponding porosities.
In order to fit the experimental data by theoretical values, the relative frequency f_j has been transformed into the cumulative one p_V by means of the formula:

$$P_V = \frac{\sum_{i=1}^{j} V_i f_i}{\sum_{i=1}^{M} V_i f_i} \tag{15}$$

The theoretical curves, obtained by (8'), (14) and (15), that best fit the experimental data, are also reported in fig. 1. For sake of simplicity only two values of porosity have been considered; one can observe, however, that they are sufficiently representative of the states labelled, respectively, as dense and loose.
On the other hand, with reference to same materials, in fig. 2 are reported the curves representing the differences between the cumulative frequencies calculated for various porosities and the cumulative frequency relative to the porosity **n** = 0.25; in the same figure, curves representing the differences between experimental "dense" and "loose" curves reported in fig. 1, can be compared.
Theoretical curves are regular bell-shaped in the semilogarithmic plot; it can thus be anticipated that there is a range of pore sizes, intermediate between maximum and minimum, for which the maximum reduction occurs in the transition from the dense to the loose state. In fact, the volume of larger voids does not undergo substantial reduction because of the limited number of them; on the other hand, further reduction of smaller voids can hard occur. Experimental results confirm, at least qualitatively, the trend depicted by theoretical curves.
It is also interesting to note that with increasing porosity the influence of large pores on frequency distribution increases so that the peak of bell curves moves progressively toward the high values of pore sizes.

FILTRATION ANALYSIS

An analysis of particulate diffusion processes occurring at the boundary between base material and filter can be started from the obvious consideration that all base particles smaller than the smallest filter void will pass through the filter, while all particles larger than the largest void will not enter the filter; particles of intermediate size will penetrate to varying depth into the filter.
Once the pore size distribution of the filter is known, one has then to compute the probability that an assigned base particle be larger than a certain pore. From a different point of view, it can be questioned how many confrontations between the base particle size and the filter void sizes are needed to find a pore smaller than the grain under consideration.
The value **m** of these proofs in given by (Silveira, 1965):

$$m = \frac{\ln(1-P_0)}{\ln(1-P_V)} \tag{16}$$

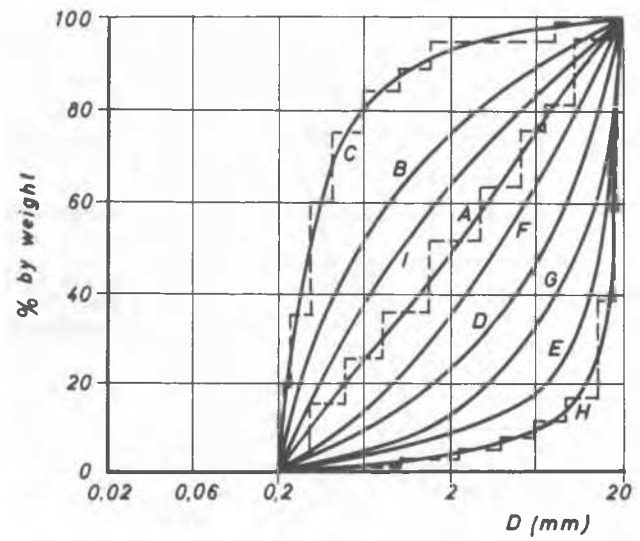

Fig. 3 - Grain size distribution curves.

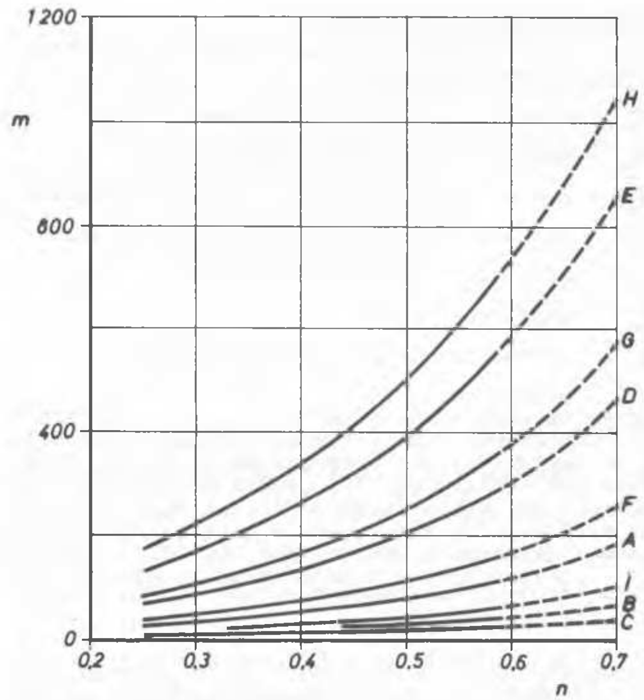

Fig. 4 - Number of confrontations versus porosity for various grain size distributions.

where p_V represents the percentage of pores smaller than the considered base particle and P_o is a confidence level.
The number m could further be transformed into the filtration length S if the average distance s of changing pore size were quantified (Wittmann, 1979). In this case, $S = m \cdot s$.
In order to examine the influence of porosity on the diffusion process, various filter materials, whose grain size distribution curves are reported in fig. 3, have been considered. The determination of number m of proofs, for each material, has been carried out with reference to only one base particle, whose volume has been assumed as equal to the smallest pore volume of the filter. The latter volume can be computed as 0,1712 D^3_{min}, where D_{min} denotes the minimum diameter of filter grains.
The results of computations, for a confidence level P_o = 0,9999 are reported in fig. 4, where the increase of number m with increasing porosity shows, at least indirectly, the modification of structural arrangement resulting from change of porosity. Then the question of the influence of porosity on filter efficiency arises.
As it is known, empirical criteria of filter design refer to some ratios determined by experiment. Widely used in practice, the "piping ratio" $r = F_{15}/B_{85}$ which, for the safety of the filter, must not exceed 5 ÷ 6.
A preliminary analysis of the physical significance of the piping ratio can be carried out if the size of the base particle considered in the above computations is now intended as just the B_{85} grain size.

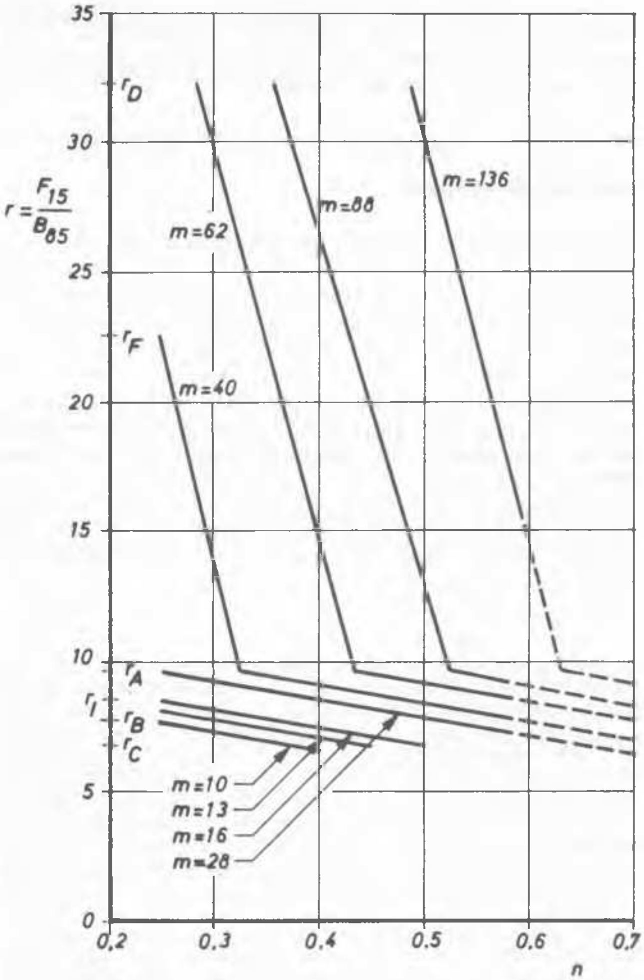

Fig. 5 - Piping ratio versus porosity for different numbers of confrontations.

In doing so, for each of filter materials reported in fig. 3, the r values can be calculated (fig. 5). Moreover, for each r value, the m values obtained for different porosities are reported in the graph. Linking the points at same m value, the curves drawn in fig. 5 are obtained.
It is easily observed that each curve shows a sudden discontinuity at a value $\bar{r}$ of the ratio r of the order of 10; for $r < \bar{r}$ (as for grain size curves labelled with A, B, C, I, in fig. 3) the number of confrontations required to arrest the considered base particle varies between 10 and 100, even for high values of porosity. Moreover, the variations of porosity do not cause large variations of the number m.
On the contrary, for $r > \bar{r}$, the higher is the piping ratio the higher becomes the number of confrontations and the more it is sensible to variations of porosity. It can thus be concluded that the existence of a critical value of piping ratio (10 in the present preliminary analysis) seems to be justified by the necessity of arresting the B_{85} particles within short distan-

ces, so that at the boundary between base material and filter a thin transition zone (sealing) composed of particles belonging to both materials can be formed.

CONCLUDING REMARKS

The fundamental principle governing the filtering action can be found in the geometric hindrance of the movement of particles of the base soil through voids of the filter. Consequently, the available theoretical formulations approach the problem by comparing the base soil particle sizes with sizes of filter pores; hence, the probable length of the path covered by an assigned base grain through filter pores can be calculated.
The method, however, suffers of limitations invalidating its safe applications to filter design. Among them, the packing model adopted in the computation of pore size distribution of the filter may be the source of significant errors.
To elude this shortcoming a theoretical model based on information theory has been built, avoiding the introduction of hypotheses on local arrangements of particles. In this way, the assembly of grains composing the porous medium can be characterized by its state probability; the resulting pore size distribution is known in Statistical Mechanics with reference to other physical problems. The model, however, still needs some refinements in order to be applicable to filter materials characterized by discontinuous grain size distribution.
The agreement of the theoretical results with some available experimental data confirms the suitability of the proposed approach, which offers the possibility of taking into account the porosity of the filter. A preliminary analysis of clogging process occurring in the filter shows that the known empirical piping ratio is susceptible of theoretical justification.

LIST OF MAIN SYMBOLS

B_{85} diameter corresponding to 85% by weight of the base material;

E entropy;

D_i diameter of filter grains;

D_{min} minimum diameter of filter grains;

f_j frequency of pores of volume V_j;

F_{15} diameter corresponding to 15% by weight of the filter material;

K constant of the entropy;

m number of confrontations between the base particle size and the filter void sizes;

n porosity of the filter;

N overall number of granulometric fractions of the filter;

N_i number of particles of diameter D_i;

N_j number of pores of volume V_j;

N_p overall number of particles of the filter;

N_v overall number of pores of the filter;

p_v cumulative frequency of pores;

P_o confidence level;

Q overall number of void fractions;

r piping ratio;

$\bar{r}$ critical value of the piping ratio;

s average distance of changing pore size;

S filtration length;

V_j volume of filter voids;

V_{min}, V_{max} minimum and maximum values of the volumes of pores;

V_{tot} total volume of the filter;

W number of micro-states corresponding to a given configuration;

Δp_i percentage by weight of particles of diameter D_i;

ΔV increment of pore volume.

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