Steady state waves in a porous elastic medium

Ondes stationnaires dans un milieu poreux élastique

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SYNOPSIS Assuming the soil composed of two distinct phases - granular structure and water - the theory of mixtures, an extension of the continuum mechanics to multiphase media, has been employed to derive the elastic wave equations in conditions of plane deformation and one-dimensional propagation. The longitudinal waves propagation equation is applied to determine the steady state response of a soil layer excited by cyclic stress or strain at the boundaries. The results, presented in dimensionless form, allow the evaluation of the amplitude and the phase angle of the displacements at the top of the layer as functions of the frequence and amplitude of the excitation, of the thickness of the layer and of the compressibility and permeability of the soil.

INTRODUCTION

In geotechnical engineering practice wave propagation is normally studied assuming soil as a single-phase medium. However, soil is a more complex material, composed of three distinct phases - granular structure, water and air. Its mechanical behaviour is controlled by the load carried by the soil structure, the rest being carried by the fluid in the voids. Hence, to predict the real behaviour of soil it is necessary to take into account the load distribution among the different phases.
The theory of mixtures seems to be a useful tool to investigate soil response in dynamic conditions. The general theoretical formulation (Truesdell and Toupin, 1960) is an extension of the continuum mechanics to multiphase media and it has been employed to derive the fundamental equations of dynamically interacting continua (Green and Naghdi, 1965; Green and Steel, 1966). Several studies have been carried on this subject and extensive reviews (Bowen, 1975) and applications to the soil (Prevost, 1980) are available.
In the theory of mixtures soil is considered constituted by superimposed interacting continuous media. Each phase is assumed to occupy simultaneously the volume as a whole.
According to this formulation the elastic waves equations in conditions of plane deformations and one-dimensional propagation have been formerly derived by the A. (Mongiovì, 1983).
In the *present paper* the *longitudinal* waves propagation equation is applied to determine the response of a layer of soil excited by cyclic stress or strain at the boundary in steady state conditions.
A layer submitted to a sinusoidal variation of base displacement or of load at the top is examined. The results allow the evaluation of the amplitude and phase angle of the displacements at the upper surface of the layer as functions of the frequence and amplitude of the excitation, of the thickness of the layer and of the compressibility and permeability of the soil.

FORMULATION OF THE PROBLEM

In the theory, assuming the interparticle voids completely filled by water, soil is considered a mixture of two constituents, the solid and liquid phases.
After a suitable definition of kinematics and of stress variables, the motion, continuity and constitutive equations are written for each constituent as for a single phase continuous medium. A term is added to the motion equations to take into account the interaction between the two phases.
In a previous report (Mongiovì, 1983) the general treatment of one-dimensional elastic waves propagation in plane strain conditions was presented. It was recognized that, if the solid skeleton is assumed to be linearly elastic and the pore fluid incompressible, the longitudinal and transversal waves are uncoupled. Furthermore, it was shown that the transversal waves propagate without relative movement between the two phases in the same way as a single-phase medium.
For this reason the study of elastic wave propagation is restricted to the longitudinal

waves.
Referring to the above-mentioned report, the equation of longitudinal wave propagation along vertical z axis directed downwards is the following

$$\frac{1}{c_p}\frac{\partial^3 w}{\partial t^2 \partial z}+\frac{1}{c_v}\frac{\partial^2 w}{\partial t\,\partial z}=\frac{\partial^3 w}{\partial z^3} \qquad (1)$$

where w is the displacement of the solid phase in z direction at time t. The propagation velocity c_p and the consolidation coefficient c_v are given by

$$c_p=\left\{\left(\rho+\frac{1-2n}{n}\rho_w\right)m_v\right\}^{-1/2} \qquad (2a)$$

$$c_v=\frac{k}{\gamma_w m_v} \qquad (2b)$$

in which ρ and ρ_w are the soil and water densities, γ_w is the unit weight of water, n is the porosity and k and m_v indicate the permeability coefficient and the volumetric compressibility.
The solution of two different cases of longitudinal waves propagation along vertical direction in a horizontal homogeneous layer of constant thickness H is presented here.
The examined cases refer to two typical situations.
In the first case the upper surface (z=0) of the layer is assumed to be free of stress and the lower surface (z=H) to oscillate with sinusoidal vertical displacements:

$$\sigma'(0,t)=0 \qquad (3a)$$

$$w(H,t)=w_b \operatorname{sen}\omega t \qquad (3b)$$

where σ' is the vertical effective stress.
In the second case the layer is supposed to be resting on a fixed base and the upper surface to be submitted to normal effective stresses varying with sinusoidal law:

$$\sigma'(0,t)=\sigma_t' \operatorname{sen}\omega t \qquad (4a)$$

$$w(H,t)=0 \qquad (4b)$$

The soil layer submitted to sinusoidal variations of displacements or stresses at the boundary is subjected to forced vibrations reaching a steady state condition. In this situation the waves become stationary, i.e. their amplitudes and phase angles are variable along z direction but constant in time t.

SOLUTION OF THE PROBLEM

For steady-state vibration the solution of equation (1) is (Bisshopp, 1959):

$$w(z,t)=u(z)\cos\omega t+v(z)\operatorname{sen}\omega t \qquad (5)$$

By substituting equation (5) into equation (1) two fifth order linear homogeneous differential equations for u(z) and v(z) are obtained, the solutions of which are:

$$u(z)=c_1\cosh az\cos bz+c_2\operatorname{senh}az\cos bz+c_3\operatorname{senh}az\operatorname{sen}bz+c_4\cosh az\operatorname{sen}bz \qquad (6a)$$

$$v(z)=c_3\cosh az\cos bz+c_4\operatorname{senh}az\cos bz-c_1\operatorname{senh}az\operatorname{sen}bz-c_2\cosh az\operatorname{sen}bz \qquad (6b)$$

where:

$$a^2=\frac{\omega^2}{2c_p^2}\left\{\left(1+\frac{c_p^4}{\omega^2 c_v^2}\right)^{1/2}-1\right\} \qquad (7a)$$

$$b^2=\frac{\omega^2}{2c_p^2}\left\{\left(1+\frac{c_p^4}{\omega^2 c_v^2}\right)^{1/2}+1\right\} \qquad (7b)$$

Sinusoidal displacements at the base of the soil layer

The upper boundary condition (3a) is:

$$w'(0,t)=u'(0)\cos\omega t+v'(0)\operatorname{sen}\omega t=0 \qquad (8)$$

in which the primes denote differentiation with respect to z.
Deriving equations (6) and imposing u'(0)=v'(0)=0, $c_2=c_4=0$ is obtained.
The lower boundary condition (3b) is:

$$w(H,t)=u(H)\cos\omega t+v(H)\operatorname{sen}\omega t=w_b\operatorname{sen}\omega t \qquad (9)$$

Then u(H)=0 and v(H)=w_b; hence:

$$c_1=-w_b\frac{\operatorname{senh}aH\operatorname{sen}bH}{\operatorname{senh}^2 aH\operatorname{sen}^2 bH+\cosh^2 aH\cos^2 bH} \qquad (10a)$$

$$c_2=w_b\frac{\cosh aH\cos bH}{\operatorname{senh}^2 aH\operatorname{sen}^2 bH+\cosh^2 aH\cos^2 bH} \qquad (10b)$$

The vertical displacement at the top of the layer is:

$$w(0,t)=c_1\cos\omega t+c_3\operatorname{sen}\omega t=w_t\operatorname{sen}(\omega t-\varphi_t) \qquad (11)$$

where the vibration amplitude w_t and the phase angle φ_t are:

$$w_t=(c_1^2+c_3^2)^{1/2} \qquad (12a)$$

$$\varphi_t=\tan^{-1}(-c_1/c_3) \qquad (12b)$$

The resonance factor R_t is defined as the ratio of the amplitude w_t at the top to the vibration amplitude w_b at the base. From equations (10) and (12a):

$$R_t = w_t / w_b = \left\{ 2/(\cosh 2aH + \cos 2bH) \right\}^{1/2} \qquad (13)$$

The phase angle between the vibrations at the top and the base of the soil layer is similarly deduced by substituting equation (10) into (12b):

$$\varphi_t = \tan^{-1}(\tanh aH \ \tan bH) \qquad (14)$$

Sinusoidal stresses at the top of the soil layer

In this case the boundary conditions are expressed by equations (4). At the top of the layer equation (4a) leads to:

$$w'(0,t) = u'(0) \cos \omega t + v'(0) \operatorname{sen} \omega t =$$

$$= m_v \sigma'(0,t) = m_v \sigma_t' \operatorname{sen} \omega t \qquad (15)$$

Hence, u'(0)=0 and $v'(0)=m_v\sigma'_t$. By performing the differentiation of equations (6), c_2 and c_4 are derived:

$$c_2 = -m_v \sigma_t' \frac{b}{a^2+b^2} \qquad (16a)$$

$$c_4 = m_v \sigma_t' \frac{a}{a^2+b^2} \qquad (16b)$$

The base layer is fixed. Thus:

$$w(H,t) = u(H) \cos \omega t + v(H) \operatorname{sen} \omega t = 0 \qquad (17)$$

From condition u(H)=v(H)=0

$$c_1 = \frac{-c_2 \operatorname{senh} aH \cosh aH - c_4 \operatorname{sen} bH \cos bH}{\cosh^2 aH \cos^2 bH + \operatorname{senh}^2 aH \operatorname{sen}^2 bH} \qquad (18a)$$

$$c_3 = \frac{c_2 \operatorname{sen} bH \cos bH - c_4 \operatorname{senh} aH \cosh aH}{\cosh^2 aH \cos^2 bH + \operatorname{senh}^2 aH \operatorname{sen}^2 bH} \qquad (18b)$$

are obtained.
The displacement amplitude w_o at the top in the absence of inertial forces is:

$$w_o = m_v \sigma_t' H \qquad (19)$$

Defining the resonance factor R_t as the ratio of the amplitude w_t at the upper surface to w_o and using equations (12), (16), (18) and (19), the expressions of R_t and the phase angle φ_t are deduced:

$$R_t = \frac{w_t}{w_o} = \frac{1}{H(a^2+b^2)^{1/2}} \; \frac{\left\{(\cosh 4aH - \cos 4bH)/2\right\}^{1/2}}{\cosh 2aH + \cos 2bH} \qquad (20)$$

$$\varphi_t = \tan^{-1}\left(\frac{b \tanh aH - a \tan bH}{a \tanh aH + b \tan bH}\right) \qquad (21)$$

DISCUSSION OF THE RESULTS

Wave damping is inversely proportional to the consolidation coefficient and in the particular case of no damping ($c_v \to \infty$) the coefficients a and b (7) reduce to:

$$a = 0 \qquad (22a)$$

$$b = \omega / c_p \qquad (22b)$$

In the case of cyclic displacement at the layer base the resonance factor R_t (13) and the phase angle φ_t (14) become:

$$R_t = |\cos bH|^{-1} \qquad (23)$$

$$\varphi_t = (i-1)\pi \qquad i = 1, 2, \ldots \qquad (24)$$

In this condition resonance ($R_t \to \infty$) occurs when:

$$\omega = (i - \tfrac{1}{2})\pi \frac{c_p}{H} \qquad i = 1, 2, \ldots \qquad (25)$$

In the case of cyclic effective stress at the top, the factor R_t (20) and the phase angle reduce to:

$$R_t = (bH)^{-1} |\tan bH| \qquad (26)$$

$$\varphi_t = (i-1)\pi \qquad i = 1, 2, \ldots \qquad (27)$$

nd the resonance circular frequency is the same as in the previous case (25).
The circular frequency of the first mode of vibration (i=1)

$$\omega = \frac{\pi}{2} \frac{c_p}{H} \qquad (28)$$

is used to obtain the solution of the problem in dimensionless form.
The coefficients a and b are rewritten:

$$a^2 = \frac{\pi^2}{2^3} \frac{\omega_o^2}{H^2} \left\{ \left(1 + 2^4 \frac{k_o^2}{\omega_o^2}\right)^{1/2} - 1 \right\} \qquad (29a)$$

$$b^2 = \frac{\pi^2}{2^3} \frac{\omega_o^2}{H^2} \left\{ \left(1 + 2^4 \frac{k_o^2}{\omega_o^2}\right)^{1/2} + 1 \right\} \qquad (29b)$$

where $\omega_o = \omega/\omega_n$ and k_o is the damping factor defined as the damping coefficient referred to the critical damping:

$$k_o = \frac{1}{2\pi} \frac{c_p H}{c_v} \qquad (30)$$

The variations of the resonance factor R_t and of the phase angle φ_t with the dimensionless frequency ω_o for different damping factors k_o are presented in Figs 1 and 2 for the first case and in Figs 3 and 4 for the second case.
In Figs 1 and 3 the resonance peak values of R_t at the odd whole numbers of ω_o decrease as k_o increases and vanish for $k_o>1$, when damping is greater than critical. At the even whole numbers

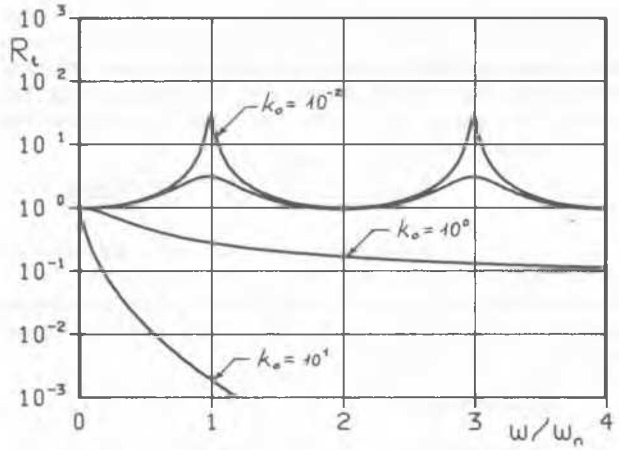

Fig. 1 - Resonance factor versus frequency in the case of cyclic displacement at the base of the layer.

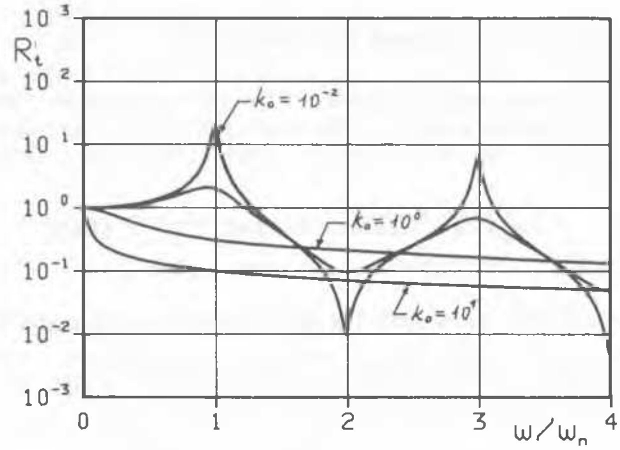

Fig. 3 - Resonance factor versus frequency in the case of cyclic stress at the top of the layer.

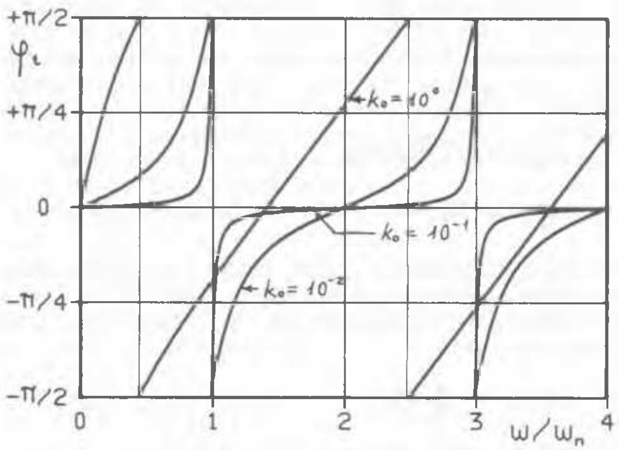

Fig. 2 - Phase angle versus frequency in the case of cyclic displacement at the base layer.

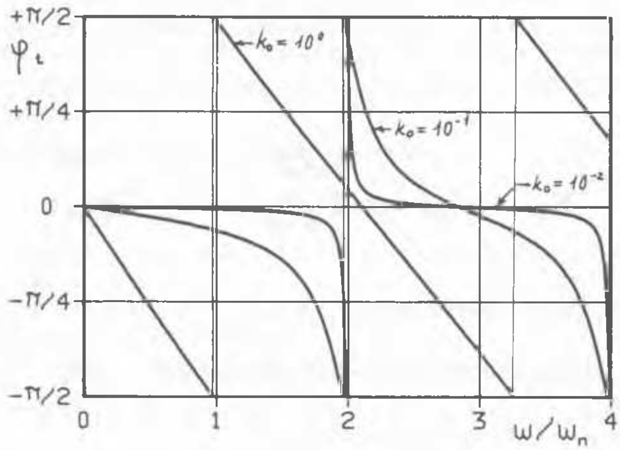

Fig. 4 - Phase angle versus frequency in the case of cyclic stress at the top of the layer.

of ω_o, in the first case (Fig 1) the resonance factor R_t has unit value in the condition of no damping and decreases as k_o increases, whereas in the second case (Fig 3) R_t has null value for k_o=0 and increases with the damping factor. In both cases the vibration amplitude decreases with increasing frequency owing to damping.

The phase angle φ_t (Figs 2 and 4) between the imposed vibration and the response at the top of the layer increases with damping.

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