Theory of Non-Linear Seepage

Théorie de la Percolation Non-Linéaire

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SYNOPSIS

The theory of seepage must be based on the system soil-water-air. It should include relevant phenomena of each of these phases. However, the physical properties of this system are usually restricted to either saturated deformable or to unsaturated undeformable porous media. In this paper an attempt is made to extend the contemporary theory to pore fluid flow in a semi-saturated deformable medium including entrapped air bubbles. Variations of the permeability and the viscosity due to pore pressure fluctuations are taken into account.

The flow process considered can be formulated in a quite simple form revealing similarity with the familiar potential equation but in terms of the exponential of the flow potential multiplied by the so-called coefficient of non-linearity, which contains all the non-linear effects concerned. This coefficient provides a convenient mean to model porous flow problems in a realistic fashion and it permits to directly evaluate the practical significance of non-linear seepage.

INTRODUCTION

The subject of this paper is the non-linear aspects of ground water flow. Considering the transport process of a pore fluid through a semi-saturated deformable porous medium one has to distinguish what flows through what. This process involves three phases: water, air and the soil skeleton. Air might be entrapped in the porous medium blocking pores, or it is conveyed as micro-bubbles by the pore water, which itself flows through a deforming (= moving) porous medium. The rather complex formulation of this flow process is based on a constitutive equation and a conservation principle for each phase, after Verruijt (1969). It can be simplified by the introduction of a mixture density for the pore water including entrapped air.

CONSTITUTIVE RELATIONS FOR RELEVANT QUANTITIES

The average density of an air-water mixture, including entrapped air pockets, air bubbles and dissolved air, is defined according to:

$$\rho' = s\rho + (1-s)\rho'' + s\omega\rho''.^{©} \qquad (1)$$

Here, s represents the saturation degree of the pore content, ρ is the water density, ρ'' the air density and ω is the air-water solubility coefficient. The water compressibility β due to the pore pressure p, defined according to:

$$\beta = d\rho/\rho dp, \qquad (2)$$

suggests to assume a similar property for the mixture:

$$\beta' = d\rho'/\rho' dp. \qquad (3)$$

Elaboration in terms of principal material properties employing Henry's solubility law, Dalton's partial pressure law, the Boyle-Mariotte law for iso-thermal conditions, and Kelvin's surface tension law leads to:

$$\beta' = \beta + \frac{(1/(1-\omega) - s)/s}{p - p_v + \frac{2\sigma}{3r}\{2 - \frac{b + \omega(1-b)}{(1-\omega)(1-s-b)}\}},$$

where p_v is the vapour pressure, σ the surface tension, r the effective bubble radius, and b the relative bulk volume of stagnant air.
At a certain pressure the surface tension term becomes negative (ω and b are rather constant), implying instability of free air bubbles.
At this particular point the mixture compressibility β' is quite a constant. Thus, requiring the term in braces to be zero, i.e.

$$s = 1 - \omega/2(1-\omega) - 3b/2,$$

results with $\omega \ll 1$ into:

$$\beta' \cong 2\omega/p_c \quad \text{for} \quad p < p_c;$$
$$\beta' = \beta \quad \text{for} \quad p > p_c; \qquad (4)$$
$$p_c = (1-s_i)p_i/\omega,$$

where s_i and p_i represent a reference state (Barends, 1980). A discontinuity in the composite compressibility β' is found at a critical pressure p_c since air bubbles become unstable and dissolve.
In fig. 1 the presented expression for the mixture compressibility is compared with several formulae given in literature.

© $s\omega\rho''$ *does not affect the saturation degree s.*

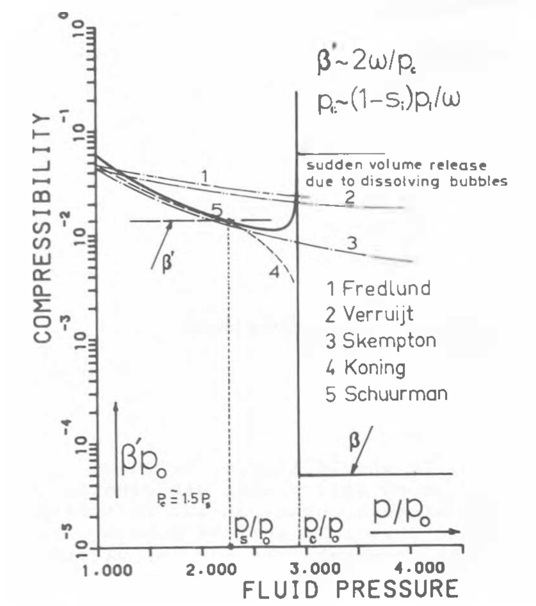

Fig. 1 The compressibility of air-water mixtures (after Barends, 1980).

The specific discharge q of this air-water mixture is related to the absolute water velocity w and the absolute soil skeleton velocity v, according to:

$$q = n(w-v). \tag{5}$$

This definition allows to introduce a constitutive relation for the generated volumetric friction force R, which obviously is connected to the relative pore fluid motion q, according to:

$$R = -\rho' g q/K, \tag{6}$$

where K represents the hydraulic permeability. The volumetric driving force H causing the porous flow can be expressed in terms of a potential Φ for irrotational flow fields:

$$H = \nabla p + \rho' g\nabla z = \rho' g\nabla\Phi. \tag{7}$$

If the mixture density ρ' is a variable and a function of the pore pressure p only, then (7) holds in isotropically permeable media. Equilibrium is satisfied by: H = R, resulting with (6) and (7) into:

$$q = -K\nabla\Phi = -(K/\rho' g)(\nabla p + \rho' g\nabla z), \tag{8}$$

which is identified as Darcy's law in the case where K is independent from q. To model non-linear flow types (ante-linear flow in fine clay's or turbulent flow in coarse beds) a linearization about K is suggested. For turbulent flow types the formula:

$$K = \sqrt{(\rho' g^2 D/AC_D|R|)}, \tag{9}$$

with C_D being a drag coefficient, D a relevant grain size and A being a coefficient related to the flow configuration, has been tested and applied in the design process of the foundation of the Oosterschelde storm surge barrier (Barends and Thabet, 1978).
The permeability K is related to the porosity n according to (Barends, 1980):

$$dK/K = \kappa de/e \;;\; e = n/(1-n). \tag{10}$$

The factor κ is to be determined by experiments; for natural situations: $3<\kappa<5$.
For large pore pressure fluctuations the kinematic viscosity ν of the pore fluid varies (slightly) and it affects the permeability. Definition of a property according to:

$$\delta = d\nu/\nu dp, \tag{11}$$

will account for this. The individual soil grain compressibility α', defined by:

$$\alpha' = -d(1-n)U/Udp;\; U\text{: bulk volume}, \tag{12}$$

is included and the last equation of state assumes a relation between the volumetric strain ε of the soil skeleton and the pore pressure p:

$$\varepsilon = \alpha(p - p_F). \tag{13}$$

Here, p_F is a reference pressure, eventually a function of time.

CONSERVATION OF THE AIR-WATER MIXTURE

The general conservation for the pore fluid states (Bear, 1972):

$$\nabla\cdot(n\rho' w) + \partial(n\rho')/\partial t = 0. \tag{14}$$

In terms of the specific discharge q this becomes, employing (5):

$$-\nabla\cdot(\rho' q) = \nabla\cdot(n\rho' v) + \partial(n\rho')/\partial t = n\rho'\frac{D\varepsilon}{Dt} + \frac{D(n\rho')}{Dt}, \tag{15}$$

where the soil substantial derivative has been used ($D/Dt = \partial/\partial t + v\cdot\nabla$). Note $D\varepsilon/Dt = \nabla\cdot v$; $\varepsilon = dU/U$. The right hand side can be expressed with (3) and (12) into:

$$-\nabla\cdot(\rho' q) = \rho'\{\frac{D\varepsilon}{Dt} + (\alpha' + n\beta')\frac{Dp}{Dt}\}. \tag{16}$$

Barends (1980) showed that in case the air is partially stagnant this must be adjusted to:

$$-\nabla\cdot(\rho' q) = \rho'\{\frac{1}{1-b}\frac{D\varepsilon}{Dt} + (\alpha' + n\beta')\frac{Dp}{Dt}\}, \tag{17}$$

where b represents the relative bulk volume of stagnant air. Application of (3), (8), (10), (11), (12) and (13) gives for (17) after some elaborations:

$$\nabla^2 p + \frac{\kappa\alpha''}{n}\nabla p\cdot\nabla p + m\frac{\partial p}{\partial z} = \frac{Dp}{c'Dt} - \frac{\rho' g\alpha}{K(1-b)}\frac{Dp_F}{Dt}, \tag{18}$$

where: $\alpha'' = \alpha + \alpha'/(1-n) - n\delta/\kappa$;
$m = \rho' g(\kappa\alpha'' + n\beta')/n$;
$c' = K/(\rho' g\{\alpha/(1-b) + \alpha' + n\beta'\})$.

Introduction of a potential Φ, according to:

$$\rho' g d\Phi = d(p - \psi p_F) + \rho' g dz,$$

or:

$$\tilde{\phi} = z + \int_{\psi p_F}^{p} (1/\rho' g)dp, \quad \psi = \frac{1}{1+\frac{\alpha'+n\beta'}{\alpha/(1-b)}}, \tag{19}$$

(completely describing the porous flow provided its irrotational character) renders (18) into:

$$\nabla^2\tilde{\phi} + m\{\nabla\tilde{\phi}\cdot\nabla\tilde{\phi} - \frac{\partial\tilde{\phi}}{\partial z}\} = \{\frac{D\tilde{\phi}}{Dt} - v\cdot\nabla z\}/c'. \tag{20}$$

The right-hand side of (20) contains the convective terms due to the motion of the soil skeleton. Rewriting (20) into:

$$\nabla^2\tilde{\phi} + m\{\nabla\tilde{\phi}\cdot\nabla\tilde{\phi} - \frac{\partial\tilde{\phi}}{\partial z}\}\{1 - v\cdot(mc'\nabla\tilde{\phi})^{-1}\} = \frac{\partial\tilde{\phi}}{c'\partial t},$$

permits to evaluate this convective effect. For practical values the following holds:

$$\frac{v}{mc'\nabla\tilde{\phi}} \simeq \frac{nv}{K\nabla\tilde{\phi}} = nv/q = 1/(\frac{w}{v} - 1). \tag{21}$$

In most cases where w>>v the convective effect due to soil deformation can be disregarded. In that case (20) becomes:

$$\nabla^2\tilde{\phi} + m\{\nabla\tilde{\phi}\cdot\nabla\tilde{\phi} - \frac{\partial\tilde{\phi}}{\partial z}\} = \frac{\partial\tilde{\phi}}{c'\partial t}. \tag{22}$$

Since through m all the non-linear aspects are included, m is called the coefficient of non-linear seepage.

THE FIELD EQUATION FOR NON-LINEAR SEEPAGE

Equation (22) is easily transformed by a new potential, referred to as the extensive potential χ:

$$\chi = \exp(m\tilde{\phi}) - 1. \tag{23}$$

For horizontal flow problems the governing field equation including the non-linearities considered becomes in terms of χ:

$$\nabla^2\chi = \frac{\partial\chi}{c'\partial t}, \tag{24}$$

which is identical to the familiar equation for

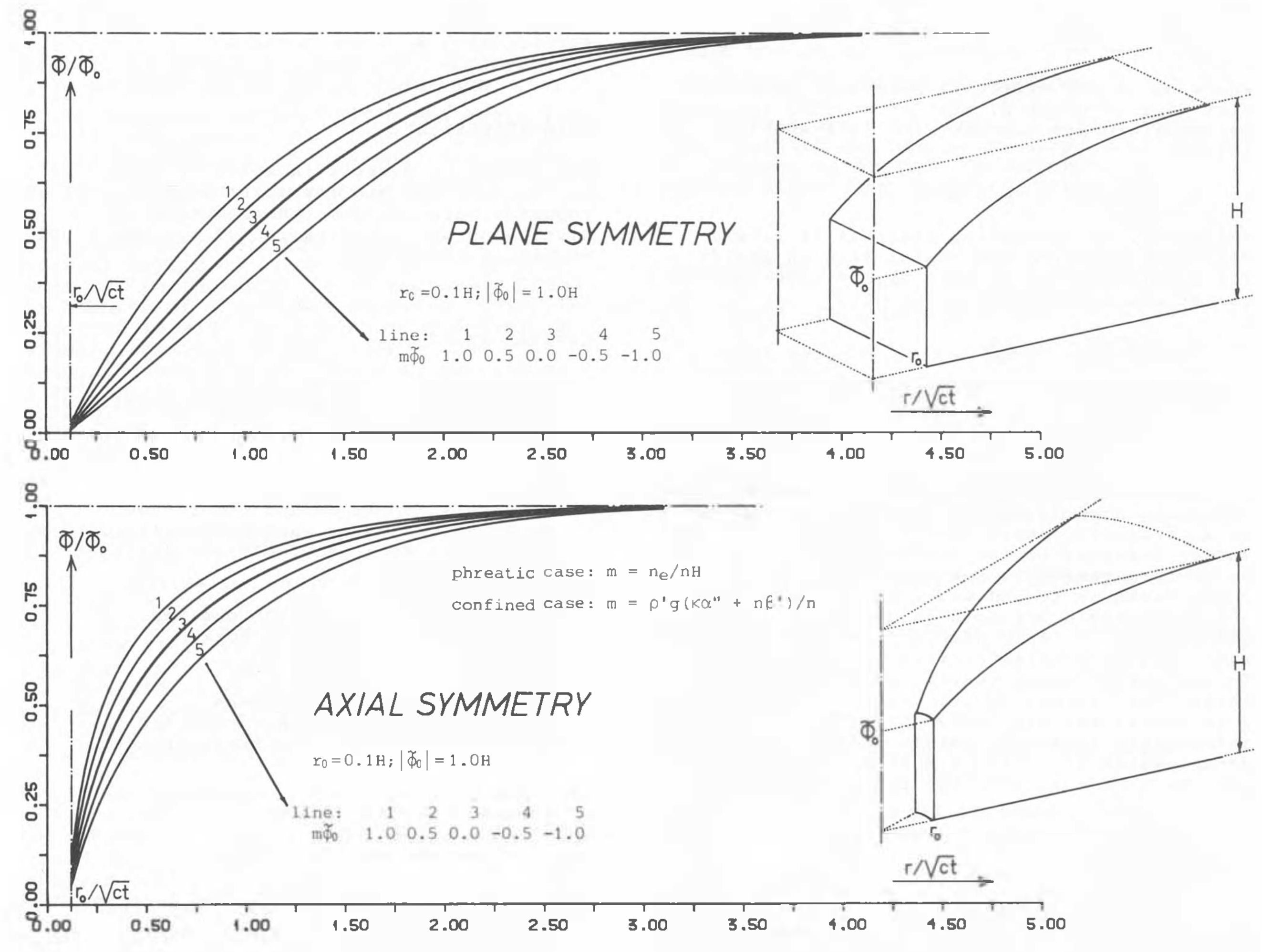

Fig. 2 Non-linear transient porous flow in a horizontal semi-infinite deformable aquifer due to a sudden drawdown at the border.

linear seepage flow.
For vertical flow equation (22) is an Euler equation. Introduction of a new coordinate: $z' = z - mc't$, will lead to a field equation similar to (24). Boundary conditions are easily expressed in terms of the extensive potential χ, and incorporation of (23) in an existing computer program for linear seepage is no problem. Hence, conventional solutions and solving methods apply equally well to non-linear seepage problems.
The field equation for phreatic flow in its extensive form is identical to (22). Consider a predominantly horizontal aquifer. Fluctuations in the pressure p averaged over the actual water height H can be expressed into:

$$d\bar{\phi} = dp/\rho'g, \qquad (25)$$

giving an extra storage possibility per unit areal surface. Consider the mass M of water per unit areal surface:

$$M = nH\rho', \qquad (26)$$

The additional mass dM becomes with (25):

$$dM = n_e\rho'd\bar{\phi} = n_e dp/g, \qquad (27)$$

where n_e is the effective porosity. Introduction of a quantity $\tilde{\beta}$, similar to (2), including phreatic storage results with (26) and (27) into:

$$\tilde{\beta} = dM/Mdp = n_e/\rho'gnH. \qquad (28)$$

Evidently, the preceding analysis is likely valid for phreatic horizontal flow in aquifers. Its result permits another significant implication of the coefficient m, to wit:

$$m = \rho'g\tilde{\beta} = n_e/nH, \qquad (29)$$

whereas the coefficient of consolidation c' reads:

$$c' = KH/n_e. \qquad (30)$$

An example of the non-linearity in seepage flow is represented in fig. 2 showing the response in a horizontal semi-infinite aquifer due to a sudden drawdown at the boundary (plane symmetry and axial symmetry). For practical values encountered in ground water the coefficient of non-linearity m yields a value in between 0.001 and 0.05 (m^{-1}) for confined flow, but it can be much larger in phreatic flow situations.
In particular non-linearity becomes manifest in large flow fields. In this regard the quantity χ is called the extensive potential. It provides information about the extent of perturbations in ground water flow. For a more detailed explanation the reader is referred to Barends (1980).

CONCLUSIONS

The presented theory is valid for coherent pore water restricting the saturation degree beyond about 0.85. Air is present in small bubbles or isolated pockets. It significantly affects the compressibility, but not beyond a specific value, since bubbles dissolve. An air bubble in pore water cannot become infinitely small. The field equation governing semi-saturated porous flow is different for stagnant or conveyed air bubbles.
The entire flow process can be expressed in terms of one single variable (potential) only if the flow is irrotational requiring the deformation behaviour of the soil skeleton to be irrotational as well. For a large class of practical problems the rotational part in the deformation can be disregarded. Pressure induced density variations of the pore fluid will not give rise to rotations in the flow provided that the medium is isotropically permeable.
Convective effects due to soil deformations are negligible except if the soil deformation velocity and the seepage discharge have similar orders of magnitude. Non-linear types of porous flow are easily incorporated in a linearized permeability. The various non-linear aspects can be included by one single coefficient.
The non-linear porous flow can be described by the traditional equation of linear porous flow, but expressed in terms of the extensive potential.
Since for most cases in practice the coefficient of non-linearity is relatively fairly small the linear theory of seepage flow provides sufficiently accurate results.

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NOTATION

A sequence of variables following after a "/" in the formulae composes a denominator:

$$\beta = d\rho/\rho dp \quad \text{"equals"} \quad \beta = \frac{d\rho}{\rho dp}.$$