Plain Plastic Flow of Granular Media

Ecoulement Plastique des Milieux Granuleux

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SYNOPSIS The plain problem for penetration of punch in disperse media is examined by means of physical and theoretical modelling. The model tests were carried out in perfectly granular heavy metal media of lead spheres. The deformation pictures of the plastic flow in quasistatic condition were fixed which deviate from the typical cases of Rankine. The image pictures and structure model of the media are described by the theory of Saint-Venant for the perfect rigid plastic material. The complete system of equations includes the stresses σ_x, σ_y and τ_{xy} and the velocity of deformation v_x and v_y. The field of stresses is made according to private solution of Sokolovskii (1948) where Mohr yield condition with experimentally-fixed function $F(\sigma,\tau)$ is applied. The admissible coaxiality of the tensor of stresses and the tensor of strain velosity allow to calculate the velocity of dessipation of the energy in non-cohesive granular media according to the equation $D_i = T_i [V_i] \geqslant 0$. The kinematic study shows the considerable influence of the energy dissipated for increase of the critical load with depth of penetration of the punch into the media.

INTRODUCTION

In our model investigations (Baloushev, 1976) we succeeded to realize improved experimental technics and experimental media of heavy metal with fine granular size uniformity, big unit weight and incompressible material, lack of cohesivity. We have quite definitely directed our attention to a soil substitute of artificial material by which one may most accurately meet the condition of the model theory.

TEST EQUIPMENT

In Fig.1 we have shown a model tank sized 1200 x 450 x 60 mm of solid steel frame and silicate glass walls with 20 mm thickness. The foundations models are 90 to 240 mm wide and are loaded by controlled way of three-axial loading device. The visible glass wall is covered by 20 x 20 mm scale mash. About 200 experiments are carried out. In different time and periods of loading and different depth d_f.

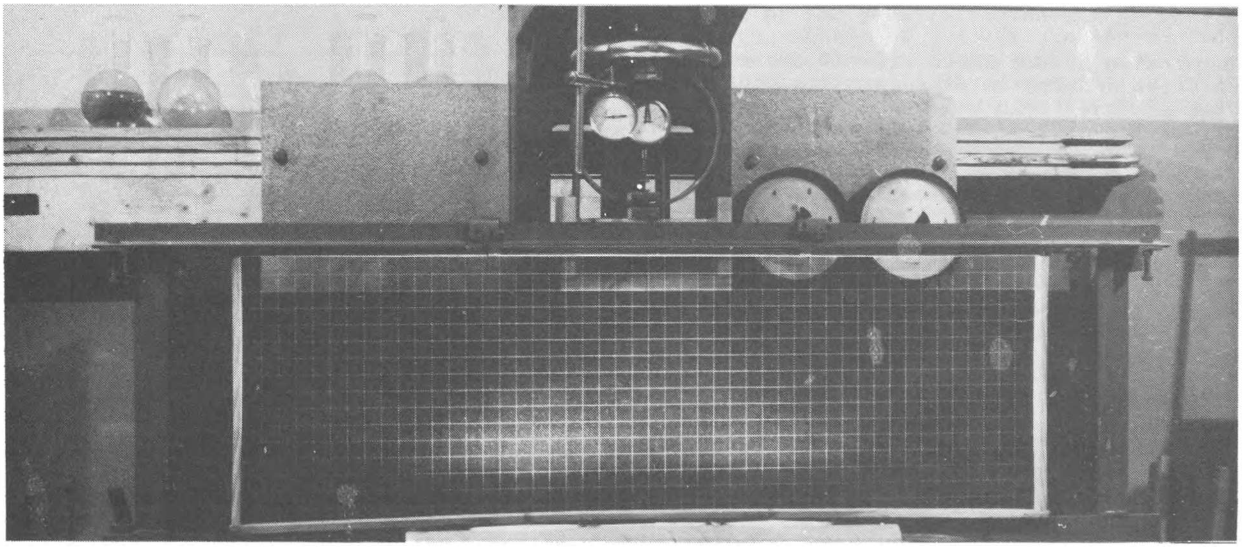

Fig.1 Test Equipment

The model media is composed by fine polished spherical granules with diametre 0,75 mm of lead with high rigidity. The granules are drily lubricated by graphite what excludes cohesion. The remaining technical data are: $\gamma = 70$ kN/m3, $\varphi = 21^{o}$, angle of wall friction in glass $\delta = 9^{o}$.

The fields of narrowed plastic deformations and fields of unlimited plastic destruction both shown in Fig.2 were studied by means of immovable photo-camera. The speed of penetration of punch-foundation was experimentally defined $v_p = 0{,}0075$ mm/s so that it may not affect kinematically the deformation picture. The higher speeds of penetration showed bigger plastic zones and this did not affect visibly the rate of the outer loading. Evidently this is the limit of quasi-static condition tolerated with the experiments.

The active part of the fields and intermediate radial field are developed close to the circle-cylindrical surfaee. The forces of loading according to the dynamometric system of the device read the moment of destruction corresponding to the picture of plastification of the material as loss of stability of the media irrespectively of that part remained between the plastic fields of the foundation. The role of the foundation breadth as factor for proportional increase of the bearing capacity was regarded under doubt (Baloushev, 1966). Only smooth foundations are examined here.

THEORETICAL ANALYSIS AND APPROACH

Structural and Mathematical Model

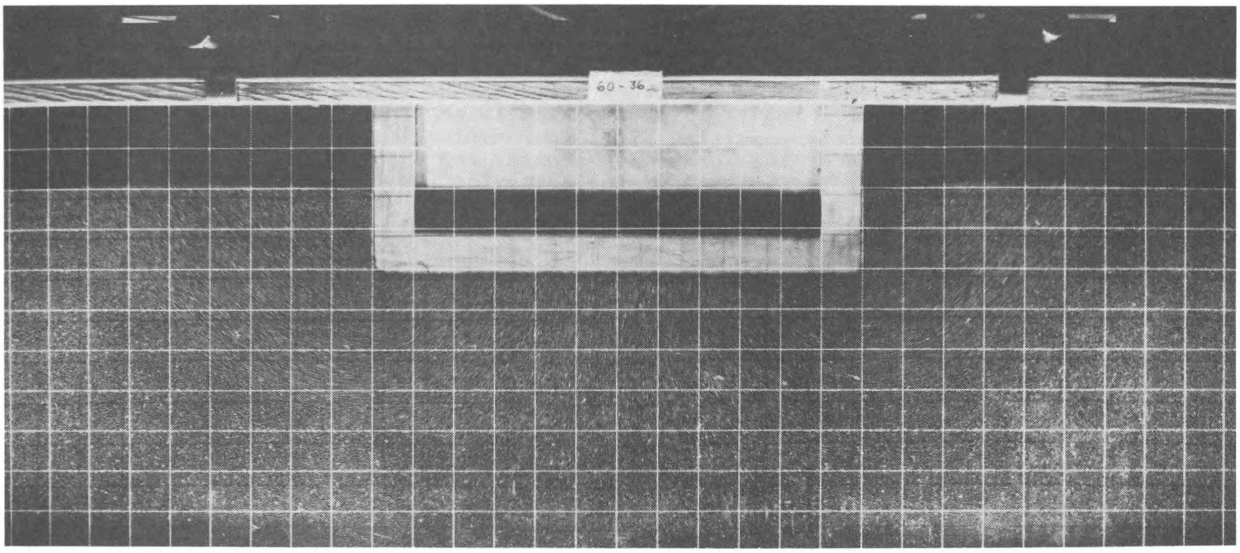

Fig.2 Test photograph picture of the plastic flow fields

TEST RESULTS

The increase of the basic size of the model-foundation results in expansion and penetration of the plastic field below the foundation, respectively outside of the same whereas it keeps the character of the form sliding surface. There is a moment when the destructed fields reach maximum size of depth below the edge of the foundation /for example at breadth 180 mm/. The increase of foundation makes more distant the two destructed fields each other without effecting substantially the character of their form and size. Similar preservation of the form of the plastic fields is obtained also in the changeable depth d_f. Fig.3 contained all pictures of deformation in general combination and shows that the depth of foundation also is not in condition to change the form and direction of development of plastic fields. They get natural continuation of the slope line of the sliding surface to the ground surface under angle close to $\pi/4$. The kinematically generalised form of the punched media at any probability is affected solely by the mechanical properties of the granular media.

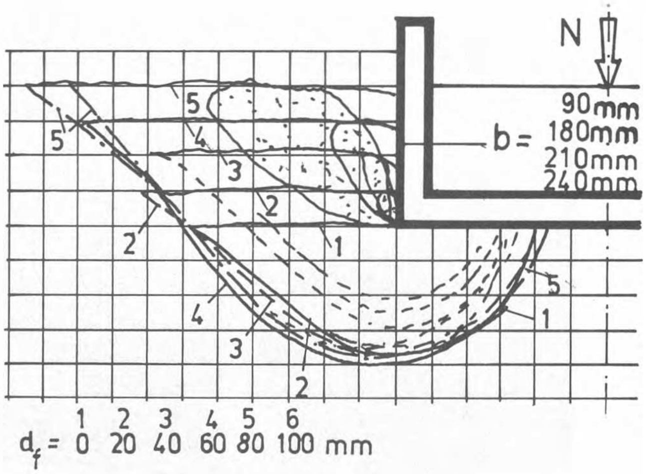

Fig.3 Plastic fields combination

The realized physical simplification of the model media as a body of fine spheric granules and pure mechanic properties proposed a good base for suitable choice of structural and mathematical model for complex decision of the statically non-determinable task as per the theory of Saint-Venant with determination of the stress fields and the fields of strain rate velocity.

We have secured condition of quasi-static flow of granular media after the plastic limit. Further we shall have in mind only the velocity of deformations that are owing to the mechanical properties of the material in treatment of the theory of plasticity. The model of the granular media is the one of the fine plastic rigid body with most typical properties: isotropy and incompressibility of material.

The problem examined falls to the plain strain problems. From the condition for voluminous incompressibility follows than any increase of deformations is clear sliding with shearing stress τ_{xy}. This is dead right at Poisson factor 0,5 and that is why the theory is applicable for the plastical fields of elasto-plastic material. The yield criterion and the equations of equilibrium in general case of the voluminous forces define the stress field:

$$\begin{aligned} &\ldots\ \partial \sigma_x / \partial x + \partial \tau_{xy} / \partial y = X \\ &..\ \partial \tau_{xy} / \partial x + \partial \sigma_y / \partial y = Y \qquad (1) \\ &..\ \tau^2 = 1/4\,(\sigma_x - \sigma_y)^2 + \tau_{xy}^2 = k^2 \end{aligned}$$

and the conditions of incompressibility of the volume and co-axiality of the principal directions of the stresses and strain velocities tensors (Hill, 1950) :

$$\begin{aligned} &\ldots\ \partial v_x / \partial x + \partial v_y / \partial y = 0 \\ &\ldots\ \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\partial v_x / \partial y + \partial v_y / \partial x}{\partial v_x / \partial x - \partial v_y / \partial y} \end{aligned} \qquad (2)$$

The system of five equations of Saint-Venant (1) and (2) provides the following possibilities:

(i) equations (2) are uniform compared to the velocities and the element time is not read.

(ii) The statical unknown quantities of the system σ_x, σ_y, τ_{xy}, v_x and v_y are determined in two individual stages: the first three stresses give statically determinable task and field of stresses; velocities may be calculated later.

(iii) The different types statically non-determinable problems in the sense of Hill (1950) are sorted out with the full system of equations and coincidence of the field of the stresses and the one of the velocities of deformation.

Mathematical Models

The plastic condition of the disperse media arises owing to the increase of tangential stress over determined critical value over certain area. This limit is attained on the spot by maximum velocity of sliding deformations at plastic invariability of the volume and co-axiality of the stress and velocity tensors. The limiting condition based on the yield criterion of Coulomb is always breached due to lack of coincidence of the genuine lines of sliding (with discontinuation of velocities) with the probable lines of Coulomb friction from the field of stress. Discontinuation of the velocities may happen only on some of the characteristics of the velocity α_v and β_v.

Mathematically avoiding of these contradictions at preserved physical sense of Coulomb friction is performed in several ways:

(a) non-co-axial models (de Jong, 1959),(Geniev, 1958),(Drescher & Jong, 1972),(Spencer, 1964),(Mandl & Luque, 1970);
(b) associated law (Drucker & Prager, 1952), (Shield, 1953,1955),(Chen, 1969),(Solovjev, 1969);
(c) incremental bonds - (Ivlev, 1967),(Weidler & Paslay, 1970);
(d) Hypotheses for microdeformation - (Rowe,1963)
(e) change of Coulomb mechanism - (Nikolaevskii, 1969).

The experimental achievements may be generalized in several conclusions: they show coaxiality of tensors of stresses and velocities of deformation (Drescher & Bujak, 1966),(Roscoe, 1970), (Baloushev, 1976); the form of destruction in curvelined surfaces tends to be circle-cylindrical one (the relative sliding of fields of the media); the role of the velocity of dilatance and experimentally observed lines of the deformations is coordinated in theoretical analysis of the velocity field (James & Bransby, 1971). Some kinematic decisions show for the groundlessness of statical models of disperse media at plastic distruction (Drescher, 1972).

The pictures obtained by us for the deformations in granular media below loaded punch determined several character moments in the formation of the plastic fields and on them it was made the scheme of the calculating model. We have ascertained big similarity with the pictures of Steenfelt (1979) whose perfect experiments and model equipment we had the chance to get acquainted with recently. The plastic fields moved in the passive pressure zone outside of the punch form sliding surface under angle to $\pi/4$. The curvelinear areas in the intermediate field below the punch edge are developed on circle-cylindrical surfaces and close to same. The sizes of the plastic fields moved tend to a determined form as well as the depth d_f does not change the character of the fields distructed

Adopted Mathematical Model

We compose the mathematical model by means of rejection of Coulomb yield criterion and introduction of Mohr yield criterion in its general type:

$$\partial F / \partial \sigma . d\sigma + \partial F / \partial \tau_m . d\tau_m = 0 \qquad (3)$$

which shows that the tangential stresses in the condition of plastic flow represent a function of normal σ stress, Fig.4, and the same endeavour to a constant critical stress K with σ_m increase, and the sliding surface to $\pi/4$ angle.

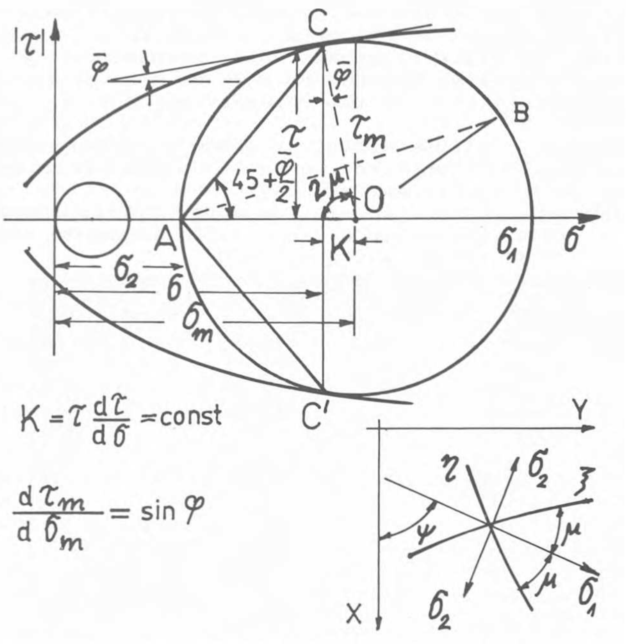

Fig.4 Yield condition of Mohr

The first traces of such a conception we met in Torre (1948). We have experimentally determined the function F (σ,τ) as a yield criterion of the type (Baloushev, 1979):

$$\ldots\ldots\ \tau^n = n.K.\sigma \qquad (4)$$

The parametre $1 \le n \le 2$ determines the remoting of the critical line from the axes of σ. The factor $K = m.\tan\varphi$ conforming to the experimental media may meet different requirements including passing to the classic case of Coulomb. The factor K must not be mixed up with the notion for internal cohesivity but like a material constant determined under conditions of free and loaded surfaces when indentation of the punch in plastic media where the solutions are functional. We met similar interpretation in Mróz (1967).

From Fig.4 is seen that the common point C of the critical curve and the stress circle of Mohr is determined of the changeable angle 2μ and the AC area gives the sliding area compared to the principal stress σ_1. The statics determinable task for the stresses is carried out as per the theory of Sokolovskii (1954) and the main system of differential equations is obtained as follows:

$$(1-A)\frac{\partial\chi}{\partial x} + B\frac{\partial\chi}{\partial y} - C\frac{\partial\psi}{\partial x} + D\frac{\partial\psi}{\partial y} = \frac{\sin^2 2\mu}{2\tau}\cdot X \qquad (5)$$

$$B\frac{\partial\chi}{\partial x} + (1-A)\frac{\partial\chi}{\partial y} + D\frac{\partial\psi}{\partial x} + C\frac{\partial\psi}{\partial y} = \frac{\sin^2 2\mu}{2\tau}\cdot Y \qquad (5)$$

with denotions: $A = \cos 2\mu.\ \cos 2\psi$

$B = \cos 2\mu.\ \sin 2\psi$

$C = \sin 2\mu.\ \sin 2\psi$

$D = \sin 2\mu.\ \cos 2\psi$

and function of Mandel $- 2d\chi = d\sigma/\tau - 2\,d\mu$. The unification of the changeable factors μ and ψ is effected by means of new changeable ones ξ and η which appear like characteristics of the field of stresses:

$$\xi = \chi(\mu) + \psi$$
$$\eta = \chi(\mu) - \psi \qquad (6)$$

STATICS SOLUTION

The yield criterion of plastic flow from the equation (4) is determined by the data of the model media and soil prototype: the unit weight of the model media is γ = 70 kN/m^3 and the average unit weight of the soil prototype is γ = 17,5 kN/m^3 and the angle of internal friction is $\varphi = 21^\circ$ for both disperse media.

The model constant of unit weights (Florin, 1959) is:

$$\alpha_\gamma = \gamma_p/\gamma_m = 0{,}25 \qquad (7)$$

We denote the parametres like function of the unit weight model constant which has leading meading also for equation (4):

$$K = m.\tan\varphi \quad ; \quad m = \sqrt{\alpha^{-3}} \quad ; \quad n = \sqrt{\alpha^{-1}} \qquad (8)$$

where: m - is parametre of the constant K

n - is parametre of parabolic curve of the yield criterion.

Boundary Conditions

All the quantities are expressed with the constant K of the material and changeable angles 2μ and 2ψ. The components of the stresses and the coordinates x and y of the point are obtained at n = 2 :

$$\left.\begin{matrix}\sigma_x\\ \sigma_y\end{matrix}\right\} = K\left(\frac{\tan 2\mu}{n}\right)^{\frac{1}{n-1}}\left(\frac{\tan 2\mu}{n} + \frac{\cos 2\mu \pm \cos 2\psi}{\sin 2\mu}\right)$$

$$\tau_{xy} = K\left(\frac{\tan 2\mu}{n}\right)^{\frac{1}{n-1}} \cdot \frac{\sin 2\psi}{\sin 2\mu} \qquad (9)$$

The equations of the coordinates are as follows:

$$x = \frac{K}{2\gamma}\left[\frac{1+\varkappa 2\cos 2\mu}{(\cos 2\mu)^2} - \left(\frac{1+\varkappa 2\cos 2\mu_o}{(\cos 2\mu_o)^2}\right)\right]$$

$$y = \pm\frac{\varkappa.K}{2\gamma}\left[\frac{\sin 2\mu}{\cos^2 2\mu} - \ln\tan(\mu+\pi/4)\right] + \bar{C} \qquad (10)$$

and in the intermediate radial zone:

$$\frac{\tan 2\mu_2}{n-1} - 2\mu_2 + \pi = \frac{\tan 2\mu_1}{n-1} - 2\mu_1 \qquad (11)$$

The angle Ψ determines the direction of the principal stress σ_1 with the vertical axle x, for the active pressure zone $\Psi = 0$ and $\varkappa = +1$, for the passive zone $\Psi = \pi/2$ and $\varkappa = -1$, and for the radial field the characteristics are $\xi \neq$ const and η = const.

The statics determinable solution with making the field of stress is shown on Fig.5

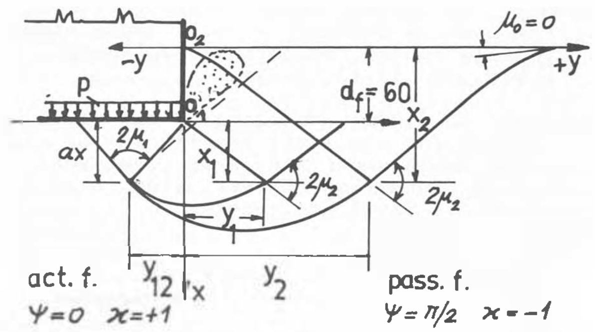

Fig.5 Field of stresses with the characteristics ξ and η

KINEMATICS SOLUTION

The theoretic field of the stresses from Fig.5 shows the probable sliding lines. As it is seen from comparison of Fig.2, Fig.3 and Fig.5 the experimentally observed strain fields get nearly complete coincidence. The orientation of one of the characteristics of the stresses under angle $\pi/4$ with the horizon with the incrcase of stress σ denotes a real coincidence with one of the real lines of distruction which line is one of discontinuation of the velocities of deformation. In theoretical sense the uniformity of the type of the characteristics from the field of the stresses with ones of the velocity field shows the endeavour to co-axiality of the stresses and the velocities tensors, one rather binding condition (Hill, 1950).

The velocity field is made as per the method of Hill (1950) or as per Sokolovskii (1969) when we have constructed the field of the stresses. The kinematics of the plastic flow of the media with condition of voluminous incompressibility and the equations of Geiringer is given with the equations:

$$2\sin 2\Psi \frac{\partial v_x}{\partial x} - \cos 2\Psi \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) = 0 \qquad (12)$$

$$2\sin 2\Psi \frac{\partial v_y}{\partial y} + \cos 2\Psi \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) = 0 \qquad (12)$$

The equations of the characteristics of the velocities have the same type:

$$\frac{dy}{dx} = -\frac{dv_x}{dv_y} = \tan\left(\Psi \mp \pi/4\right) \qquad (13)$$

where $\mu \approx \pi/4$, ortogonality of the characteristics obligatory as per the conditions of Geiringer.

By the admission that we come nearer to coaxiality when the leading angle μ of the sliding areas comes close to $\mu = \pi/4$ by increase of σ stresses we can put to a general plan the field of stresses, the field of velocities and the hodograph of the velocities on Fig.6 a, b and c.

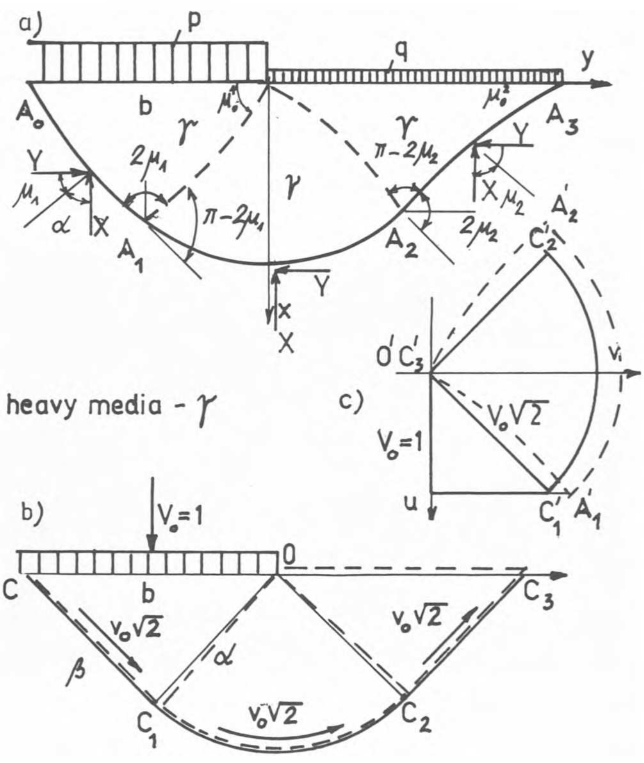

Fig.6 a) The field of stresses
b) The field of velocities and
c) The hodographs.

VELOCITY OF ENERGY DISSIPATION

In conformity with the theorems of Drucker & Greenberg, Prager (1951) the velocities of the stresses in the beginning of the plastic flow and the velocities of the elastic deformations are zeroes. The velocities of the plastic deformation are admitted for velocities of full deformation in the integral of virtual work.

Provided $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$ and $\dot{\gamma}_{xy}$ are the deformation velocities the dissipation energy is equal to:

$$D_R = \int_R (\sigma_x . \dot{\varepsilon}_x + \sigma_y . \dot{\varepsilon}_y + \tau . \dot{\gamma}).dA \geq 0 \quad (14)$$

The closed plastic area R may be examined for dissipated energy by means of the surface forces under contour B as per Prager & Hodge (1956) with Fig.7 and equations:

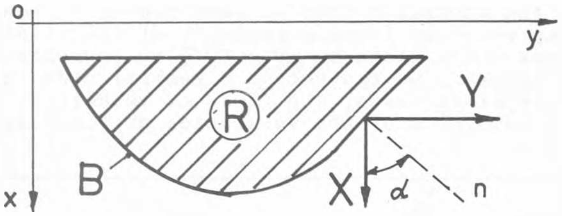

Fig.7

$$X = \sigma_x.\cos\alpha + \tau.\sin\alpha$$

$$Y = \sigma_y.\sin\alpha + \tau.\cos\alpha \quad (15)$$

$$\int_R (\sigma_x . \dot{\varepsilon}_x + \sigma_y . \dot{\varepsilon}_y + \tau . \dot{\gamma})dA = \int_B (X.v_x + Y.v_y)ds$$

The fields of the stresses and strain velocities are determined by means of all σ_x, σ_y, τ_{xy} and v_x, v_y as functions of the system x,y of the continuous differential field.

CONCLUSION

The theoretical model was tested in numeral examples and compared to the results of the experiment. The bearing capacity for the model on b = 180 mm has the following values:

(i) At d_f = 0 mm from the statics and kinematics solution:

$$p_{st} = p_{kin} = 0{,}063 \text{ MN/m}^2$$

(ii) For depth d_f = 60 mm:
statics solution - p_{st} = 0,096 MN/m^2
kinematics solution - p_{kin} = 0,148 MN/m^2
$m_{en} = p_{kin} / p_{st} = 1{,}54$

(iii) Experimental results:
b = 180 mm, d_f = 0 mm, p_k = 0,082 MN/m^2
b = 180 mm, d_f = 60 mm, p_k = 0,1435 MN/m^2

The increase of the depth of foundation rises the role of the energy dissipated and its reading in the kinematic solution gives answer with increased bearing capacity. The mathematic model with yield criterion of Mohr represented a possibility for individual solutions in separate phases of the task, as well as for complete kinematic study of the solutions. The parabolic function from (4) for the concrete model media may be modified for any other media by means of unit weight model constant α_γ, including a natural one. The kinematic study became possible thanks to the theoretic construction of plastic fields by shape and size. The heavy model media showed fields of plastic flow different from the well-known conditions of Rankine and that is what directed our attention to co-axiality of the models.

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