Three Methods of Slope Stability Analysis

Trois Méthodes de Calcul de Stabilité des Talus

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SYNOPSIS. The 1st part of the paper (Fedorovsky) gives a variational method for the slope stability analysis based on the assumption that Coulomb law and Kötter differential equation are valid for stresses along the rupture line which separates the moving soil from the immobile one. Therefore the stresses and safety factor can be found for any potential rupture surface. The critical rupture surface must give an extremal value of the safety factor if the displaced soil mass is in equilibrium.

The 2nd part of the paper (Freiberg) deals with the plane problem for heterogeneous slope weakened along certain surfaces and thus divided into individual part. The value and orientation of forces applied to these parts (and consequently, the safety factor) are determined from the limit equilibrium condition and some plausible assumptions.

The 3rd part (Vasiljev) is also dedicated to the heterogeneous slopes with weak surfaces, 3-D shape of the displaced soil mass and of the stress distribution is considered as well. The safety factor is computed from the equilibrium condition for moments applied to the kinematically virtual sheared soil mass. A quantative example of calculation is given.

1. A good many of soilbase failure cases (ordinary and anchor footings, retaining walls, slopes) feature one or more narrow rupture s strips separating the moving soil mass from the immobile one. The deformations of both masses are negligible as compared to the deformations along the rupture surfaces. Therefore the soil medium can be adopted to be rigid plastic with the assumption that the rupture surface corresponding to slip lines in the plane problem are characterized by a certain limit condition. If said condition is Coulomb law then stresses along rupture line comply with Kötter differential equation.

The essential part of the variational method is that stresses along a sufficiently smooth arbitrary rupture line can be obtained by integrating the above equation with allowance made for the boundary conditions. The equilibrium conditions of the sheared soil mass are also valid. The expression for the critical load or safety factor can be derived from these conditions (depending on the problem defined). Then follows variational approach, i.e. the virtual line is singled out of all other which meets the boundary conditions, provides for the equilibrium of the displaced soil mass and corresponds to the extremal value of the specified parameter (safety factor for slope stability). This can be linked up with the appropriate principle of the limit equilibrium theory.

Basically, the variational approach originates from Coulomb technique of strength analysis of foundations that could not be, however, used because the stress distribution along the failure surface had been unknown for a long period of time. Gersevanov (1923) was the first to overcome this obstactle having, in fact, resorted to a rather abstract model of soil. Kopachi (1955) was more radical to refuse any kind of limitations for the stress distribution (beside the equilibrium of the displaced soil mass) so that both the failure surface and stresses along it are obtained by the variational technique. The equations thus produced are somewhat simpler than those proposed by us, nevertheless, both methods mutually complement rather than exclude each other, same as statical and kinematical approaches in the theory of limit equilibrium (being the particular cases of those).

Kötter equations were also earlier used in foundation strength analysis but for a fixed failure surface shape (e.g., Brinch Hansen, 1957) rather than in combination with the variational approach.

We consider the simplest case of limit equilibrium for a homogeneous slope with a boundary $\bar{y}(x)$ (Fig.1). Coulomb condition is fulfilled along rupture line $y(x)$, i.e.

$$\tau = (\sigma f + c)/k, \tag{1}$$

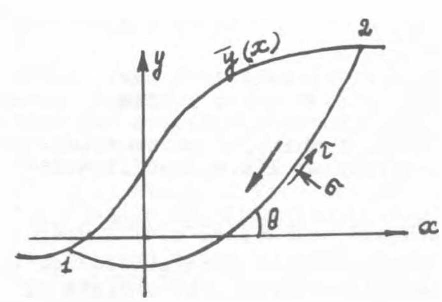

Fig.1. Scheme of the slope

where k is a diminutive factor of soil strength parameters with the extremal value equal to slope safety factor. Making allowance for (1) we put down Kötter equation as follows:

$$\varphi_1 \equiv \sigma' - 2(\sigma f + c)\theta' - \frac{\gamma k}{k^2+f^2}\left(f - k\,tg\,\theta\right) = 0 \tag{2}$$

with θ as the slope of y(x) curve

$$\varphi_2 \equiv y' - tg\,\theta = 0. \tag{3}$$

Equilibrium conditions of the moving soil mass with (1) taken into account are the following:

$$J_1 = \int_{x_1}^{x_2} f_1 dx \equiv \int_{x_1}^{x_2} (\sigma f + c - k\sigma\, tg\,\theta)dx = 0 \tag{4}$$

$$J_2 = \int_{x_1}^{x_2} f_2 dx \equiv \int_{x_1}^{x_2} [(\sigma f + c)tg\,\theta + k\sigma - k\gamma(\bar{y} - y)]dx = 0 \tag{5}$$

$$J_3 = \int_{x_1}^{x_2} f_3 dx \equiv \int_{x_1}^{x_2} \{y(\sigma f + c - k\sigma\, tg\,\theta) - x[(\sigma f + c)tg\,\theta + k\sigma - k\gamma(\bar{y} - y)]\}dx = 0 \tag{6}$$

The boundary conditions express the fact that there is maximal, in point 1, and minimal, in point 2, limit stress state.

$$y_i = \bar{y}(x_i); \quad \sigma_i = \pm c/\sqrt{f^2+k^2}; \tag{7}$$

$$\theta_i = arc\,tg\,\bar{y}_i' + \frac{1}{2} arc\,tg\frac{f}{k} \mp \frac{\pi}{4}; \quad i = 1, 2$$

where the upper sign is valid when i=1 while the lower when i=2.

The problem under consideration is to find the system corresponding to the extremum of the functional

$$J = k \tag{8}$$

among the systems of parameter k and functions y, θ and σ(x) that comply with conditions (2)-(7).

This is Bolza problem (Hestenes,1956). Its solution is the unconditional extremal of the functional $\int_{x_1}^{x_2} F(x, y, \theta, \sigma, \lambda, \mu)dx$ with

$$F = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \mu_1(x)\varphi_1 + \mu_2(x)\varphi_2 \tag{9}$$

We thus obtain the system of ordinary differential equations of the 1st order that includes equations (2),(3) and Euler equations

$$(F_{y'})' - F_y = 0; \quad (F_{\theta'})' - F_\theta = 0; \quad (F_{\sigma'})' - F_\sigma = 0 \tag{10}$$

Cauchy problem can be solved for this system of five equations involving five unknown functions y, θ, σ, μ_1 and μ_2. The obtained solution depends on six constants: x_1, $\mu_1(x_1)$, $\mu_2(x_1)$, λ_2, λ_3, k (λ_1 can be adopted to be equal to 1).

Selecting these parameters we meet two boundary conditions for θ and σ ,three integral equilibrium conditions (4)-(6) and the transversality condition which,if we take into account the extremality of k(i.e.dk=o),becomes

$$\left[(F - y'F_{y'} - \theta'F_{\theta'} - \sigma'F_{\sigma'})dx + F_{y'}dy + F_{\theta'}d\theta + F_{\sigma'}d\sigma\right]_1^2 = 0 \tag{11}$$

The above method can be generalized for the case of forces distributed over surface or volume (these forces being distinct from the weight proper). The question of definition of the safety factor arises however, because if instead of the above decrease of strength parameters adopted for this purpose the increase of applied loads is done then the extremals and the final results would be otherwise.

This method lends itself to analysing heterogeneous slope as well. To this end a "refraction law" for rupture lines crossing boundaries between homogeneous zones is necessary. This law can be developped from the condition of continuity of stresses σ_n and τ_{nt} at the boundary (Fig.2). We define α_i as the angle between a rupture line and a boundary, σ_i as normal stress along such a line, $H_i = c_i/f_i$ and $\varphi_i = arc\,tg\,(f_i/k)$ as strength parameters. Then

$$\Delta\sigma_n = \frac{\sigma_i + H_i}{\cos^2\varphi_i}\left[1 - \sin\varphi_i \sin(\varphi_i + 2\alpha_i)\right]\Big|_1^2 = 0 \tag{12}$$

$$\Delta\tau_{nt} = \frac{\sigma_i + H_i}{\cos^2\varphi_i}\sin\varphi_i \cos(\varphi_i + 2\alpha_i)\Big|_1^2 = 0 \tag{13}$$

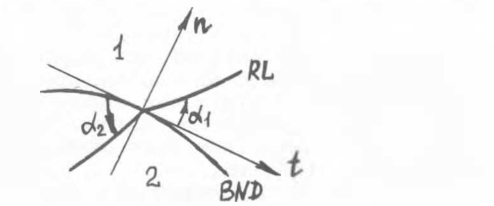

Fig.2. Rupture line crossing the boundary of different soil properties

It while plotting an extremal we approach a boundary we can, thanks to (12) and (13),obtain the values of θ and σ on the other side of the boundary; with continuity of y, μ_1 and μ_2 taken account of, this done we can further the plotting.

2. A method for analysing the stability of soil slope having a virtual failure prism crossed by faults(slackened contacts between layers, disturbed zones, cracks) which are potential shear surfaces. The method is based on the solution of the limit equilibrium plane problem for a failure prism divided by arbitrarily oriented surfaces into a finite number of interacting elements. The general case of this problem has been solved for the slope with heterogeneous cohesive soil and curvilinear free surface approximated by a broken line (Fig.3). The stability of the failure prism is assessed in compliance with soviet regulations by condition

$$n_c N_p \le m R / K_н \tag{14}$$

with N_p and R as design values of the generalized shearing force and ultimate resistance force; $K_н$ as reliability factor; n_c as load combination factor,m as working conditions factor.

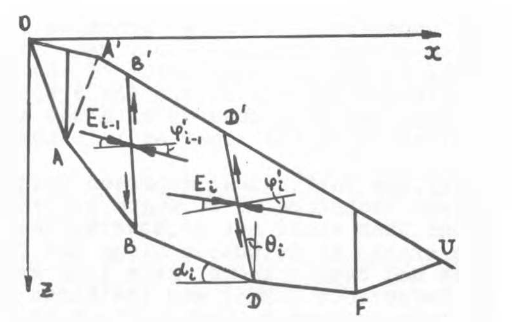

Fig.3. Failure prism elements and inclination of interaction forces

We single out the i-th element of the failure prism (BB'DD') having oriented the ordinate axis parallel to the force of gravity (Fig.3). Beside external forces reactive stresses along the shear surface in the subsoil are applied to the element as well as the forces of interaction with adjoining elements. It is assumed that on achieving the limit equilibrium the limit stress state characterized by Coulomb criterium is realized either at the base of the element or at the side boundaries BB' and DD' which are weakened surfaces:

$$\tau' = \sigma' tg\varphi' + c' \qquad (15)$$

where tg φ' and c' correspond to friction ratio and cohesion along the weakend surface respectively.

A system of equilibrium equations (16) and (17) for an element of the prism is compiled by projecting the forces on the direction of soil reaction at the base of the element P_i and on the normal to it. The equation for moments of forces (18) is compiles relative to point "m" of the soil reaction application:

$$\sum_{k=\mathrm{I},\mathrm{II}} (-1)^k \left[c_i'^{(k)} h_i^k \cos(\alpha_i - \varphi_i + \theta_i^k) - E_i^k \sin(\alpha_i - \varphi_i - \varphi_i'^{(k)} + \theta_i^k)\right] + P_i + c_i l_i \sin\varphi_i - Q_{z_i} \cos(\alpha_i - \varphi_i) + Q_{x_i} \sin(\alpha_i - \varphi_i) = 0 \qquad (16)$$

$$\sum_{k=\mathrm{I},\mathrm{II}} (-1)^{k+1} \left[c_i'^{(k)} h_i^k \sin(\alpha_i - \varphi_i + \theta_i^k) + E_i^k \cos(\alpha_i - \varphi_i - \varphi_i'^{(k)} + \theta_i^k)\right] - c_i l_i \cos\varphi_i + Q_{z_i} \sin(\alpha_i - \varphi_i) + Q_{x_i} \cos(\alpha_i - \varphi_i) = 0 \qquad (17)$$

$$\sum_{k=\mathrm{I},\mathrm{II}} \left\{(-1)^{k+1} E_i^k \left[a_i^k \cos(\theta_i^k - \varphi_i'^k) - (-1)^k x_i^k \sin(\theta_i^k - \varphi_i'^k)\right] - c_i'^k h_i^k \left(x_i^k \cos\theta_i^k + (-1)^k a_i^k \sin\theta_i^k\right)\right\} + Q_{x_i} z_i - Q_{z_i} b_i = 0 \qquad (18)$$

Here Q_{x_i} and Q_{z_i} are components of the resultant design load Qi for gravity forces, hydrostatics, filtration, seismic and other volume and surface forces; tg φ_i and c_i -soil parameters at the base of the element. The components of interaction forces at the top E_i^{I} and at the bottom E_i^{II} facets of the element are the resultants of normal and friction forces acting on the planes of division; tg $\varphi_i'^{(\mathrm{I},\mathrm{II})}$ and $c_i'^{(\mathrm{I},\mathrm{II})}$ - soil parameters along the weakened surfaces; $\theta_i^{\mathrm{I},\mathrm{II}}$ and $h_i^{\mathrm{I},\mathrm{II}}$ -the inclinations of the weakened surfaces to the vertical and their lengths; $a_i^{\mathrm{I},\mathrm{II}}$, b_i, $x_i^{\mathrm{I},\mathrm{II}}$, z_i arms of the forces (Fig.4).

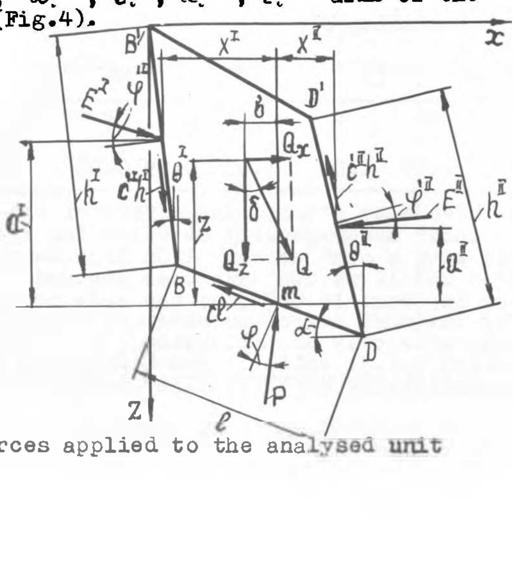

Fig.4.Forces applied to the analysed unit

The components of the interaction forces E_i^{II} for the elements all over the failure prism are determined from eq.(17) starting from the upper (first) element where component $E_i^{\mathrm{I}} = 0$. The vector forces of interaction between adjoining elements are equal to each other, i.e. $E_i^{\mathrm{I}} = E_{i-1}^{\mathrm{II}}$; for the lower (last) element $E_n^{\mathrm{II}} = 0$. The stability of the failure prism is assessed as a whole when breaking down eq. (17) for the force E_n into expressions for R and N_p by grouping items with tgφ, c , tgφ', c' and without them:

$$R = A_n + B_n + C_n\{A_{n-1} + B_{n-1} - B'_{n-1} + C_{n-1}\{\cdots + C_2\{A_1 - B'_1\{\cdots\}, \qquad (19)$$

$$N_p = D_n + C_n\{D_{n-1} + \cdots + C_2\{D_1\}\cdots\}; \qquad (20)$$

where $A_k = [tg\varphi(Q_z\cos\alpha - Q_x\sin\alpha) + cl]_k \cos\varphi_k$,

$$B_k = c'_{k-1} h_{k-1} \sin(\alpha_k - \varphi_k + \theta_{k-1}); \quad B'_k = [c'h\sin(\alpha - \varphi + \theta)]_k, \qquad (21)$$

$$C_k = \cos(\alpha_k - \varphi_k - \varphi'_{k-1} + \theta_{k-1}) / \cos(\alpha - \varphi - \varphi' + \theta)_{k-1},$$

$$D_k = [(Q_z\sin\alpha + Q_x\cos\alpha)\cos\varphi]_k$$

The compatibility of the solutions of (17) the initial equation for stability assessment, and (18), the equation for the moments of forces, is achieved when unknown values a_i^{II} and b_i (arms of forces in equation (18)) are linked up by some hypothesis that complies with some plausible level of location of the interaction forces between the elements (no tension along the division planes) and to the real character of reactive stress distribution at the bases of the elements. As verifying solutions using the above relations have shown all conditions of the limit equilibrium are always met if the interaction forces are applied at the level a_i within $(1/3 \div 2/3)h_i$ range.

The limit state along the weakened surfaces, boundaries between the elements, when the whole failure prism is in the limit equilibrium is assessed considering (15) by the following condition:

$$\tau_i' \geqslant \sigma_i' tg\varphi_i' + c_i' \qquad (22)$$

When some weakened surfaces do not feature the limit state (condition (22) is not fulfilled) the prism is divided into interacting elements both by the weakened surfaces and by vertical planes. For those dividing planes where shear stresses do not achieve limit values the inclination of the interaction force is determined by an iterative procedure, assuming its constancy over the whole failure prism, from the limit equilibrium condition considering known angles of inclination of the interaction forces along other dividing surfaces. This procedure is similar to known technique proposed by Mozhevitinov (1970) for the case, when the inclination of the interaction force is constant over the whole failure prism.

The above method enables to estimate the slope stability for all failure prism elements in sequence starting from the top one. Moreover, if while determing an interaction force component E_i^{I} (for descending shear surfaces) its value for some prism element becomes negative then it shows that the mass overlaying the given division surface does not need any retaining force (i.e. the part of the prism is stable) In this case the magnitude of the interaction force for the given element is adopted to be equal to zero and the stability of the whole overlaying mass is to be assessed separately.

The proposed method has been used for analysing the stability of slopes formed of heterogeneous soils with weakened surfaces dividing the failure prism. Slope stability factors have been obtained with account made of the effect of the weakened surfaces. Said factors were more exact than those supplied by exsisting methods (Mozhevitinov,1970; Fredlund, 1977).

3. If all three dimensions are considered then stability is never underestimated. If the slope is formed of soils that are heterogeneous as to their deformation and strength properties then it is necessary to consider its actual physical state which makes internal enteraction forces E u Q able to redistribute reactive forces σ_k and τ_k in a various degree between the parts of a heterogeneous shear surface (Fig.5) and thus to increase stability when they concentrate at the areas with most resistant soils, or to decrease it if the redistribution is of a reverse character (Vasiljev,1974).

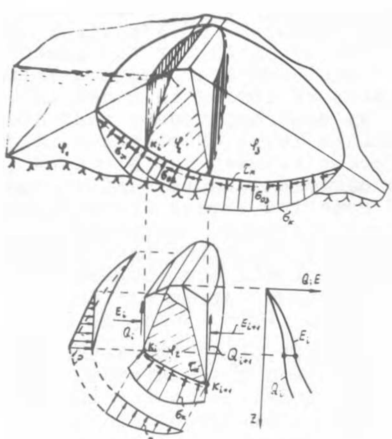

Fig.5. Redistribution of reactive stresses along a heterogeneous shear surface resulting from the effect of internal interaction forces Q and E between units

This method for stability assessment determines the physical state by a tentative analysis of stress-strain behaviour based on elasto-plastic solutions (Bugrov,1978). All this contributes to the plausibility of the results and makes it possible to analyse the effect of soil strength and deformation properties on stability.

The method is grounded on the following assumptions. The shear surface shape is adopted to be in compliance with the kinematical scheme of the virtual displacement of the rigid failure body while the generatrix of the shear surface is chosen so that it is possible to incorporate the failure body in the soil prism with low strength parameters or in the ravine. The most dangerous shear surface shape is then found by varying the values of parameters.

Stability is assessed through comparison of the moments of critical and applied forces determined relative to the given location of axis x'. To determine the moment of critical forces the failure body is divided into rigid units each supported by a part of the shear surface that is homogeneous as to its strength properties. The conditions of limit and actual equilibrium are analysed for each individual unit.

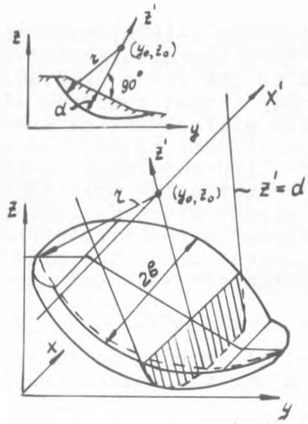

Fig.6. Scheme of the failure body

To save on computer time the interaction forces between units are determined from diagrams and tables of Q and E (Fig.5). Each value of Q and E substitutes for and distributed along the whole soil column resting upon the elementary shear area at intersection of soil strata boundaries.

Limit stress $\tau = \sigma_k tg\varphi + C$ are assumed to be applied to planes normal to the axis while critical stresses $\tau_k = \sigma_k tg\varphi_k + C_k$ to planes normal to the vector of the resultant moment of active forces $M_a = M_{ax} + M_{ay} + M_{az}$. For a given σ_k distribution law, e.g. $\sigma_k = \sigma_0 h \cos\alpha$, with σ_0 as stress intensity coefficient, h as the thickness of a soil layer under a shear area and α as the inclination of the area to axis y the problem boils down to determination of σ_0, φ_k and C_k from the condition of its actual (critical) equilibrium.

By solving the system expressions have been obtained for computing σ_0 in the form $A\sigma_0^2 + B\sigma_0 + D = 0$ with A,B and D as dimension coefficients that depend on the strength parameters, magnitude and direction of active and reactive forces.

Siburua dam landslide stability have been analysed by the above method said landslide having been resulted from floating a part of the slope. The stability have been estimated on the assumption that shear occured along the surface generated by a curve $z' = d(x'/b)^n$. The dimensions of the landslide were known to be 2b=22m and 2b/d=8,5 and soil parameters were $\varphi = 22^o$ and c=3kPa (Wolfskiel,Lambe,1967). The analysis has displayed that limit state of the sliding body occurs when r=9,5m and n=4. Table 1 gives computed results for various widths of the landslide 2b.

Table 1

2b,m	18		22		46	600
n	2	5	2	5	5	plane problem
k	1.05	1.01	1.02	0.99	0.95	0.92

We have also studied the effect of the deformational heterogeneity of soils for a designed dam with a central core 120m high. Mean deformation moduli values have been assumed as follows: for the core 14.8MPa, for the side prisms 40MPa. For the most dangerous shear surface in the case of slowly consolidating core we have obtained K=1.31 while it was K=1.41 for a rapidly consolidating core (Vasiljev,1976).