Deviatoric Stress Strain Theory for Soils

Théorie Effort-Déformation de Distorsion des Sols

E.JUAREZ-BADILLO Research Prof., Graduate School of Eng., Nat. Univ. of Mexico, and Consultant, General Direction of Technical Services, Ministry of Public Works, Mexico

SYNOPSIS A general non linear deviatoric stress-strain theory for soils is presented. Theory relates fundamental stresses to natural effective shear strains. Shearing resistance and deviatoric behaviour are incorporated into one single equation. The theory is applied to the triaxial compression and extension drained tests in normally consolidated clays and to the compression branch of the standard onedimensional consolidation of clays. It is found a relationship among the angle of shearing resistance ϕ, the coefficient of compressibility γ, the coefficient of shear deformability μ and the value of the coefficient K_0 in onedimensional consolidation. Comparison between theory and experimental data on Weald clay is made. Theory anticipates a unique compression and extension deviatoric curve. This is experimentally so up to 50% of the failure deviator stress. Theory duplicates experimental data of the triaxial extension test after 50% of the failure deviator stress. To values up to 50%, the experimental data suggest higher values of the potential angle of shearing resistance at the start of the triaxial tests.

"Nature is nonlinear"

INTRODUCTION

New concepts of deformation, applicable to infinitesimal and finite deformation, have been introduced(Juarez-Badillo-1974a, b). The concept of effective natural shear deformation was presented and the idealized shear in only one direction, only two directions and a particular three dimensional case were analysed. This last case was considered in relation to the compression and extension triaxial tests. Let fig 1 represent a cylinder which is subjected to a compression or to an extension test . Let x be the inclination of a family of parallel planes, covering the whole cylinder, whose shear deformation will be considered. Let us also consider the symmetric planes inclined π-x with respect to the positive x_3 axis. If now these planes undergo an infinitesimal effective shear deformation $d\bar{\eta}$, it was shown that the corresponding deviatoric axial strain de_a is given by

$$de_a = d\bar{\eta} \sin 2x \tag{1}$$

where the upper sign is to be used for the compression test and the lower sign for the extension test.

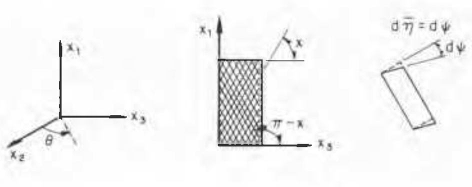


Fig 1 Cylinder in a triaxial test

In actual triaxial tests there are effective shears taking place in all possible planes. In all sets of symmetric planes inclined x, $0 \le x \le \pi/2$, and all $\theta, 0 \le \theta \le \pi$. Consequently, the resultant deviatoric axial strain is postulated to be given by

$$de_a = \mp\int_0^{\pi} d\theta \int_0^{\pi/2} d\bar{\eta} \sin 2x \cdot dx = \mp\pi \int_0^{\pi/2} d\bar{\eta} \sin 2x \cdot dx \tag{2}$$

If $d\varepsilon_1$, $d\varepsilon_2$ and $d\varepsilon_3$ are the total instantaneous longitudinal strains along the principal axes, the instantaneous volumetric strain $d\varepsilon_v$ is given by

$$d\varepsilon_v = d\varepsilon_1 + d\varepsilon_2 + d\varepsilon_3 \tag{3}$$

The instantaneous isotropic component of the instantaneous strain tensor, $d\varepsilon$, is given by

$$d\varepsilon = \frac{d\varepsilon_1 + d\varepsilon_2 + d\varepsilon_3}{3} = \frac{d\varepsilon_v}{3} \tag{4}$$

and the instantaneous deviatoric components will be

$$\begin{aligned} de_1 &= d\varepsilon_1 - d\varepsilon \\ de_2 &= d\varepsilon_2 - d\varepsilon \\ de_3 &= d\varepsilon_3 - d\varepsilon \end{aligned} \tag{5}$$

Note that $de_1 + de_2 + de_3 = 0$.

For standard triaxial tests $d\varepsilon_2 = d\varepsilon_3 = d\varepsilon_r$ (radial) and $d\varepsilon_1 = d\varepsilon_a$ (axial) and integrating the resulting equations we get

$$\varepsilon_v = \varepsilon_a + 2\varepsilon_r \tag{6}$$

$$\varepsilon = \frac{\varepsilon_a + 2\varepsilon_r}{3} = \frac{\varepsilon_v}{3} \tag{7}$$

and

$$\begin{aligned} e_a &= \varepsilon_a - \varepsilon \\ e_r &= \varepsilon_r - \varepsilon = -\tfrac{1}{2} e_a \end{aligned} \tag{8}$$

since $e_a + 2e_r = 0$.

All strains dealt with in this paper are natural strains, that is,

$$\varepsilon_v = \int_{V_o}^{V} \frac{dV}{V} = \ln \frac{V}{V_o} \tag{9}$$

and

$$\varepsilon_a = \int_{x_{10}}^{x_1} \frac{dx_1}{x_1} = \ln \frac{x_1}{x_{10}} \tag{10}$$

where V stands for volume.

Furthermore, the effective general natural shear strain $\bar{\eta}$ corresponding to a fixed direction x in physical space is given by

$$\bar{\eta} = \int_0^{\bar{\eta}} d\bar{\eta} = \int_0^{\psi} (d\psi)_x \tag{11}$$

where $(d\psi)_x$ stands for the infinitesimal change of angle of the normal to the lines ocupying the fixed direction x in physical space.

Volumetric behaviour of clays has been dealt with elsewhere (Juarez-Badillo-1963, 1965, 1969b, 1975). Shearing resistance has already been dealt with as well(Juarez-Badillo 1969a, 1975). Non linear theories were developed in terms of equivalent consolidation pressures and fundamental stresses.

This paper is restricted to deviatoric behaviour of soils and a practical application to normally consolidated clays is made. Delay effects are not considered.

FUNDAMENTAL LAW OF SHEAR BEHAVIOUR

The basic ideas that govern the whole development are:

1. Any distortion(change in form) of a body requires effective shear strains.
2. Effective shear strains uniquely define distortion.
3. Effective shear strains do not produce volumetric change.

Inverse of statements 1 and 2 are not true, that is:

4. Effective shear strains may not produce distortion. The result may be only pure rotation (Juarez-Badillo-1974b).
5. Distortion *does not* define effective shear strains. There may exist infinite spectra of effective shears producing the same distortion. (Juarez-Badillo-1974b).

Let σ_n and τ be the fundamental normal stress and the shearing stress in the horizontal planes of the "sample" of fig 2. Consider the ideal case that only horizontal planes may undergo effective shear strains under a change in stresses. For "normally consolidated" samples fundamental normal stresses are equal to effective normal stresses.

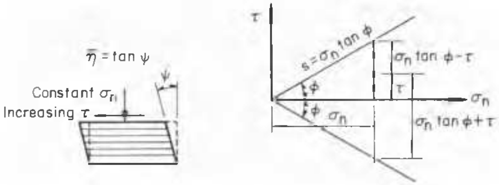


Fig 2 Effective shear in "only" horizontal planes

Another basic idea is:

6. Any change in τ and/or σ_n produces a change in the effective shear $\bar{\eta}$. Except for $\tau=0$ and variable σ_n where $\bar{\eta} = 0$.

For constant σ_n and increasing τ the following stress-strain law is postulated:

If ϕ (assumed constant) is the angle of shearing resistance, then to a change $d\tau$ corresponds a change in effective shear strain $d\bar{\eta}$ that is directly proportional to $\frac{d\tau}{\sigma_n}$ and inversely proportional to $(\tau_{max} - \tau)$, more basically to $(\frac{\tau}{\sigma_n})_{max} - \frac{\tau}{\sigma_n} = \tan\phi - \frac{\tau}{\sigma_n}$. Accordingly, the following relationship is postulated

$$d\bar{\eta} = \mu \frac{1}{\tan\phi - \frac{\tau}{\sigma_n}} \frac{d\tau}{\sigma_n} \tag{12}$$

Multiplying numerator and denominator by $\cot\phi$

$$d\bar{\eta} = \mu \frac{1}{1 - \cot\phi \cdot \frac{\tau}{\sigma_n}} \cot\phi \frac{d\tau}{\sigma_n} \tag{13}$$

In eq 13, $\cot\phi \frac{\tau}{\sigma_n}$ varies between 0 and 1 and furthermore

$$\left[\frac{d\bar{\eta}}{\cot\phi \frac{d\tau}{\sigma_n}}\right]_{\cot\phi \frac{\tau}{\sigma_n} = 0} = \mu \tag{14}$$

The coefficient of proportionality μ will be referred to as the "coefficient of shear deformability" or, briefly, the "shear coefficient". If, only for this section, it is written

$$x = \cot\phi \frac{\tau}{\sigma_n} \tag{15}$$

then eqs 13 and 14 may be written as

$$d\bar{\eta} = \mu \frac{1}{1-x} dx \tag{16}$$

and

$$\left[\frac{d\bar{\eta}}{dx}\right]_{x=0} = \mu \tag{17}$$

For a complete study of relationships of the type of eq 16 the following more general expression will be considered

$$d\bar{\eta} = \mu \left(\frac{1}{1-x}\right)^{\nu} dx \tag{18}$$

where, again, eq 17 holds.

Integration of eq 18 for $\nu = 0$, 1, 2 and 3 provides

For $\nu = 0$ $\quad d\bar{\eta} = \mu\, dx$

and $\quad \bar{\eta} = \mu x \quad$ or $\quad \frac{\bar{\eta}}{\mu} = x \qquad (19)$

For $\nu = 1$ $\quad d\bar{\eta} = \mu \frac{dx}{1-x}$

and $\quad \bar{\eta} = \mu \ln \frac{1}{1-x} \quad$ or $\quad \frac{\bar{\eta}}{\mu} = \ln \frac{1}{1-x} \qquad (20)$

For $\nu = 2$ $\quad d\bar{\eta} = \mu \left(\frac{1}{1-x}\right)^2 dx$

and $\quad \bar{\eta} = \mu \frac{x}{1-x} \quad$ or $\quad \frac{\bar{\eta}}{\mu} = \frac{x}{1-x} \qquad (21)$

For $\nu = 3$ $\quad d\bar{\eta} = \mu \left(\frac{1}{1-x}\right)^3 dx$

and $\quad \bar{\eta} = \mu \frac{x(2-x)}{2(1-x)^2} \quad$ or $\quad \frac{\bar{\eta}}{\mu} = \frac{x(2-x)}{2(1-x)^2} \qquad (22)$

Fig 3 shows the graphs of eqs 19 to 22.

The above integrals and all others in this paper were obtained using Mathematical and Integrals Tables (Peirce, B.O.-1929) and (Hodgman, C.D.-1941).

For both increasing τ and σ_n the following fundamental law of shear behaviour is postulated (compare with eq 13)

$$d\bar{\eta} = \mu \left[\frac{1}{1 - \cot\phi \frac{\tau}{\sigma_n}} \cot\phi \frac{d\tau}{\sigma_n} - \frac{1}{1 + \cot\phi \frac{\tau}{\sigma_n}} \cot\phi \frac{\tau}{\sigma_n} \frac{d\sigma_n}{\sigma_n}\right] \tag{23}$$

For this eq 23 we have again

$$\left[\frac{d\bar{\eta}}{\cot\phi\,\frac{d\tau}{\sigma_n}}\right]_{\cot\phi\,\frac{\tau}{\sigma_n}=0}=\mu \tag{24}$$

It is observed that as $d\left(\frac{\tau}{\sigma_n}\right)=\frac{d\tau}{\sigma_n}-\frac{\tau}{\sigma_n}\frac{d\sigma_n}{\sigma_n}$, then the quantity $-\frac{\tau}{\sigma_n}\frac{d\sigma_n}{\sigma_n}$ that enters in eq 23 is the complement of the total differential of $\frac{\tau}{\sigma_n}$. Note also the similarity between $1-\cot\phi\frac{\tau}{\sigma_n}$ and $1+\cot\phi\frac{\tau}{\sigma_n}$. The first quantity is a measure of the "distance" to failure when such a distance is decreasing (increasing τ) while the second one is the "distance" to the "symmetrical failure condition" when the distance to failure is increasing (increasing σ_n) (See figs 2 and 4).

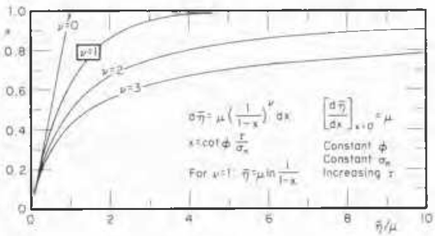


Fig 3. Fundamental law of shear behaviour ($\nu=1$) in relation to other similar expressions

For decreasing τ the quantity $(1-\cot\phi\frac{\tau}{\sigma_n})^{-1}$ should be substituted by $-(1+\cot\phi\frac{\tau}{\sigma_n})^{-1}$. Similarly, for decreasing σ_n the quantity $-(1+\cot\phi\frac{\tau}{\sigma_n})^{-1}$ should be substituted by $(1-\cot\phi\frac{\tau}{\sigma_n})^{-1}$. For normally consolidated soils with increasing σ_n and τ the fundamental normal stress σ_n is equal to the effective normal stress and for drained tests σ_n is equal to the total normal stress.

For a better mathematical appreciation of the resulting stress-strain curves the following expression, corresponding to $\nu=2$, will also be considered

$$d\bar{\eta}=\mu\left[\left(\frac{1}{1-\cot\phi\frac{\tau}{\sigma_n}}\right)^2\cot\phi\frac{d\tau}{\sigma_n}-\left(\frac{1}{1+\cot\phi\frac{\tau}{\sigma_n}}\right)^2\cot\phi\frac{\tau}{\sigma_n}\frac{d\sigma_n}{\sigma_n}\right] \tag{25}$$

For eq 25 we have again that eq 24 holds.

Application of eq 23 to preconsolidated clays will be the subject of another paper.

COMPRESSION AND EXTENSION DRAINED TESTS

In compression drained tests, axial increased, and extension drained tests, radial stress increased, a normally consolidated sample is at every instant in a normally consolidated state if preconsolidation due to delay effects is disregarded. Both types of tests will be analyzed simultaneously. Whenever a double sign appears the upper one will refer to the compression test and the lower one to the extension test. Applying eq 23 to the planes inclined χ it is found (the same is true for the symmetrical planes inclined $\pi-\chi$), with reference to fig 4, and where σ_1 and σ_3 are the major and minor compressive principal stresses.

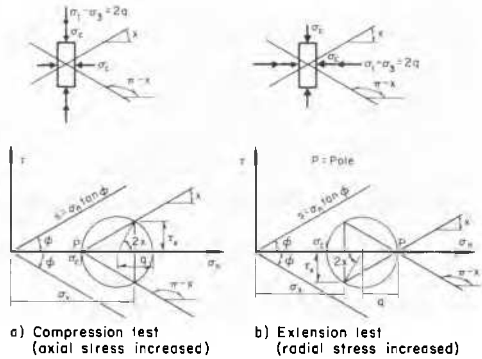


Fig 4. Triaxial drained tests. Normally consolidated clay

If

$$\sigma_1-\sigma_3=\sigma_1-\sigma_c=2q \tag{26}$$

then

$$\tau_x=q\sin 2x\quad\therefore\quad d\tau_x=\sin 2x\,dq$$
$$\sigma_x=\sigma_c+q(1\pm\cos 2x)\quad\therefore\quad d\sigma_x=(1\pm\cos 2x)\,dq \tag{27}$$

and the quantities entering eq 23 are then given by

$$\frac{\tau_x}{\sigma_x}=\frac{q\sin 2x}{\sigma_c+q(1\pm\cos 2x)}=\frac{\sin 2x}{1+(1\pm\cos 2x)\frac{q}{\sigma_c}}\frac{q}{\sigma_c} \tag{28}$$

$$\frac{d\tau_x}{\sigma_x}=\frac{\sin 2x\,dq}{\sigma_c+q(1\pm\cos 2x)}=\frac{\sin 2x}{1+(1\pm\cos 2x)\frac{q}{\sigma_c}}\frac{dq}{\sigma_c} \tag{29}$$

$$\frac{d\sigma_x}{\sigma_x}=\frac{(1\pm\cos 2x)\,dq}{\sigma_c+q(1\pm\cos 2x)}=\frac{1\pm\cos 2x}{1+(1\pm\cos 2x)\frac{q}{\sigma_c}}\frac{dq}{\sigma_c} \tag{30}$$

Using the symbols

$$A=1\pm\cos 2x\qquad B=\cot\phi\sin 2x \tag{31}$$

and

$$q_c=\frac{q}{\sigma_c} \tag{32}$$

Eqs 28 to 30 may be written in the modified form

$$\cot\phi\,\frac{\tau_x}{\sigma_x}=\frac{B}{1+Aq_c}q_c \tag{33}$$

$$\cot\phi\,\frac{d\tau_x}{\sigma_x}=\frac{B}{1+Aq_c}dq_c \tag{34}$$

$$\frac{d\sigma_x}{\sigma_x}=\frac{A}{1+Aq_c}dq_c \tag{35}$$

Introducing eqs 33 to 35 into eq 23 it is obtained

$$d\bar{\eta}_x=\mu\left[\frac{1}{1-\frac{B}{1+Aq_c}q_c}\frac{B}{1+Aq_c}dq_c-\frac{1}{1+\frac{B}{1+Aq_c}q_c}\frac{BA}{(1+Aq_c)^2}q_c\,dq_c\right]$$

Simplifying this equation we get

$$d\bar{\eta}_x=\mu\left[\frac{B\,dq_c}{1+(A-B)q_c}-\frac{AB\,q_c\,dq_c}{[1+(A+B)q_c](1+Aq_c)}\right] \tag{36}$$

Introducing eq 36 into eq 2 we obtain

$$de_a=\mp\pi\mu\int_0^{\pi/2}\left[\frac{B\,dq_c}{1+(A-B)q_c}-\frac{AB\,q_c\,dq_c}{[1+(A+B)q_c](1+Aq_c)}\right]\sin 2x\,dx \tag{37}$$

Integrating eq 37 from $q_c=0$ to $q_c=q_c$ we can write

$$e_a=\mp\pi\mu\int_0^{\pi/2}\sin 2x\left\{\int_0^{q_c}\frac{B\,dq_c}{1+(A-B)q_c}-\int_0^{q_c}\frac{AB\,q_c\,dq_c}{[1+(A+B)q_c](1+Aq_c)}\right\}dx \tag{38}$$

The integrals in q_c of eq 38 are of the form (Peirce, B.O -1929)

$$\int\frac{dx}{a+bx}=\frac{1}{b}\ln(a+bx) \tag{39}$$

and

$$\int \frac{x\,dx}{(a+bx)(a'+b'x)} = \frac{1}{ab'-a'b}\left[\frac{a}{b}\ln(a+bx) - \frac{a'}{b'}\ln(a'+b'x)\right] \qquad (40)$$

Applying eq 39 to the first integral in eq 38

$$\int_0^{q_c} \frac{B\,dq_c}{1+(A-B)q_c} = \frac{B}{A-B}\left[\ln\langle 1+(A-B)q_c\rangle\right]_0^{q_c}$$

then

$$\int_0^{q_c} \frac{B\,dq_c}{1+(A-B)q_c} = \frac{B}{A-B}\ln[1+(A-B)q_c] \qquad (41)$$

Applying eq 40 to the second integral in eq 38

$$\int_0^{q_c} \frac{AB\,q_c\,dq_c}{[1+(A+B)q_c](1+Aq_c)} = \frac{AB}{A-(A+B)}\left[\frac{1}{A+B}\ln\langle 1+(A+B)q_c - \frac{1}{A}\ln(1+Aq_c)\right]_0^{q_c}$$

then

$$\int_0^{q_c} \frac{AB\,q_c\,dq_c}{[1+(A+B)q_c](1+Aq_c)} = -\frac{A}{A+B}\ln[1+(A+B)q_c] + \ln(1+Aq_c) \qquad (42)$$

Introducing eqs 41 and 42 into eq 38

$$e_a = \mp\pi\mu\int \sin 2x\left\{\frac{B}{A-B}\ln[1+(A-B)q_c] + \frac{A}{A+B}\ln[1+(A+B)q_c] - \ln(1+Aq_c)\right\}dx \qquad (43)$$

The last term in this integral is, from eqs 31

$$\int_0^{\pi/2} \sin 2x \ln(1+Aq_c)\,dx = \int_0^{\pi/2} \ln[1+q_c(1\pm\cos 2x)]\sin 2x\,dx \qquad (44)$$

This integral is of the form(Peirce,B.O-1929)

$$\int \ln x\,dx = x\ln x - x \qquad (45)$$

Applying eq 45 to eq 44 we obtain

$$\int_0^{\pi/2} \ln[1+q_c(1\pm\cos 2x)]\sin 2x\,dx = \mp\frac{1}{2q_c}\Big\{[1+q_c(1\pm\cos 2x)]\cdot\ln[1+q_c(1\pm\cos 2x)] - [1+q_c(1\pm\cos 2x)]\Big\}_0^{\pi/2} \qquad (46)$$

For the compression test(upper signs) we then obtain

$$\int_0^{\pi/2} \ln[1+q_c(1+\cos 2x)]\sin 2x\,dx = -\frac{1}{2q_c}\left\{-1-(1+2q_c)\ln(1+2q_c)+1+2q_c\right\}$$

and

$$\int_0^{\pi/2} \ln[1+q_c(1+\cos 2x)]\sin 2x\,dx = \frac{1+2q_c}{2q_c}\ln(1+2q_c) - 1 \qquad (47)$$

For the extension test we similarly obtain,

$$\int_0^{\pi/2} \ln[1+q_c(1-\cos 2x)]\sin 2x\,dx = \frac{1}{2q_c}\left\{(1+2q_c)\ln(1+2q_c)-1-2q_c+1\right\}$$

and

$$\int_0^{\pi/2} \ln[1+q_c(1-\cos 2x)]\sin 2x\,dx = \frac{1+2q_c}{2q_c}\ln(1+2q_c) - 1 \qquad (48)$$

Both integrals have then the same value given by eq 47 or 48. Eq 43 can then be written

$$e_a = \mp\mu\pi I \qquad (49)$$

where, from eqs 31, 43, 47, 48 and 49

$$I = \int_0^{\pi/2} \frac{\cot\phi\sin 2x}{1+\cos 2x-\cot\phi\sin 2x}\sin 2x\ln[1+q_c(1+\cos 2x-\cot\phi\sin 2x)]\,dx + \int_0^{\pi/2} \frac{1+\cos 2x}{1+\cos 2x+\cot\phi\sin 2x}\sin 2x\ln[1+q_c(1+\cos 2x+\cot\phi\sin 2x)]\,dx - \frac{1+2q_c}{2q_c}\ln(1+2q_c)+1 \qquad (50)$$

The integrals in eq 50 have the same value if + cos 2x is substituted by -cos 2x (see also fig 4) and the deviatoric axial natural strain is then given by eqs 49 and 50 for both, compression and extension drained tests, negative in the first case, and positive in the second case.

The author has been unsuccesfull in finding a closed form for the integrals in eq 50. It would be very good to know if they really exist.

Eq 50 can be calculated once for all, for different values of the angle of internal friction ϕ if it is written in a normalized form.

For this purpose we get, from fig 4 and eq 32

$$\sin\phi = \frac{q_{max}}{\sigma_c + q_{max}} = \frac{q_{c\,max}}{1+q_{c\,max}} \qquad (51)$$

then

$$q_{c\,max} = \frac{\sin\phi}{1-\sin\phi} \qquad (52)$$

Variable y is defined by(where eq 52 is used)

$$y = \frac{q_c}{q_{c\,max}} = \frac{1-\sin\phi}{\sin\phi}q_c \qquad (53)$$

then

$$q_c = q_{c\,max}\,y = \frac{\sin\phi}{1-\sin\phi}y \qquad (54)$$

Introducing eq 54 into eq 50 we obtain

$$I = \int_0^{\pi/2} \frac{\cot\phi\sin 2x}{1+\cos 2x-\cot\phi\sin 2x}\sin 2x\ln\left[1+\frac{\sin\phi}{1-\sin\phi}y(1+\cos 2x-\cot\phi\sin 2x)\right]dx + \int_0^{\pi/2} \frac{1+\cos 2x}{1+\cos 2x+\cot\phi\sin 2x}\sin 2x\ln\left[1+\frac{\sin\phi}{1-\sin\phi}y(1+\cos 2x+\cot\phi\sin 2x)\right]dx - \frac{1+2\frac{\sin\phi}{1-\sin\phi}y}{2\frac{\sin\phi}{1-\sin\phi}y}\ln\left[1+2\frac{\sin\phi}{1-\sin\phi}y\right]+1 \qquad (55)$$

Eq 55 shows a type of "symmetry" that facilitates mathematical calculation in a computer. Relative simplification can be gained using the identity $1+\cos 2x = \cot x \sin 2x$. Some of the curves obtained appear in fig 5.

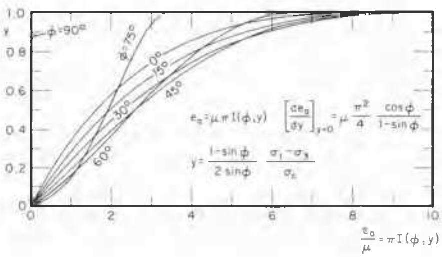


Fig 5 Theoretical deviatoric stress-strain curves for $\nu=1$

Eq 55, for $\phi=0$, reduces to

In the two integrals of eq 55 we have

$$\lim_{\phi\to 0} \frac{\cot\phi\sin 2x}{1+\cos 2x-\cot\phi\sin 2x} = -1 \qquad (56)$$

$$\lim_{\phi\to 0} \frac{1+\cos 2x}{1+\cos 2x+\cot\phi\sin 2x} = 0 \qquad (57)$$

$$\lim_{\phi\to 0} \frac{\sin\phi}{1-\sin\phi}y(1+\cos 2x\mp\cot\phi\sin 2x) = \mp y\sin 2x \qquad (58)$$

In the third term of eq 55, we have

$$\lim_{\phi\to 0} \ln\left[1+2\frac{\sin\phi}{1-\sin\phi}y\right] = 0 \qquad (59)$$

Furthermore, applying L'Hospital's rule

$$\lim_{\phi\to 0} \frac{\ln\left(1+2\frac{\sin\phi}{1-\sin\phi}y\right)}{2\frac{\sin\phi}{1-\sin\phi}y} = \lim_{x\to 0}\frac{\ln(1+xy)}{xy} = \lim_{x\to 0}\frac{\frac{y}{1+xy}}{y} = 1 \qquad (60)$$

Introducing eqs 56 to 60 into eq 55

$$I_{\phi=0} = \int_0^{\pi/2} \sin 2x\ln[1-y\sin 2x]\,dx \qquad (61)$$

Eq 55, for $\phi=90°$, it is evident from eq 23 that

$$I_{\phi=90°} = 0 \qquad (62)$$

Note that $I_{\phi=0}$ given by eq 61 is a "virtual" curve since for $\phi=0$, $q_{c\,max}=0$. Eq 61 has only a mathematical interest.

At the origin, for $q_c=0$ or $y=0$, the "slope" or rate of deviatoric deformability may be cal culated as follows.

From eq 37, for $q_c=0$, it can be written

$$\left[d e_a\right]_{q_c=0} = \mp \pi\mu \int_0^{\pi/2} B\, dq_c \sin 2x\, dx \tag{63}$$

Using eqs 31, then

$$\left[\frac{de_a}{dq_c}\right]_{q_c=0} = -\pi\mu \cot\phi \int_0^{\pi/2} \sin^2 2x\, dx = \mp \pi\mu \cot\phi \left[\frac{\pi}{4}\right] \tag{64}$$

where the value of the integral was obtained from (Peirce, B.O.-1929).

Then

$$\left[\frac{de_a}{dq_c}\right]_{q_c=0} = \mp \frac{\pi^2}{4}\mu \cot\phi \tag{65}$$

In terms of the normalizing variable y, introducing eq 54 into eq 65 it is obtained

$$\left[\frac{de_a}{dy}\right]_{y=0} = \mp \frac{\pi^2}{4}\mu \cot\phi \frac{\sin\phi}{1-\sin\phi}$$

and then

$$\left[\frac{de_a}{dy}\right]_{y=0} = \mp \frac{\pi^2}{4}\mu \frac{\cos\phi}{1-\sin\phi} \tag{66}$$

The curves of fig 5 satisfy eq 66.

Eq 65 can be generalized to include overconsolidated clays. The initial isotropic fundamental pressure is now equal to the initial equivalent consolidation pressure σ_e corresponding to the initial consolidation pressure σ_c. Consequently, if q_e is defined by (compare with eq 32)

$$q_e = \frac{q}{\sigma_e} \tag{67}$$

Eqs 63 to 65 are valid if substitution of q_c by q_e is made, and it can then be written

$$\left[\frac{de_a}{dq_e}\right]_{q_e=0} = \mp \frac{\pi^2}{4}\mu \cot\phi \tag{68}$$

where as already noted, the upper sign is to be used for compression tests and the lower sign for extension tests.

As, from eqs 67 and 32

$$q_e = \frac{q}{\sigma_e} = \frac{q}{\sigma_c}\frac{\sigma_c}{\sigma_e} = q_c \frac{\sigma_c}{\sigma_e} \tag{69}$$

then, introducing eq 69 into eq 68 we obtain

$$\left[\frac{de_a}{dq_c}\right]_{q_c=0} = \mp \frac{\pi^2}{4}\mu \cot\phi \frac{\sigma_c}{\sigma_e} \tag{70}$$

Eq 70 includes eq 65 since for normally consolidated soils $\sigma_e = \sigma_c$. From eqs 26 and 32 it can be written

$$q_c = \frac{q}{\sigma_c} = \frac{1}{2}\frac{\sigma_1 - \sigma_3}{\sigma_c} \tag{71}$$

then eq 70 can be written in the form

$$\left[\frac{de_a}{d\frac{\sigma_1-\sigma_3}{\sigma_c}}\right]_{\frac{\sigma_1-\sigma_3}{\sigma_c}=0} = \mp \frac{\pi^2}{8}\mu \cot\phi \frac{\sigma_c}{\sigma_e} \tag{72}$$

Eq 72 is useful in practice. It gives the "slope" of the deviatoric stress-total axial strain for all types of undrained triaxial tests and for the compression and extension drained tests with J_1= constant ($J_1=\sigma_1+\sigma_2+\sigma_3$= first invariant of the total stress tensor).

For draines tests from eqs 7 and 8, it can be written

$$\frac{de_a}{d\frac{\sigma_1-\sigma_3}{\sigma_c}} = \frac{d\varepsilon_a}{d\frac{\sigma_1-\sigma_3}{\sigma_c}} - \frac{d\varepsilon}{d\frac{\sigma_1-\sigma_3}{\sigma_c}} = \frac{d\varepsilon_a}{d\frac{\sigma_1-\sigma_3}{\sigma_c}} - \frac{1}{3}\frac{d\varepsilon_v}{d\frac{\sigma_1-\sigma_3}{\sigma_c}} \tag{73}$$

At the origin $\left(\frac{\sigma_1-\sigma_3}{\sigma_c}=0\right)$ the volumetric component is given by (Juarez-Badillo-1965, 1969b, 1975)

$$d\varepsilon_v = \frac{dV}{V} = -\gamma \frac{d\sigma_e}{\sigma_e} \tag{74}$$

For normally consolidated soils in increasing σ_c triaxial tests $\sigma_e=\sigma_c$ and

$$\frac{dV}{V} = -\gamma \frac{d\sigma_c}{\sigma_c} \tag{75}$$

For overconsolidated soils in all types of triaxial tests and for normally consolidated soils in decreasing σ_c triaxial tests

$$\frac{dV}{V} = -\gamma \frac{d\sigma_e}{\sigma_e} = -\gamma\rho \frac{d\sigma_c}{\sigma_c} = -\gamma_p \frac{d\sigma_c}{\sigma_c} \tag{76}$$

where γ and γ_p are the compressibility and expansion coefficients respectively and ρ is the expansion-compressibility ratio.

For increasing (+) and decreasing (-) axial stress

$$\frac{d\sigma_c}{\sigma_c} = \pm \frac{1}{3}\frac{d(\sigma_1-\sigma_3)}{\sigma_c} \tag{77}$$

For increasing (+) and decreasing (-) radial stress

$$\frac{d\sigma_c}{\sigma_c} = \pm \frac{2}{3}\frac{d(\sigma_1-\sigma_3)}{\sigma_c} \tag{78}$$

For J_1= constant triaxial tests

$$\frac{d\sigma_c}{\sigma_c} = 0 \tag{79}$$

Introducing eqs 77 to 79 into eqs 75 or 76 the volumetric component at the origin in eq 73 can be found. One gets an expression of the type

$$\frac{d\varepsilon_v}{d\frac{\sigma_1-\sigma_3}{\sigma_c}} = -c\gamma \quad \text{or} \quad -c\gamma_p \tag{80}$$

where c may take the values $0, \pm\frac{1}{3}$ or $\pm\frac{2}{3}$ and γ or γ_p is used according to the type of the considered standard triaxial test.

An application to Weald clay will later be discussed.

Eqs 49 and 55 give the deviatoric axial strain for the compression and extension drained tests using the fundamental law of shear behaviour given by eq 23 into eq 2. If eq 25 ($\nu=2$) is used as the fundamental law instead, the expression obtained from eq 2 is (Appendix)

$$e_a = \mp \mu_2 \pi J \tag{81}$$

where μ_2 is the shear coefficient associated to $\nu=2$ and

$$\begin{aligned}
J = {} & \frac{\sin\phi}{1-\sin\phi} y \int_0^{\pi/2} \frac{\cot\phi \sin 2x}{1+\frac{\sin\phi}{1-\sin\phi} y\left(1+\cos 2x-\cot\phi \sin 2x\right)} \sin 2x\, dx \\
& + \int_0^{\pi/2} \frac{(1+\cos 2x)\cot\phi \sin 2x}{(1+\cos 2x-\cot\phi \sin 2x)^2} \sin 2x \ln\left[1+\frac{\sin 2x}{1-\sin 2x} y\left(1+\cos 2x-\cot\phi \sin 2x\right)\right] dx \\
& - \frac{\sin\phi}{1-\sin\phi} y \int_0^{\pi/2} \frac{(1+\cos 2x)\cot\phi \sin 2x}{\left[1+\cos 2x-\cot\phi \sin 2x\right]\left[1+\frac{\sin\phi}{1-\sin\phi} y\left(1+\cos 2x-\cot\phi \sin 2x\right)\right]} \sin 2x\, dx \\
& + \int_0^{\pi/2} \frac{(1+\cos 2x)\cot\phi \sin 2x}{(1+\cos 2x+\cot\phi \sin 2x)^2} \sin 2x \ln\left[1+\frac{\sin 2x}{1-\sin 2x} y\left(1+\cos 2x+\cot\phi \sin 2x\right)\right] dx \\
& - \frac{\sin\phi}{1-\sin\phi} y \int_0^{\pi/2} \frac{(1+\cos 2x)\cot\phi \sin 2x}{\left[1+\cos 2x+\cot\phi \sin 2x\right]\left[1+\frac{\sin\phi}{1-\sin\phi} y\left(1+\cos 2x+\cot\phi \sin 2x\right)\right]} \sin 2x\, dx
\end{aligned} \tag{82}$$

Compare with eq 55.

Some of the curves obtained appear in fig 6. Compare with fig 5.

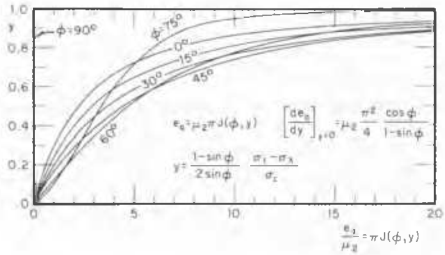


Fig 6 Theoretical deviatoric stress-strain curves for ν=2

In eq 82, for $\phi=0$, all integrals, except the first one, reduce to zero and, using eq 58 for first integral, it is obtained

$$J_{\phi=0} = y \int_0^{\pi/2} \frac{\sin^2 2x}{1 - y \sin 2x}\, dx \tag{83}$$

Eq 82, for $\phi=90°$, from eq 25, it is evident that

$$J_{\phi=90°} = 0 \tag{84}$$

Eq 83, like eq 61, gives also a "virtual curve" since for $\phi=0$, $q_{c\,max}=0$.

At the origin, for $q_c=0$ (eq 54) or $y=0$, the "slope" or rate of deviatoric deformability is given by eqs 65 and 66 and by eqs 70 and 72.

ONEDIMENSIONAL CONSOLIDATION TEST

In the virgin compression branch of the one-dimensional consolidation curve the sample is in a normally consolidated state. In this test to every decrease in height of the sample corresponds a volumetric strain and a distorsional strain as well. In this test $\varepsilon_1=\varepsilon_a$ and $\varepsilon_2=\varepsilon_3=\varepsilon_r=0$ and we have, from eqs 6

$$d\varepsilon_v = \frac{dV}{V} = d\varepsilon_a \tag{85}$$

and then, from eqs 7 and 8

$$de_a = d\varepsilon_a - \frac{1}{3} d\varepsilon_a = \frac{2}{3} d\varepsilon_a \tag{86}$$

Combining eqs 85 and 86 it is obtained

$$d\varepsilon_v = \frac{3}{2} de_a \tag{87}$$

Let σ_v be the vertical normal stress, and $K_0\sigma_v$ (K_0 = constant) be the radial normal stress. From fig 7 it can be written

$$\tau_x = \frac{1-K_0}{2} \sigma_v \sin 2x \tag{88}$$

and $$\sigma_x = \frac{1+K_0}{2}\sigma_v + \frac{1-K_0}{2}\sigma_v \cos 2x$$

$$\therefore\ \sigma_x = \left[\frac{1+K_0}{2} + \frac{1-K_0}{2}\cos 2x\right]\sigma_v \tag{89}$$

and the quantities entering eq 23 are then given by

$$\frac{\tau_x}{\sigma_x} = \frac{(1-K_0)\sin 2x}{1+K_0+(1-K_0)\cos 2x} = \frac{\sin 2x}{K+\cos 2x} \tag{90}$$

$$\frac{d\tau_x}{\sigma_x} = \frac{(1-K_0)\sin 2x}{1+K_0+(1-K_0)\cos 2x}\frac{d\sigma_v}{\sigma_v} = \frac{\sin 2x}{K+\cos 2x}\frac{d\sigma_v}{\sigma_v} \tag{91}$$

$$\frac{d\sigma_x}{\sigma_x} = \frac{d\sigma_v}{\sigma_v} \tag{92}$$

where

$$K = \frac{1+K_0}{1-K_0} \tag{93}$$

Introducing eqs 90 to 92 into eq 23 it is obtained

$$d\bar{\eta}_x = \mu\left[\frac{1}{1-\frac{\cot\phi\sin 2x}{K+\cos 2x}}\frac{\cot\phi\sin 2x}{K+\cos 2x} - \frac{1}{1+\frac{\cot\phi\sin 2x}{K+\cos 2x}}\frac{\cot\phi\sin 2x}{K+\cos 2x}\right]\frac{d\sigma_v}{\sigma_v}$$

Simplifying somewhat this equation we get

$$d\bar{\eta}_x = \mu\left[\frac{\cot\phi\sin 2x}{K+\cos 2x-\cot\phi\sin 2x} - \frac{\cot\phi\sin 2x}{K+\cos 2x+\cot\phi\sin 2x}\right]\frac{d\sigma_v}{\sigma_v} \tag{94}$$

Eq 94 can be written as

$$d\bar{\eta}_x = \mu\frac{2(\cot\phi\sin 2x)^2}{(K+\cos 2x)^2-(\cot\phi\sin 2x)^2}\frac{d\sigma_v}{\sigma_v} \tag{95}$$

Writing $1-\cos^2 2x$ for $\sin^2 2x$

$$d\bar{\eta}_x = 2\mu\frac{\cot^2\phi(1-\cos^2 2x)}{(K+\cos 2x)^2-\cot^2\phi(1-\cos^2 2x)}\frac{d\sigma_v}{\sigma_v} \tag{96}$$

and writing $\operatorname{cosec}^2\phi$ for $1+\cot^2\phi$

$$d\bar{\eta}_x = 2\mu\frac{\cot^2\phi(1-\cos^2 2x)}{K^2-\cot^2\phi+2K\cos 2x+\operatorname{cosec}^2\phi\cos^2 2x}\frac{d\sigma_v}{\sigma_v} \tag{97}$$

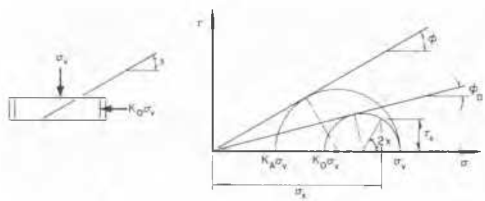
Introducing eq 97 into eq 2 (compression test) we get

$$de_a = \pi\mu\frac{d\sigma_v}{\sigma_v}\int_0^{\pi/2}\frac{\cot^2\phi(1-\cos^2 2x)}{K^2-\cot^2\phi+2K\cos 2x+\operatorname{cosec}^2\phi\cos^2 2x}\, d(\cos 2x) \tag{98}$$

since $-2\sin 2x\, dx = d(\cos 2x)$.

On the other hand, the infinitesimal volumetric strain for this type of test can be expressed by (Juarez-Badillo-1965,1969b and 1975)

$$d\varepsilon_v = \frac{dV}{V} = -\gamma\frac{d\sigma_v}{\sigma_v} \tag{99}$$



$$K_A = \frac{1-\sin\phi}{1+\sin\phi} \qquad K_0 = \frac{1-\sin\phi_0}{1+\sin\phi_0} \qquad n_0 = \frac{\sin\phi_0}{\sin\phi},\ 0<n_0<1$$

Fig 7 Onedimensional consolidation test

Introducing eqs 98 and 99 into eq 87 it is obtained

$$\frac{\gamma}{\mu} = -\frac{3}{2}\pi\int_0^{\pi/2}\frac{\cot^2\phi(1-\cos^2 2x)}{K^2-\cot^2\phi+2K\cos 2x+\operatorname{cosec}^2\phi\cos^2 2x}\, d(\cos 2x) \tag{100}$$

If

$$x' = \cos 2x \tag{101}$$

then eq 100 can be written as

$$\frac{\gamma}{\mu} = \frac{3}{2}\pi\cot^2\phi\int_{-1}^{1}\frac{1-x'^2}{K^2-\cot^2\phi+2Kx'+\operatorname{cosec}^2\phi\, x'^2}\, dx' \tag{102}$$

The integrals in eq 102 are of the form (peirce, B.O).-1929)

$$\int\frac{dx}{X} = \frac{2}{\sqrt{q}}\tan^{-1}\frac{2cx+b}{\sqrt{q}} \tag{103}$$

and

$$\int\frac{x^2}{X}dx = \frac{x}{c} - \frac{b}{2c^2}\ln X + \frac{b^2-2ac}{2c^2}\int\frac{dx}{X} \tag{104}$$

where

$$X = a + bx + cx^2 \tag{105}$$

and

$$q = 4ac - b^2 \tag{106}$$

Combining eqs 103 and 104 we get

$$\int_{-1}^{1}\frac{1-x^2}{X}dx = \left[\frac{b}{2c^2}\ln X + \frac{2c^2-b^2+2ac}{c^2}\frac{1}{\sqrt{q}}\tan^{-1}\frac{2cx+b}{\sqrt{q}} - \frac{x}{c}\right]_{-1}^{1}$$

$$= \frac{b}{2c^2}\ln\frac{a+b+c}{a-b+c} + \frac{2c^2-b^2+2ac}{c^2}\frac{1}{\sqrt{q}}\left(\tan^{-1}\frac{b+2c}{\sqrt{q}} - \tan^{-1}\frac{b-2c}{\sqrt{q}}\right) - \frac{2}{c} \tag{107}$$

But from (Peirce, B.O.-1929)

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy} \tag{108}$$

Applying eq 108 to the corresponding term of eq 107 we get

$$\tan^{-1}\frac{b+2c}{\sqrt{q}} - \tan^{-1}\frac{b-2c}{\sqrt{q}} = \tan^{-1}\frac{\frac{4c}{\sqrt{q}}}{1+\frac{b^2-4c^2}{q}} = \tan^{-1}\frac{4c\sqrt{q}}{q+b^2-4c^2}$$

$$= \tan^{-1}\frac{4c\sqrt{q}}{4ac-4c^2} = \tan^{-1}\frac{\sqrt{q}}{a-c} \tag{109}$$

where eq 106 has been used.
Introducing eq 109 into eq 107 we get

$$\int_{-1}^{1}\frac{1-x^2}{X}dx = \frac{b}{2c^2}\ln\frac{a+b+c}{a-b+c} + \frac{2c^2-b^2+2ac}{c^2}\frac{1}{\sqrt{q}}\tan^{-1}\frac{\sqrt{q}}{a-c} - \frac{2}{c} \tag{110}$$

The quantities in eq 110 from eqs 102, 105 and 106 are

$$\frac{b}{2c^2} = \frac{2K}{2\operatorname{cosec}^4\phi} = K\sin^4\phi \tag{111}$$

$$\frac{a+b+c}{a-b+c} = \frac{K^2-\cot^2\phi+2K+\operatorname{cosec}^2\phi}{K^2-\cot^2\phi-2K+\operatorname{cosec}^2\phi} = \frac{K^2+2K+1}{K^2-2K+1} = \left(\frac{K+1}{K-1}\right)^2 \tag{112}$$

where the following identity has been used

$$1+\cot^2\phi = \operatorname{cosec}^2\phi \tag{113}$$

And (where eq 113 is also used)

$$\frac{2c^2-b^2+2ac}{c^2} = \frac{2\operatorname{cosec}^4\phi-4K^2+2(K^2-\cot^2\phi)\operatorname{cosec}^2\phi}{\operatorname{cosec}^4\phi}$$

$$= \frac{2\operatorname{cosec}^2\phi+2K^2\operatorname{cosec}^2\phi-4K^2}{\operatorname{cosec}^4\phi}$$

$$= 2\sin^2\phi\,(1+K^2-2K^2\sin^2\phi) \tag{114}$$

$$\sqrt{q} = \sqrt{4(K^2-\cot^2\phi)\operatorname{cosec}^2\phi-4K^2}$$

$$= 2\sqrt{K^2(\operatorname{cosec}^2\phi-1)-\cot^2\phi\operatorname{cosec}^2\phi}$$

$$= 2\cot\phi\sqrt{K^2-\operatorname{cosec}^2\phi} \tag{115}$$

where again eq 113 has been used.
Using also eqs 114 and 115 we then have

$$\frac{2c^2-b^2+2ac}{c^2}\frac{1}{\sqrt{q}} = \sin^2\phi\,\frac{1+K^2-2K^2\sin^2\phi}{\cot\phi\sqrt{K^2-\operatorname{cosec}^2\phi}} \tag{116}$$

and (using also eq 113)

$$\frac{\sqrt{q}}{a-c} = \frac{2\cot\phi\sqrt{K^2-\operatorname{cosec}^2\phi}}{K^2-\cot^2\phi-\operatorname{cosec}^2\phi} = \frac{2\cot\phi\sqrt{K^2-\operatorname{cosec}^2\phi}}{1+K^2-2\operatorname{cosec}^2\phi} \tag{117}$$

$$\frac{2}{c} = \frac{2}{\operatorname{cosec}^2\phi} = 2\sin^2\phi \tag{118}$$

Introducing eqs 111,112,116,117 and 118 into eq 110 we get

$$\int_{-1}^{1}\frac{1-x^2}{X}dx = 2\sin^2\phi\left[K\sin^2\phi\ln\frac{K+1}{K-1} + \frac{1+K^2-2K^2\sin^2\phi}{2\cot\phi\sqrt{K^2-\operatorname{cosec}^2\phi}}\tan^{-1}\frac{2\cot\phi\sqrt{K^2-\operatorname{cosec}^2\phi}}{1+K^2-2\operatorname{cosec}^2\phi} - 1\right] \tag{119}$$

introducing eq 119 into eq 102 we get

$$\frac{\gamma}{\mu} = 3\pi\cos^2\phi\left[K\sin^2\phi\ln\frac{K+1}{K-1} + \frac{1+K^2-2K^2\sin^2\phi}{2\cot\phi\sqrt{K^2-\operatorname{cosec}^2\phi}}\tan^{-1}\frac{2\cot\phi\sqrt{K^2-\operatorname{cosec}^2\phi}}{1+K^2-2\operatorname{cosec}^2\phi} - 1\right] \tag{120}$$

where K is given by eq 93, that is, the ratio of the compressibility and shear coefficients, γ/μ, is given by eq 120 in terms of the angle of shearing resistance ϕ and the at rest coef ficient of earth pressure K_o. More bassically, the coefficient K_o is given in an implicit form by eq 120 as a function of the fundamental coefficient ϕ and the ratio of the fundamental coefficients γ/μ.

Coefficient K_o in eq 120 is substituted by a new normalizing parameter n_o defined, from fig 7, by

$$n_o = \frac{\sin\phi_o}{\sin\phi} = \frac{1-K_o}{(1+K_o)\sin\phi} \tag{121}$$

where ϕ_o is defined by

$$K_o = \frac{1-\sin\phi_o}{1+\sin\phi_o} = \frac{1-n_o\sin\phi}{1+n_o\sin\phi} \tag{122}$$

in a similar form as is defined the active pressure coefficient K_A in terms of ϕ

$$K_A = \frac{1-\sin\phi}{1+\sin\phi} \tag{123}$$

Introducing eqs 121 and 122 into eq 93 it is obtained

$$K = \frac{1+K_o}{1-K_o} = \frac{1+\frac{1-\sin\phi_o}{1+\sin\phi_o}}{1-\frac{1-\sin\phi_o}{1+\sin\phi_o}} = \frac{2}{2\sin\phi_o} = \frac{1}{\sin\phi_o}$$

$$\therefore\ \frac{1}{K} = n_o\sin\phi \tag{124}$$

The quantities in eq 120 are then iqual to

$$\frac{K+1}{K-1} = \frac{1+\frac{1}{K}}{1-\frac{1}{K}} = \frac{1+n_o\sin\phi}{1-n_o\sin\phi} \tag{125}$$

$$\frac{1+K^2-2K^2\sin^2\phi}{2\cot\phi\sqrt{K^2-\operatorname{cosec}^2\phi}} = \frac{1+\frac{1}{K^2}-2\sin^2\phi}{2\frac{1}{K}\cot\phi\sqrt{1-\frac{\operatorname{cosec}^2\phi}{K^2}}}$$

$$= \frac{1-2\sin^2\phi+n_o^2\sin^2\phi}{2n_o\sqrt{1-n_o^2}\cos\phi} \tag{126}$$

$$\frac{2\cot\phi\sqrt{K^2-\operatorname{cosec}^2\phi}}{1+K^2-2\operatorname{cosec}^2\phi} = \frac{2\frac{1}{K}\cot\phi\sqrt{1-\frac{\operatorname{cosec}^2\phi}{K^2}}}{1+\frac{1}{K^2}-2\frac{\operatorname{cosec}^2\phi}{K^2}}$$

$$= \frac{2n_o\sqrt{1-n_o^2}\cos\phi}{1-2n_o^2+n_o^2\sin^2\phi} \tag{127}$$

Introducing eqs 124 to 127 into eq 120 we finally get

$$\frac{\gamma}{\mu} = 3\pi\cos^2\phi\left[\frac{\sin\phi}{n_o}\ln\frac{1+n_o\sin\phi}{1-n_o\sin\phi} + \frac{1-2\sin^2\phi+n_o^2\sin^2\phi}{2n_o\sqrt{1-n_o^2}\cos\phi}\tan^{-1}\frac{2n_o\sqrt{1-n_o^2}\cos\phi}{1-2n_o^2+n_o^2\sin^2\phi} - 1\right] \tag{128}$$

Fig 8 shows graphs of K_o as function of n_o for different values of ϕ. Eq 122. This fig also shows the graphs of the empirical relations $K_o = 1-\sin\phi$ and $K_o = 0.95-\sin\phi$. It is noted that $K_o = 1$ for $n_o=0$ and $K_o = K_A$ for $n_o=1$

Fig 9 show graphs of $\frac{\gamma}{\mu} = f(\phi, n_o)$, eq 128. It is noted that in eq 128 as tha argument of $\tan^{-1}$ is, for all values of ϕ different from 90°, an increasing function of n_o from 0 to ∞ and later on an increasing function form $-\infty$ to 0, then the value of $\tan^{-1}$ increases first from o to $\frac{\pi}{2}$ and later on it is to be taken as an increasing function from $\frac{\pi}{2}$ to π.

For $\phi=0$, eq 128 reduces to

$$\left[\frac{\gamma}{\mu}\right]_{\phi=0} = 3\pi\left[\frac{1}{2n_o\sqrt{1-n_o^2}}\tan^{-1}\frac{2n_o\sqrt{1-n_o^2}}{1-2n_o^2} - 1\right] \tag{129}$$

For $\phi=90°$, eq 128 reduces to

$$\left[\frac{\gamma}{\mu}\right]_{\phi=90°} = 0 \tag{130}$$

For $n_o=0$, using L'Hospital's rule in eq 128, it can be shown that

$$\lim_{n_o\to 0}\frac{1}{n_o}\ln\frac{1+n_o\sin\phi}{1-n_o\sin\phi} = 2\sin\phi \tag{131}$$

$$\lim_{n_o\to 0}\frac{1}{2n_o\sqrt{1-n_o^2}\cos\phi}\tan^{-1}\frac{2n_o\sqrt{1-n_o^2}\cos\phi}{1-2n_o^2+n_o^2\sin^2\phi} = 1 \tag{132}$$

Introducing eqs 131 and 132 into eq 128 we get

$$\left[\frac{\gamma}{\mu}\right]_{n_o=0} = 3\pi\cos^2\phi[2\sin^2\phi+(1-2\sin^2\phi)-1]=0 \qquad (133)$$

Finally, for $n_o = 1$, the coefficient of $\tan^{-1}$ tends to ∞ and then we have

$$\left[\frac{\gamma}{\mu}\right]_{n_o=1} = \infty \qquad (134)$$

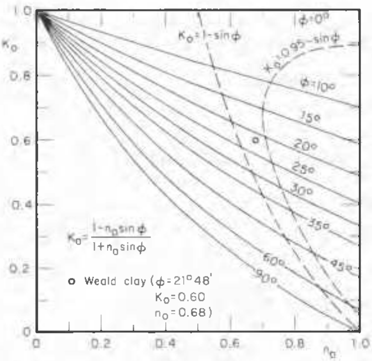


Fig 8 Graphs of $K_o = f(\phi, n_o)$

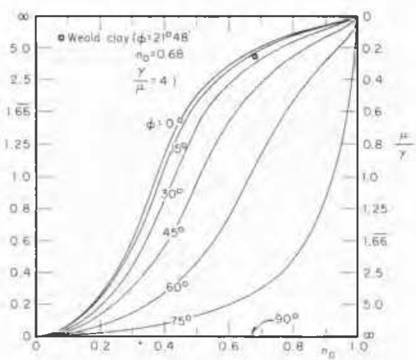


Fig 9 Graphs of $\frac{\gamma}{\mu} = f(\phi, n_o)$

PRACTICAL APPLICATION

The whole theory presented above is now applied to the experimental data of Weald clay. The data of triaxial tests performed at Imperial College, University of London, was kindly made available to the author by J.D. Henkel.

From earlier work, previously mentioned, it was found for Weald clay

$$\begin{aligned} &\phi = 21°48' \quad (\tan\phi = 0.4)\\ &\gamma = 0.06\\ &p = 1/3 \qquad (\gamma_p = 0.02) \end{aligned} \qquad (135)$$

For undrained tests (Henkel J.D. and Sowa V.A.-1963) report a shearing resistance envelope inclined ϕ_u =25.9° and they also report a K_o=0.59 ± 0.02. This angle ϕ_u corresponds to an angle of shearing resistance, with yielding planes at 45° (Juarez-Badillo-1969a), of $\phi=\tan^{-1}(\sin\phi_u)$ = 23.6°. For our value ϕ=21° 48' it is probable a little higher value of K_o. It is then assumed K_o= 0.60.

For ϕ= 21°48' and K_o=0.60 corresponds, from eq 121 and fig 8, a value n_o=0.68. For ϕ=21° 48' and n_o=0.68 corresponds, from eq 128 and fig 9, a value $\frac{\gamma}{\mu} = 4$. From eq 135 we then have $\mu = \frac{0.06}{4} = 0.015$ and we can write for Weald clay

$$\begin{aligned} &K_o = 0.60\\ &n_o = 0.68\\ &\mu = 0.015 \quad (\gamma/\mu = 4) \end{aligned} \qquad (136)$$

Application of eqs 49 and 55 with ϕ=21°48' and μ=0.015 provide the theoretical points (o) shown in fig 10. This fig 10 shows in discontinuous lines the stress-strain curves for drained compression (axial stress increased) and drained extension (radial stress increased) tests in normally consolidated Weald clay. The continuous lines are the corresponding deviatoric curves which were obtained subtracting the isotropic component strain to the total axial strain, eq 8. The isotropic components were obtained from the corresponding curves not included in this paper (Juarez-Badillo-1965, 1969b, 1975). The experimental and theoretical tangents at origin are also noted. Fig 10 also shows, for comparison, the theoretical points obtained from eqs 81 and 82 for ν=2 and using μ_2=0.008. The strength is, from eqs 26, 32, 52 and 135

$$\left(\frac{\sigma_1-\sigma_3}{\sigma_c}\right)_f = \frac{2\sin\phi}{1-\sin\phi} = 1.18 \qquad (137)$$

Experimental values of $\left(\frac{\sigma_1-\sigma_3}{\sigma_c}\right)_f$ were 1.16 and 1.18 for compression and extension drained tests repectively.

Fig 10 shows coincidence of deviatoric compression and extension curves for $\frac{\sigma_1-\sigma_3}{\sigma_c}$ up to 50% of the strength. For higher values the compression test shows higher deviatoric strains. Theoretical points fall on the extension deviatoric curve for $\frac{\sigma_1-\sigma_3}{\sigma_c}$ grater than 50% of the strength and for values of $\frac{\sigma_1-\sigma_3}{\sigma_c}$ up to 50%, theory overestimates deviatoric strains.

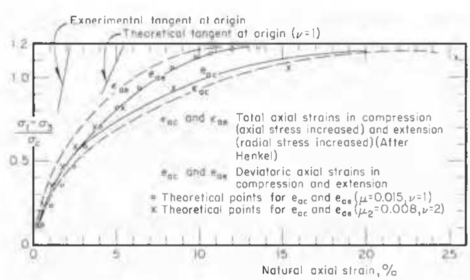


Fig 10 Drained triaxial tests. Normally consolidated Weald clay

Theoretical tangent at origin from eq 72 is

$$\left[\frac{de_a}{d\,\frac{\sigma_1-\sigma_3}{\sigma_c}}\right]_{\frac{\sigma_1-\sigma_3}{\sigma_c}=0} = \frac{\pi^2}{8}\times 0.015\times 2.5 = 4.6\,\% \tag{138}$$

Experimental tangent at origin, from fig 10, is 1.85%. If this value is introduced in eq 72 it is obtained

$$\frac{\pi^2}{8}\mu\cot\phi = 1.85\,\% \tag{139}$$

and for μ=0.015 we get

$$\left[\cot\phi\right]_{origin} = 1 \quad \therefore \quad \left[\phi\right]_{origin} = 45^\circ \tag{140}$$

Eq 72 can be written

$$\left[\frac{de_a}{d\,\frac{\sigma_1-\sigma_3}{\sigma_c}}\right]_{\frac{\sigma_1-\sigma_3}{\sigma_c}=0} \cdot \frac{\sigma_e}{\sigma_c} = \frac{\pi^2}{8}\mu\cot\phi \tag{141}$$

Application of eq 141 to the totality of the stress-strain curves of drained and undrained compression and extension tests is shown in Table I. The symbols used are

$$(\mathring{\varepsilon})_{Theor} = \left[\frac{d\varepsilon}{d\,\frac{\sigma_1-\sigma_3}{\sigma_c}}\right]_{Theoretical} = \frac{1}{3}\left[\frac{d\varepsilon_v}{d\,\frac{\sigma_1-\sigma_3}{\sigma_c}}\right]_{Theor} \tag{142}$$

$$(\mathring{\varepsilon})_{Exp} = \left[\frac{d\varepsilon}{d\,\frac{\sigma_1-\sigma_3}{\sigma_c}}\right]_{Experimental} = \frac{1}{3}\left[\frac{d\varepsilon_v}{d\,\frac{\sigma_1-\sigma_3}{\sigma_c}}\right]_{Exp} \tag{143}$$

$$\mathring{\varepsilon}_a = \frac{d\varepsilon_a}{d\,\frac{\sigma_1-\sigma_3}{\sigma_c}} \tag{144}$$

$$\mathring{e}_a = \frac{de_a}{d\,\frac{\sigma_1-\sigma_3}{\sigma_c}} \tag{145}$$

TABLE I. TRIAXIAL TESTS. DATA AT ORIGIN. WEALD CLAY

TRIAXIAL TESTS		O.C.R. σ_p/σ_c	O.C.F. σ_e/σ_c	$(\mathring{\varepsilon})_{Theor}$ %	$(\mathring{\varepsilon})_{Exp}$ %	$\mathring{\varepsilon}_a$ %	$\mathring{e}_a$ %	$\frac{\sigma_e}{\sigma_c}\mathring{e}_a$ %	Average value %
Drained compression tests	Axial stress increased	1.0	1.0	-0.7	-0.3	-2.0	-1.7	-1.7	-1.5
		1.7	1.5	-0.2	-0	-1.0	-1.0	-1.5	
		2.0	1.6	-0.2	-0	-1.0	-1.0	-1.6	
		2.7	2.0	-0.2	-0	-0.5	-0.5	-1.0	
		4.0	2.6	-0.2	-0	-0.5	-0.5	-1.3	
		8.0	4.1	-0.2	-0	-0.5	-0.5	-2.0	
		12.0	5.2	-0.2	-0	-0.25	-0.25	-1.3	
		24.0	7.0	-0.2	-0	-0.25	-0.25	-1.7	
	Radial stress decreased	1.0	1.0	+0.4	+0.1	-0.8	-0.9	-0.9	-2.0
		1.7	1.5	+0.4	+0.2	-0.8	-1.0	-1.5	
		2.0	1.6	+0.4	+0.2	-0.6	-0.8	-1.3	
		4.0	2.6	+0.4	+0.3	-0.4	-0.7	-1.8	
		8.0	4.1	+0.4	+0.3	-0.3	-0.6	-2.5	
		12.0	5.2	+0.4	+0.3	-0.3	-0.6	-3.1	
		24.0	7.0	+0.4	+0.3	-0.3	-0.6	-4.2	
	J_1 constant	1.0	1.0	0	-0.2	-2.5	-2.3	-2.3	-2.0
		4.0	2.6	0	+0	-0.8	-0.8	-2.1	
		12.0	5.2	0	+0	-1.4	-1.4	-7.3	
Drained extension tests	Radial stress increased	1.0	1.0	-1.3	-0.9	+1.0	+1.9	+1.9	+2.5
		2.0	1.6	-0.4	-0.1	+0.5	+0.6	+1.0	
		4.0	2.6	-0.4	-0.2	+0.5	+0.7	+1.8	
		8.0	4.1	-0.4	-0.1	+0.5	+0.6	+2.5	
		12.0	5.2	-0.4	-0.2	+0.5	+0.7	+3.6	
		24.0	7.0	-0.4	-0.1	+0.5	+0.6	+4.2	
	Axial stress decreased	1.0	1.0	+0.2	+0	+1.0	+1.0	+1.0	+2.0
		1.7	1.5	+0.2	+0	+1.0	+1.0	+1.5	
		2.0	1.6	+0.2	+0.1	+1.0	+0.9	+1.4	
		4.0	2.6	+0.2	+0.1	+1.0	+0.9	+2.3	
		8.0	4.1	+0.2	+0.1	+0.5	+0.4	+1.6	
		12.0	5.2	+0.2	+0.1	+0.5	+0.4	+2.1	
		24.0	7.0	+0.2	+0.1	+0.5	+0.4	+2.8	
	J_1 constant	1.0	1.0	0	-0	+1.2	+1.2	+1.2	+2.0
		4.0	2.6	0	+0	+0.8	+0.8	+2.1	
		12.0	5.2	0	+0.2	+1.0	+0.8	+4.2	
Undrained compression tests	Axial stress increased	1.0	1.0	-	-	-2.0	-2.0	-2.0	-1.5
		1.7	1.5	-	-	-0.5	-0.5	-0.8	
		2.0	1.6	-	-	-0.5	-0.5	-0.8	
		2.7	2.0	-	-	-0.5	-0.5	-1.0	
		4.0	2.6	-	-	-0.5	-0.5	-1.3	
		8.0	4.1	-	-	-0.5	-0.5	-2.0	
		12.0	5.2	-	-	-0.25	-0.25	-1.3	
		24.0	7.0	-	-	-0.5	-0.5	-3.5	
Undrained extension tests	Axial stress decreased	1.0	1.0	-	-	+0.5	+0.5	+0.5	+1.0
		1.7	1.5	-	-	+0.5	+0.5	+0.8	
		2.0	1.6	-	-	+0.5	+0.5	+0.8	
		4.0	2.6	-	-	+0.25	+0.25	+0.7	
		8.0	4.1	-	-	+0.25	+0.25	+1.0	
		12.0	5.2	-	-	+0.25	+0.25	+1.3	
		24.0	7.0	-	-	+0.25	+0.25	+1.7	

Table I shows the overconsolidation ratios $OCR = \frac{\sigma_p}{\sigma_c}$, where σ_p is the preconsolidation pressure, and the corresponding experimental overconsolidation factors $OCF = \frac{\sigma_e}{\sigma_c}$. Alternatively the OCF can be estimated from (Juarez-Badillo-1965, 1969, 1975)

$$OCF = (OCR)^{1-\rho} = (OCR)^{2/3} \tag{146}$$

Theoretical isotropic strain slopes, eq 142, were obtained from eqs 4 and 80

$$(\overset{\circ}{\varepsilon})_{Theor} = \frac{1}{3}\left[\frac{d\varepsilon_v}{d\,\frac{\sigma_1 - \sigma_3}{\sigma_c}}\right]_{Theor} = -\frac{1}{3}\, c\,(\gamma \text{ or } \gamma_p) \tag{147}$$

where

$$c = 0, \pm\frac{1}{3}, \pm\frac{2}{3} \tag{148}$$

depending of the type of drained triaxial test as discussed above. The values obtained using eq 135 appear in Table I.

Experimental isotropic strain slopes, eq 143, we obtained from the corresponding experimental curves. Theoretical values are, on the average, about twice the experimental values. Complete theoretical curves appear in (Juarez-Badillo-1969b).

Total axial strain slopes, eq 144, were obtained from the corresponding experimental curves. Deviatoric axial strain slopes, eq 145, were obtained subtracting the experimental isotropic strain slopes from the total axial strain slopes, eq 73. Finally, the product of the deviatoric axial strain slopes and the corresponding OCF, eq 141, were obtained. Average values for each type of test, discarding the highest and lowest values, were calculated and rounded off to 0.5%. Over all average value for drained tests is 2.0% while for undrained tests is 1.25%. These values support eq 139 and we are forced to conclude that, at the origin, for μ=0.015, eq 140 is true for the totality of stress-strain curves of triaxial tests on Weald clay.

DISCUSSION

The theory developed above and its application to Weald clay bring forward some important points to be elucidated in the future. First, theory anticipates a unique deviatoric curve for drained compression(axial stress increased) and drained extension (radial stress increased) tests on normally consolidated clays. For Weald clay this was experimentally so up to 50% of the failure deviator stress. Later on compression test showed higher strains. Can this difference be explained by some effects, like "anisotropy" and non homogeneity of the clay samples, not considered in the theory? This should be elucidated experimentally. Second, it is clear that, in this approach, parameter ν=1, eqs 20 and 23 and fig 10, for the fundamental law of shear behaviour is the only one to be considered. Third, smaller strains at the start of triaxial tests is a behaviour that requires further experimental evidence before further theoretical efforts are made to explain them.

With respect to the second consideration above we can still add that for ν=0 in eq 17, eq 23 reduces to

$$d\bar{\eta} = \mu \cot\phi\, d\left(\frac{\tau}{\sigma_n}\right) \tag{149}$$

Introducing eq 90, corresponding to the one dimensional consolidation test, into eq 149 we get

$$d\bar{\eta} = \mu \cot\phi\, d\,\frac{\sin 2\chi}{K + \cos 2\chi} = 0 \tag{150}$$

that is, there would not be any distortion and, correspondingly, nor any consolidation.

The experimental fact that K= constant, eq 93 in the onedimensional consolidation test then turns ν=0 as an impossible value for ν in this theoretical approach. Fractional values of ν are also not considered.

Highly desirable are experimental data on the relationship among the angle of shearing resitance ϕ and the compressibility and shear coefficients γ and μ. Does μ depend on ϕ? What is the range of variation of η_o and γ/μ (eq 128)? On this respect from fig 8, for ϕ between 15° and 30°, for K_o between 0.5 and 0.7 and also for K_o between 0.95 -sinϕ and 1-sinϕ, we get values of η_o between 0.6 and 0.75 and from fig 9, for the above intervals of ϕ and η_o, we get values of γ/μ between 2.5 and 5 (μ/γ between 0.2 and 0.4). So we may conclude that most common values of η_o and γ/μ are

$$\eta_o = 0.67 \pm 0.07 \tag{151}$$

$$\frac{\gamma}{\mu} = 4 \pm 1 \tag{152}$$

From eq 152 the compressibility and shear deformability ratio is about 4 and then both coefficients are not independent of each other. How far this is so? Assuming it is so the shear coefficient can be estimated from the compressibility which in turn can be estimated from the liquidlimit w_L from(Juarez-Badillo-1975)

$$\gamma \doteq 0.0016\,(w_L - 10) \tag{153}$$

and introducing eq 153 into eq 152 we get

$$\mu \doteq 0.0004\,(w_L - 10) \tag{154}$$

Wave propagation requires tangent at origin Young's modulus E_o under undrained conditions. From eq 141 using eq 140 we get

$$\left[\frac{d e_a}{d(\sigma_1 - \sigma_3)}\right]_{\sigma_1 - \sigma_3 = 0} = \frac{\pi^2}{8}\,\frac{\mu}{\sigma_e} \tag{155}$$

and then

$$E_o = \left[\frac{d(\sigma_1 - \sigma_3)}{d e_a}\right]_{\sigma_1 - \sigma_3 = 0} = \frac{8}{\pi^2}\,\frac{\sigma_e}{\mu} \tag{156}$$

In terms of σ_c and the overconsolidation factor σ_e/σ_c or the overconsolidation ratio σ_p/σ_c

$$E_o = \frac{8}{\pi^2}\,\frac{\sigma_c}{\mu}\,\frac{\sigma_e}{\sigma_c} \tag{157}$$

and introducing eq 146 into eq 157

$$E_o = \frac{8}{\pi^2}\,\frac{\sigma_c}{\mu}\left(\frac{\sigma_p}{\sigma_c}\right)^{1-\rho} \tag{158}$$

Eq 158 for Weald clay using eqs 135 and 136 would read

$$E_o = 54\,\sigma_c\,(OCR)^{2/3} \tag{159}$$

For undrained tests form Table I it appears that E_o is even grater. For an "Average value" of 1.0% corresponds (compare eqs 155 and 156)

$$E_o = 100\,\sigma_c\,(OCR)^{2/3} \tag{160}$$

Further evidence of the applicability of eq 157 is needed, specially for the difference between drained and undrained tests showed by Table I.

CONCLUSIONS

The most important conclusions and recomendations are as follows:

1. Deviatoric or distortional (change in form)behaviour of soils is the macroscopic result of a complete three dimensional spectrum of infinitesimal effective shears taking place in all possible planes in a flat physical space (compare eqs 1,2,37 and 38).
2. Infinitesimal effective shears are due to a change in shear stress and/or normal fundamental stress.
3. The fundamental law of shear behaviour given by eq 23 is postulated. The shear coefficient μ is presented.
4. Integration of the infinitesimal effective general shear strains give eqs 49 and 55 for the deviatoric axial strains of compression (axial stress increased) and extension(radial stress increased) tests on normally consolidated clays. Theory anticipates a unique curve for both types of tests. This requires further experimental verification. See fig 10.
5. Experimental curves on Weald clay for all types of triaxial tests (compression and extension, drained and undrained, normally consolidated and preconsolidated) show smaller strains at the start of the tests than those predicted by theory. This fact suggest a potential angle of shearing resistance $\phi=45°$ at the start of the tests. This is somewhat disturbing.
6. Eq 128 (fig 9) relates the compressibility shear coefficient ratio $\frac{\gamma}{\mu}$ to the angle of shearing resistance ϕ and the parameter $\eta_o = f(\phi, K_o)$.
7. Experimental evidence indicates strong relationship between the compressibility coefficient γ and the shear coefficient μ: $\gamma \doteq 4\mu$, Eq 152.
8. Tangent at origin Young's modulus E_o for wave propagation purposes given by eq 156 is proposed.

ACKNOWLEDGEMENTS

Deviatoric behaviour research was started by the author while he was at the University of Texas, at Austin, as a Visiting Professor in 1966. Since then the author has been working on this subject with the continued and uninterrupted support of the Faculty of Engineering (Graduate and Research Divisions) of the National University of Mexico and of the Ministry of Public Works.

The author acknowledges to D. J. Henkel the experimental data of Weald clay, to the General Direction of Systems Engineering, Ministry of Public Works, the computations of eqs 55, 82 and 128 (figs 5, 6 and 9) and to his colleague E. Aztegui T. T. the computation of Eq 128 (fig 9).

REFERENCES

Henkel D.J. and Sowa V.A. (1963), *"The Influence of Stress History on Stress Paths in Undrained Triaxial Tests on Clay"*, Laboratory Shear Testing of Soils, ASTM Special Technical Publication N° 361, pp. 280-291.

Hodgman, C.D. (1941), Mathematical Tables from Handbook of Chemistry and Physics, VII Ed., Chemical Rubber Publish. Co., Cleveland, Ohio.

Juarez-Badillo, E. (1963), *"Pore Pressure Functions in Saturated Soils"*, NRC-ASTM Symposium of "Laboratory Shear Testing of Soils", Ottawa, Canada, STP 361, pp. 226-249.

Juarez-Badillo, E. (1965), *"Compressibility of Soils"*, Fifth Symposium of the Civil and Hydraulic Engineering Department on "Behaviour of Soil Under Stress", Indian Institute of Science, Bangalore, India, Vol I, pp. A2/1-35.

Juarez-Badillo, E. (1969a), *Failure Theory for Clays"*, Seventh Int. Conf. on Soil Mechanics and Foundation Engineering, Mexico, Vol. I, pp. 203-213.

Juarez-Badillo, E. (1969b), *"Pore Pressure and Compressibility Theory for Saturated Clays"*, Specialty Session N° 12 on Advances in Consolidation Theories for Clays, University of Waterloo, Canada, pp. 99-116.

Juarez-Badillo, E.(1974a), *"Theory of Natural Deformation"*, First Australian Conference on Engineering Materials, The University of New South Wales, Kensington, Australia, pp. 441-465.

Juarez-Badillo, E. (1974b), *"Natural Shear Deformation"*, First Australian Conference on Engineering Materials, The University of New South Wales, Kensington, Australia, pp. 467-495.

Juarez-Badillo, E. (1975), *"Constitutive Relationships for Soils"*, Symposium on Recent Developments in the Analysis of Soil Behaviour and their Application to Geotechnical Structures, The University of New South Wales, Kensington, Australia, pp. 231-257.

Peirce, B. O. (1929), A Short Table of Integrals, III Revised Ed., Ginn and Co. Boston.

APPENDIX.- *OBTAINMENT OF EQS 81 AND 82*

Eqs 81 and 82 can be obtained as follows
Introducing Eqs 33 to 35 into eq 25 it is obtained

$$d\bar{\eta}_x = \mu_2 \left[\left(\frac{1}{1 - \frac{B}{1+Aq_c} q_c} \right)^2 \frac{B}{1+Aq_c} dq_c - \left(\frac{1}{1 + \frac{B}{1+Aq_c} q_c} \right)^2 \frac{BA}{(1+Aq_c)^2} q_c \, dq_c \right] \qquad \text{(A-1)}$$

Simplifying this equation we get

$$d\bar{\eta}_x = \mu_2 \left\{ \frac{B(1+Aq_c)}{[1+(A-B)q_c]^2} dq_c - \frac{BA}{[1+(A+B)q_c]^2} q_c \, dq_c \right\} \qquad \text{(A-2)}$$

Introducing eq A-2 into eq 2 we obtain

$$de_a \doteq \pi \mu_2 \int_0^{\pi/2} B \left\{ \frac{1+Aq_c}{[1+(A-B)q_c]^2} dq_c - \frac{Aq_c}{[1+(A+B)q_c]^2} dq_c \right\} \sin 2x \, dx \qquad \text{(A-3)}$$

Integrating eq A-3 from $q_c = 0$ to $q_c = q_c$

$$e_a \doteq \pi \mu_2 \int B \sin 2x \left\{ \int_0^{q_c} \frac{(1+Aq_c)\, dq_c}{[1+(A-B)q_c]^2} - \int_0^{q_c} \frac{Aq_c \, dq_c}{[1+(A+B)q_c]^2} \right\} dx \qquad \text{(A-4)}$$

The integrals in q_c of eq A-4 are of the form (Peirce, B.O.-1929)

$$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)} \qquad \text{(A-5)}$$

and

$$\int \frac{x\, dx}{(a+bx)^2} = \frac{1}{b^2} \left[\ln(a+bx) + \frac{a}{a+bx} \right] \qquad \text{(A-6)}$$

Applying eq A-5 to eq A-4 we get

$$\int_0^{q_c} \frac{dq_c}{[1+(A-B)q_c]^2} = \left\{ -\frac{1}{(A-B)[1+(A-B)q_c]} \right\}_0^{q_c} = \left\{ \frac{1}{A-B} - \frac{1}{(A-B)[1+(A-B)q_c]} \right\}$$

$$= \frac{q_c}{1+(A-B)q_c} \qquad \text{(A-7)}$$

Applying eq A-6 to eq A-4 we get

$$\int_0^{q_c}\frac{q_c\,dq_c}{[1+(A-B)q_c]^2}=\frac{1}{(A-B)^2}\left\{\ln[1+(A-B)q_c]+\frac{1}{1+(A-B)q_c}\right\}_0^{q_c}$$

$$=\frac{1}{(A-B)^2}\left\{\ln[1+(A-B)q_c]+\frac{1}{1+(A-B)q_c}-1\right\}$$

$$=\frac{1}{(A-B)^2}\left\{\ln[1+(A-B)q_c]-\frac{(A-B)q_c}{1+(A-B)q_c}\right\} \qquad \text{(A-8)}$$

and, similarly

$$\int_0^{q_c}\frac{q_c\,dq_c}{[1+(A+B)q_c]^2}=\frac{1}{(A+B)^2}\left\{\ln[1+(A+B)q_c]-\frac{(A+B)q_c}{1+(A+B)q_c}\right\} \qquad \text{(A-9)}$$

Introducing eqs A-7 to A-9 into eq A-4

$$e_a=\mp\pi\mu_2\int_0^{\pi/2}B\sin 2x\left\{\frac{q_c}{1+(A-B)q_c}+\frac{A}{(A-B)^2}\left[\ln\langle 1+(A-B)q_c\rangle-\frac{(A-B)q_c}{1+(A-B)q_c}\right]\right.$$

$$\left.-\frac{A}{(A+B)^2}\left[\ln\langle 1+(A+B)q_c\rangle-\frac{(A+B)q_c}{1+(A+B)q_c}\right]\right\}dx \qquad \text{(A-10)}$$

Eq A-10 can be written as

$$e_a=\mp\mu_2\pi J \qquad \text{(A-11)}$$

where, from eqs 31, A-10 and A-11

$$J=\int_0^{\pi/2}\frac{q_c\cot\phi\sin 2x}{1+q_c(1+\cos 2x-\cot\phi\sin 2x)}\sin 2x\,dx$$

$$+\int_0^{\pi/2}\frac{(1+\cos 2x)\cot\phi\sin 2x}{(1+\cos 2x-\cot\phi\sin 2x)^2}\sin 2x\ln[1+q_c(1+\cos 2x-\cot\phi\sin 2x)]\,dx$$

$$-\int_0^{\pi/2}\frac{q_c(1+\cos 2x)\cot\phi\sin 2x}{[1+\cos 2x-\cot\phi\sin 2x][1+q_c(1+\cos 2x-\cot\phi\sin 2x)]}\sin 2x\,dx$$

$$-\int_0^{\pi/2}\frac{(1+\cos 2x)\cot\phi\sin 2x}{(1+\cos 2x+\cot\phi\sin 2x)^2}\sin 2x\ln[1+q_c(1+\cos 2x+\cot\phi\sin 2x)]\,dx$$

$$+\int_0^{\pi/2}\frac{q_c(1+\cos 2x)\cot\phi\sin 2x}{[1+\cos 2x+\cot\phi\sin 2x][1+q_c(1+\cos 2x+\cot\phi\sin 2x)]}\sin 2x\,dx \qquad \text{(A-12)}$$

Introducing the normalyzing parameter Y, eq 53, into eq A-12, eq 82 is obtained.