Variational Approach to Slope Stability

Le Calcul des Variations Appliqué à la Stabilité des Talus

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SYNOPSIS A generalized limit equilibrium approach, based on variational calculus, is applied to slope stability calculations. The problem is defined as a search for the minimum value of the usual factor of safety with respect to strength. No assumptions with respect to the shape of the failure line $y(x)$, or the normal stress distribution along it $\sigma(x)$ are made or implied. Instead, the most dangerous combination of $y(x)$ and $\sigma(x)$ is looked for. The method is exact in the sense that all equations of equilibrium are satisfied.

INTRODUCTION

There are numerous methods currently available for the stability analysis of plane slopes. Most are of the limiting equilibrium (L.E.) type. These methods differ from each other not in their fundamental approach, but rather in the simplifying assumptions used to obtain solutions. It will be shown that these assumptions are not only arbitrary but also unnecessary. A general solution can be based on the L.E. approach using the concept of factor of safety, and the method of variational calculus.

FORMULATION OF THE PROBLEM

A mass of soil is considered to be in a state of L.E. if:

a) Coulomb's yield condition is satisfied along a potential slip surface $y(x)$

$$\tau(x) = c + \sigma(x) \tan \phi \qquad (1)$$

where: $\tau(x)$ and $\sigma(x)$ are the distribution of tangential and normal stress along $y(x)$; c and ϕ are the cohesion and friction angle respectively.

b) The equations of horizontal, vertical, and moment equilibrium are satisfied for the sliding mass

$$\int_{\ell}(\tau\cos\alpha-\sigma\sin\alpha)\,d\ell=0 \qquad (2.1)$$

$$\int_{\ell}(\tau\sin\alpha+\sigma\cos\alpha)\,d\ell-\int_{x_1}^{x_2}\gamma(\bar{y}-y)\,dx=0 \qquad (2.2)$$

$$\int_{\ell}[(\tau\cos\alpha-\sigma\sin\alpha)y-(\tau\sin\alpha+\sigma\cos\alpha)x]\,d\ell-\int_{x_1}^{x_2}\gamma x(\bar{y}-y)\,dx=0 \qquad (2.3)$$

where: $\tan\alpha$ is the slope of the potential slip surface,
ℓ is the arc length along $y(x)$,
γ is unit weight of soil,
x_1, x_2 are the end points of $y(x)$,
$\bar{y}(x)$ is the slope surface (Fig. 1).

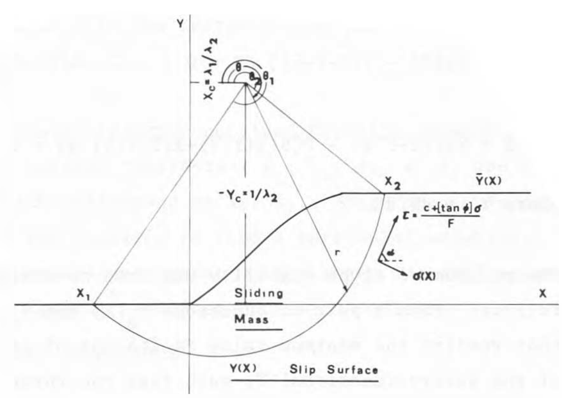

Fig. 1 Slope of uniform soil.

In general there are two ways by which a soil mass of a given geometry can be "brought" to a state of L.E.:

1) by increasing an externally applied load,
2) by adjusting its strength parameters.

The first possibility usually corresponds to a problem of bearing capacity, and has an obvious physical meaning. The second one corresponds to "replacement" of the soil strength parameters by artificial ones $\bar{c}, \overline{\tan\phi}$ for which a state of L.E. is realized.

There are many possible ways by which the strength parameters can be adjusted in order to realize a state of L.E. It is customary, however, to adjust both parameters by a single factor F in the following manner

$$\bar{c} = c/F \quad \ldots \quad (3.1)$$

$$\overline{\tan\phi} = \tan\phi/F \quad \ldots \quad (3.2)$$

The quantity F required to bring a mass of soil to a state of L.E. depends on the choice of $y(x)$ and $\sigma(x)$, and therefore is a functional. This functional is termed the safety - functional, to be distinguished from the factor of safety Fs of a given slope, which is the minimum value of F.

$$Fs = \min F[y(x), \sigma(x)] = F[y_e(x), \sigma_e(x)] \quad \ldots \quad (4)$$

where $y_e(x)$ and $\sigma_e(x)$ are the critical slip surface and critical normal stress distibution, respectively.

It is convenient to introduce the following non-dimensional quantities: $N = c/\gamma H$, $S = \sigma/\gamma H$, $\psi = \tan\phi$, $Y = y/H$, $\bar{Y} = \bar{y}/H$, $X = x/H$ where H is the height of the slope. Substituting Eqs.(3) and (1) into Eqs.(2) and using the non-dimensional quantities defined above, the equations of L.E. reduce to the following

$$\int_{X_1}^{X_2} [(N + S\psi) - FSY']\, dX = 0 \quad \ldots \quad (5.1)$$

$$\int_{X_1}^{X_2} [(N + S\psi)Y' - F(\bar{Y}-Y-S)]\, dX = 0 \quad \ldots \quad (5.2)$$

$$\int_{X_1}^{X_2} [(N + S\psi)(Y-Y'X) - F(S(X+Y'Y)-X(\bar{Y}-Y))]\, dX = 0 \quad (5.3)$$

where $Y' = dY/dX$.

The problem of slope stability can than be stated as follows: Find a pair of functions $Y_e(X)$ and $S_e(X)$ that realize the minimum value Fs (factor of safety) of the safety-functional F, such that the three equations of L.E. (Eqs.5) are satisfied. This problem will be solved using the method of variational calculus. Consequently,the only restrictions placed on the shape of the functions $Y_e(X)$ and $S_e(X)$ are those concerned with continuity and differentiality. No other restrictions are needed for application of the method; nor are any called for by the nature of the problem.

All other L.E. methods contain some arbitrary assumptions with regards to $Y_e(X)$ or $S_e(X)$ or both. For example the classical Taylor method is based on the assumptions that $Y_e(X)$ is a circular arc. In the slice methods that satisfy all conditions of equilibrium (e.g. the Morgenstern - Price, or Janbu methods) $Y_e(X)$ is free of assumptions but $S_e(X)$ is implied by assumptions regarding the direction or line of action of the interslice forces. Similarly, in the variational method presented by Garber (1973) $Y_e(X)$ is derived but $S_e(X)$ is implied by the assumption of a hypothetical failure mechanism. Any assumption with respect to $Y_e(X)$ or $S_e(X)$ must lead to an unconservative estimate of Fs as compared to that obtained from the present approach since they search for min F over a narrow class of functions.

The only attempt, known to the authors,to investigate slope stability without arbitrary assumptions on $Y_e(X)$ or $S_e(X)$ was performed by Kopacsy (1955, 1957, 1961) and recently reiterated by Chen and Snitbhan (1975). In this attempt the minimization was done with respect to the weight of the sliding mass, rather than the safety-functional. The factor of safety was not defined at all; consequently, the equations of L.E. as written by Kopacsy are valid for slopes on the verge of failure only. However, if it is known in advance that a slope is on the verge of failure, no analysis is called for. Therefore Kopacsy's presentation has little practical value. The present analysis was motivated by Kopacsy's analysis and overcomes the inconsistencies of his approach by introducing the factor of safety.

MATHEMATICAL ANALYSIS

A direct application of standard variational techniques to the problem as presented in this work is impossible since the quantity to be minimized, F, appears in all three Eqs.(5). To overcome this difficulty, Eq. (5.1) is rewritten as

$$F = \frac{\int_{X_1}^{X_2} (N + S\psi)\, dX}{\int_{X_1}^{X_2} SY'\, dX} \quad \ldots \quad (6)$$

The problem can therefore be considered as one of finding the minimum value of F given by Eq.(6), subject to the integral constraints Eqs. (5.2) and (5.3). Eqs.(5.2) or (5.3) could have been used for the expression of F and the remaining two equilibrium equations used as constraints without affecting the final results. This is the consequence of the reciprocity principle of variational calculus (Bolza;1973, p.228).

It can be shown (Petrov;1968, p.144) that the functions $Y_e(X)$ and $S_e(X)$ realizing the minimum ratio of the two integrals in Eq.(6) are the same as those that realize the minimum value of the standard functional R, defined by

$$R = \int_{X_1}^{X_2} [(N + S\psi) - FsSY'] \, dX \quad \ldots\ldots\ldots\ldots (7)$$

subject to the condition min R = 0.

It is easy to see that this condition is equivalent to the satisfaction of the equation for horizontal equilibrium for the critical pair $Y_e(X)$ and $S_e(X)$. The remaining two equations of equilibrium, Eqs.(5.2) and (5.3), must be valid for every pair of Y(X) and S(X), including the critical pair $Y_e(X)$ and $S_e(X)$. Therefore the original problem can be transformed to the following standard isoperimetric problem of the calculus of variations: Find the minimum value of the functional R, Eq.(7), subject to the following set of integral constraints

$$\int_{X_1}^{X_2} [(N + S_e\psi) - FsS_eY_e'] \, dX = 0 \quad \ldots\ldots\ldots\ldots (8.1)$$

$$\int_{X_1}^{X_2} [(N + S_e\psi)Y_e' - Fs(\bar{Y}-Y_e-S_e)] \, dX = 0 \quad \ldots\ldots (8.2)$$

$$\int_{X_1}^{X_2} [(N + S_e\psi)(Y_e-Y_e'X) - Fs(S_e(X+Y_e'Y_e)-X(\bar{Y}-Y_e))] \, dX = 0 \quad (8.3)$$

The solution to this problem can be obtained using the method of Lagrange's undetermined multipliers. Consistent with this method an auxiliary function G is introduced as follows

$$G = [(N + S_e\psi) - FsS_eY_e'] + \lambda_1[(N + S_e\psi)Y_e' - Fs(\bar{Y}-Y_e-S_e)] + \lambda_2[(N + S_e\psi)(Y_e-Y_e'X) - Fs(S_e(X+Y_e'Y_e)-X(\bar{Y}-Y_e))] \quad (9)$$

where λ_1,λ_2 are the Lagrange's multipliers.
It can be shown (Petrov;1968, p.40) that the function G may be multiplied by a constant without affecting the final results. This, and the fact that the functions under the integrals in Eqs.(7) and (8.1) are similar, make it possible to use only two Lagrange multipliers, although there are three integral constraints.

The functions $Y_e(X)$ and $S_e(X)$ that constitute the solution to the problem are obtained from Euler's differential equations for the function G. The Euler differential equations are

$$\frac{d}{dX}\left[\frac{\partial G}{\partial S_e'}\right] - \frac{\partial G}{\partial S_e} = 0 \quad \ldots\ldots\ldots\ldots (10.1)$$

$$\frac{d}{dX}\left[\frac{\partial G}{\partial Y_e'}\right] - \frac{\partial G}{\partial Y_e} = 0 \quad \ldots\ldots\ldots\ldots (10.2)$$

Since G is independent of S_e' and dependent on S_e linearly, Eq.(9), the first Euler equation, Eq.(10.1), is a first order differential equation in Y_e only. Solving this equation, and substituting the result into the second Euler equation, Eq.(10.2), a first order differential equation in S_e is obtained. The solutions for Y_e and S_e obtained this way are

$$r_e = A \exp(\psi\theta/Fs) \quad \ldots\ldots\ldots\ldots (11.1)$$

$$S_e = \begin{cases} B\exp(-2\psi\theta/Fs) - \dfrac{AFs(3\psi\cos\theta+Fs\sin\theta)\exp(\psi\theta/Fs)}{Fs^2 + 9\psi^2} - \dfrac{N}{\psi} & (\psi \neq 0) \quad (11.2) \\ B-A\sin\theta-2N\theta/Fs & (\psi = 0) \quad (11.3) \end{cases}$$

where r and θ are polar coordinates that are related to the coordinates X and Y by the transformation

$$X = \frac{\lambda_1}{\lambda_2} - r\cos\theta = X_c - r\cos\theta \quad \ldots\ldots\ldots\ldots (12.1)$$

$$Y = -\frac{1}{\lambda_2} + r\sin\theta = Y_c + r\sin\theta \quad \ldots\ldots\ldots\ldots (12.2)$$

The constants A and B are constants of integration, and (X_c,Y_c) is the center of the polar coordinate system (Fig. 1).

The general solution obtained, Eqs.(11), depends on seven unknown constants: X_c, Y_c, Fs, A, B, and θ_1, θ_2 which correspond to X_1, X_2. Therefore seven equations are required to find a particular solution. These equations consist of the three integral equations, Eqs.(8)*, and four boundary conditions.

BOUNDARY CONDITIONS

The function Y_e has to satisfy the geometrical boundary conditions

$$Y_e(X_1) = \bar{Y}(X_1) \quad \ldots\ldots\ldots\ldots (13.1)$$

$$Y_e(X_2) = \bar{Y}(X_2) \quad \ldots\ldots\ldots\ldots (13.2)$$

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*Substituting the general solution, Eqs.(11), and the coordinate transformation, Eqs.(12), into the integral relations, Eqs.(8), it is possible to execute the integral operations so that these equations become algebraic.

Using the transformation Eqs.(12) and the solution for r_e,Eq.(11.1),it is possible to express these equations in terms of the seven basic unknowns mentioned previously. Two additional boundary conditions are still required. It would be convenient if at the end points of Y_e, the values of the functions Y'_e and σ_e were equal to those obtained graphically from Mohr circle and failure envelope. There are, however, four such conditions (two at each end point), and therefore it is impossible to satisfy them all. This limitation is inherent in the L.E. methods, since in all such methods equilibrium is satisfied only globally, for the whole sliding mass, but not at each point.

Instead of the physical boundary conditions, one may choose the points X_1 and X_2 in such a way as to minimize Fs. This is consistent with the variational approach that seeks a lower bound on Fs. In order to achieve the critical selection of X_1 and X_2, a set of variational boundary conditions, known as the transversarity conditions has to be satisfied. These conditions are written as (Elsgolc;1962, p.75)

$$\left(G-Y'_e\frac{\partial G}{\partial Y'_e}-S'_e\frac{\partial G}{\partial S'_e}\right)\bigg|_{X=X_i}\delta X+\frac{\partial G}{\partial Y'_e}\bigg|_{X=X_i}\delta Y_e+\frac{\partial G}{\partial S'_e}\bigg|_{X=X_i}\delta S_e=0 \quad . . . (14)$$

where X_i is X_1 or X_2 and δ is the variational operator. For the present problem this equation becomes

$$N(\bar{Y}(X_1)+\tan\theta_1)+S_e(\theta_1)[\bar{Y}'(X_1)(\psi-Fs\ \tan\theta_1)+(Fs+\psi\tan\theta_1)]=0 \quad (15.1)$$

$$N(\bar{Y}(X_2)+\tan\theta_2)+S_e(\theta_2)[\bar{Y}'(X_2)(\psi-Fs\ \tan\theta_2)+(Fs+\psi\tan\theta_2)]=0 \quad (15.2)$$

These two equations complete the system of equations required for specification of a particular solution to the problem. The seven equations are the three integral relations, Eqs.(8), two geometrical boundary conditions, Eqs.(13), and two variational boundary conditions, Eqs.(15). By proper substitution it is possible to reduce this system of seven equations in seven unknowns to four non-linear simultaneous algebraic-transcendental equations in four unknowns. This reduced system must be solved numerically, possibly using a computer.

SUMMARY

A generalized limit equilibrium approach is applied to the evaluation of the stability of slopes. It is shown that the critical slope surface has the form of log-spiral, as in the case using limit plasticity. The critical form of the normal stress distribution is derived rather than assumed as is usually done in L.E. calculations. The solution of the problem reduces to the solution of four simultaneous equations. The approach does not utilize any constitutive equation; instead the most conservative estimate of Fs, consistent with Coulomb's yield condition and overall equilibrium is obtained.

REFERENCES

Bolza, O.(1973), "Lectures on the Calculus of Variations", Chelsea Publishing Company, New York.

Chen, W. and Snitbhan N. (1975), "On Slip Surface and Slope Stability Analysis", Soils and Foundations, Japan, Vol. 15, No. 3, pp. 41-49.

Elsgolc, L.(1962), "Calculus of Variations", Pergamon press, London-Paris-Frankfurt.

Garber, M.(1973), "Variational Method for Investigating the Stability of Slopes", Soil Mechanics and Foundation Engineering, New York, Vol. 10, No. 1, pp.77-79.

Kopacsy, J.(1955), " Über die brudkflachen und bruckspannungen in den erdbauter", Gedenkbuck für Prof. Dr. J. Jaky, editor K. Szechy, Budapest, pp.81-99.

Kopacsy, J.(1957), "Three Dimensional Stress Distribution and Slip Surface in Earth Work at Rupture" Proc. of the 4th Int. Conf. on Soil Mech. and Found. Eng.,Vol. I., London, pp.339-342.

Kopacsy, J.(1961), "Distribution des contraintes a la rupture, forme de la surface de glissement et hauteur theorique des talus", Proc. of the 5th Int. Conf. on Soil Mech. and Found. Eng., Vol. II., Paris, pp.641-650.

Petrov, I.(1968), "Variational Methods in Optimum Control Theory", Academic Press, New York and London.