The Calculus of Variations and the Stability of Slopes

Le Calcul des Variations et la Stabilité des Talus

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SYNOPSIS.- This work shows the possibilities of application of the calculus of variations to slope stability analysis.After generalizing the classical Euler equations and the natural and transversality conditions to functionals defined as a quotient of integrals,the determination of the critical sliding line(the one giving the minimun safety factor)is performed very easily. The mathematical statement of the slope stability problem is established with generality and then applied to the case of the Janbu's method obtaining very interesting results for the case of an homogeneous soil and compiling adimensional charts which allow the stability number to be obtained as a function of the geometry of the slope and the strength characteristics of the soil. The method is later generalized to the case of several layers,for which mathematical reasons lead to sliding lines partially coincident with the boundary line between layers.In this case the calculus of variations gives also the solution as it is shown with several examples.

1.- INTRODUCTION

Methods used at the present time for analizing the stability of slopes consist of determinig the factors of safety associated with a series of sliding lines previously defined by the engineer and then obtaining the minimun of these,which is taken as the safety factor of the slope. But due to the fact that this trial procedure does not cover all the possible curves it is obvious that other lines could lead to a smaller factor of safety.However since the number of curves which are actually used in the trial is very high, it can be hoped that the difference between the absolute minimum and the computed one would be small enough not to give serious consequences.

To summarize,the analysis of stability of a slope by limiting equilibrium criteria reduces to the search for the critical slip line in the sense of giving the minimum safety factor,and to the calculation of this value.

The common procedure of searching for this line,which has been previously described,is arduous and becomes terribly cumbersome owing the high number of calculations actually involved. For these and other reason it would be very useful to have a method available to provide this sliding line Without the above shortcomings.

The method presented here meets these requirements allowing a solution of the problem by the use of the calculus of variations theory. The results obtained Show the method to be promising.

2.- MATHEMATICAL BACKGROUND

The functional defining the factor of safety of a slope can be expressed (figure 1) as:

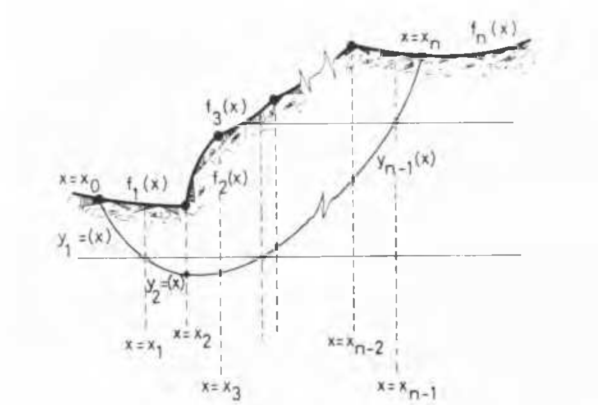

Fig. 1 Analitical definition of the slope

$$F=\frac{\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} F_i(x,y_i,y_i')\,dx}{\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} G_i(x,y_i,y_i')\,dx} \qquad (1)$$

where $y_i=y_i(x)$,$(i=1,2,\ldots,n)$ represents the sliding curve in the interval (x_i,x_{i-1}),and the functions F_i and G_i are related with the shear strength and the actual shear stress of the soil respectively.

Consequently,the determination of the safety factor of a given slope coincides with the problem of determining the minimum value which takes functional(1).

The analysis of this kind of functional is not generally studied in the existent literature then, before applying the theory of the Calculus of Variations to this case, a generalization has been done by the authors with the help of the differentiable applications theory.

It can be proved that the EULER's equations associated to this new problem are:

$$F = \frac{\dfrac{\partial F_i}{\partial y_i} - \dfrac{d}{dx}\left(\dfrac{\partial F_i}{\partial y'_i}\right)}{\dfrac{\partial G_i}{\partial y_i} - \dfrac{d}{dx}\left(\dfrac{\partial G_i}{\partial y'_i}\right)} \quad ; \; (i=1,2,\ldots n) \qquad (2)$$

and in consequence the curve which gives the minimun factor of safety will have to satisfy this equation.
In order to have a well defined problem, or, in other words, a problem with a unique solution, more conditions must be given, because the solution of equation (2) depends on several arbitrary constants.
From the Calculus of Variations theory it is also known that, in order to obtain an extreme for the functional (1) the following natural boundary conditions must be satisfied at the points x_i where there is a change in the analytical expresion of the profile of the slope $(f_i(x))$. These equations are:

$$\left.\frac{\partial F_i}{\partial y'_i} - F\frac{\partial G_i}{\partial y'_i}\right|_{x=x_i} = \left.\frac{\partial F_{i+1}}{\partial y'_{i+1}} - F\frac{\partial G_{i+1}}{\partial y'_{i+1}}\right|_{x=x_i} \qquad (3)$$

In addition the transversality conditions must be satisfied in x_0 and x_n and at the points, x_j, with a change of stratus. In this case transversality equations become:

$$(F_1-FG_1)+(f'_1-y'_1)\left.\left(\frac{\partial F_1}{\partial y'_1} - F\frac{\partial G_1}{\partial y'_1}\right)\right|_{x=x_0} = 0 \qquad (4a)$$

$$(F_n-FG_n)+(f'_n-y'_n)\left.\left(\frac{\partial F_n}{\partial y'_n} - F\frac{\partial G_n}{\partial y'_n}\right)\right|_{x=x_n} = 0 \qquad (4b)$$

$$(F_j-FG_j)+(\varphi'_j-y'_j)\left.\left(\frac{\partial F_j}{\partial y'_j} - F\frac{\partial G_j}{\partial y'_j}\right)\right|_{x=x_j} = \qquad (4c)$$

$$(F_{j+1}-FG_{j+1})+(\varphi'_{j+1}-y'_{j+1})\left.\left(\frac{\partial F_{j+1}}{\partial y'_{j+1}} - F\frac{\partial G_{j+1}}{\partial y'_{j+1}}\right)\right|_{x=x_j}$$

3.- APPLICATION TO THE JANBU'S METHOD

In order to clearly understand the ideas expressed in the previous paragraphs and to show power of the method it will be applied to the Janbu's method.

In this case, the formula which provides the factor of safety, F, corresponding to a given sliding curve and neglecting shearing forces between slices is:

$$F = \frac{\sum_{i=1}^{n} (c+\Delta w_i \, tg\phi)\,\Delta x_i\,(1+tg^2\beta_i)\left(1+\frac{tg\phi\, tg\beta_i}{F}\right)}{\sum_{i=1}^{n} \Delta w_i \;\; tg\,\beta_i} \qquad (5)$$

where Δw_i is the weigth of the i-th slice; β_i the inclinations of the sliding curve; Δx_i the width of the i-th slice; c the cohesion, ϕ the angle of internal friction and γ the unit weigth of the soil.

If the width of the slices reduce to zero the following equations for F_i and G_i are obtained:

$$F_i = \frac{\left[c+(y_i-f_i)\,\gamma\, tg\,\phi\right]\;(1+y_i'^2)}{1+\dfrac{tg\,\phi\,\cdot\,y'_i}{F}} \qquad (6)$$

$$G_i = \gamma(y_i - f_i)\, y'_i \qquad (7)$$

3.1.- HOMOGENEOUS SOIL

Let us consider an homogeneous soil(figure 2)

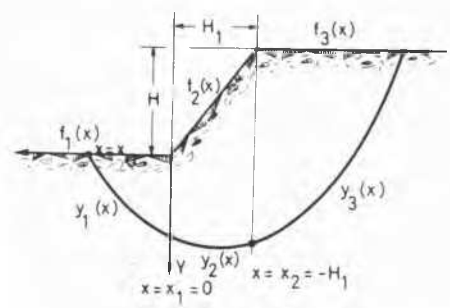

Fig. 2.- Analitical definition of a rectilinear slope.

The functions defining the slope profile are:

$$f_1\;(x) = 0 \qquad x > 0$$

$$f_2\;(x) = \frac{H}{H_1}\;x \qquad -H_1 < x < 0 \qquad (8)$$

$$f_3\;(x) = -H \qquad x < -H_1$$

Let us assume also a continuous sliding line:

$$y_i\;(x_i) = y_{i-1}\;(x_i) \qquad (0 \neq i \neq n) \qquad (9)$$

which together with condition (3) gives:

$$y'_i\;(x_i) = y'_{i-1}\;(x_i) \qquad (10)$$

at the points where there is not a change of stratus.

Due to the tact that equations (2) are of second order the number of equations and unknowns are as follows:

a) <u>Deep sliding line (figure 3)</u>

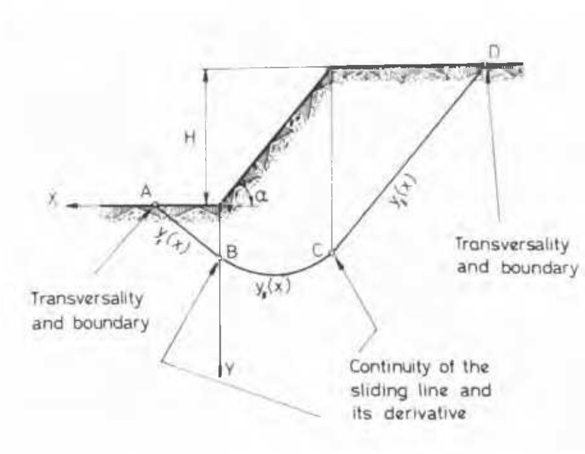

Fig. 3.- Equations defining the problem of a deep line

<u>Equations</u>

2.- Transversality conditions (points A and D)
2.- Boundary conditions (points A and D)
2.- Continuity of y_i (points B and C)
2.- Continuity of y_i' (points B and C)
1.- Equation defining F (equation (1))

<u>Unknowns</u>

6.- Constans of integration
2.- Abscisses of A and D
1.- Safety factor F

b) <u>Lines passing through the toe of the slope (figure 4)</u>

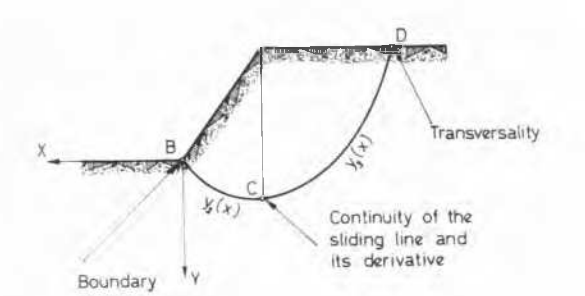

Fig. 4.- Equations defining the problem of a line passing through the toe

<u>Equations</u>

1.- Boundary condition (point B)
1.- Continuity of y_i (point C)
1.- Continuity of y_i' (point C)
1.- Transversality (point D)
1.- Boundary condition (point D)
1.- Equation defining F (equation (1))

<u>Unknowns</u>

4.- Constants of integration
1.- Abscissa of the point D

1.- Safety factor F

3.1.1.- SOIL WHITHOUT FRICTION

3.1.1.1.- <u>Deep sliding line</u>

The Euler equations in this case are:

$$F = -\frac{2\,c\,y_i''}{\gamma\, f_i'} \qquad (i = 1,2,3) \tag{11}$$

which upon integration and taking into account that

$$f_1'(x) = f_3'(x) = 0 \tag{12}$$

leads to

$$\begin{aligned} y_1 &= B_1\,x + D & x_0 \geq x > 0 \\ y_2 &= -\frac{x^2}{4NH_1} + B_2\,x + D_2 & 0 \geq x \geq -H_1 \\ y_3 &= B_3\,x + D_3 & -H_1 > x \geq x_3 \end{aligned} \tag{13}$$

where

$$N = \frac{c}{F\,\gamma\,H} \tag{14}$$

is the stability number.

Making use of the first 8 equations listed be fore results

$$\begin{aligned} B_1 &= B_2 = -1 \\ B_3 &= 1 \\ D_1 &= D_2 = x_0 \\ D_3 &= -H_1 - H - x_0 \\ N &= 1/4 \end{aligned} \tag{15}$$

and entering these values in equation (1) finally results:

$$N = \frac{-H_1 - 3H - 6x_0}{-8H_1 - 12H - 24x_0} \tag{16}$$

The compatibility of this equation and N=1/4 leads to $x_0 \to \infty$, which means a deep line.

3.1.1.2.- <u>Sliding line passing through the toe</u>

An analogous procedure for this new case leads to

$$\frac{x_3}{H} = \frac{-\left(1+\frac{3}{H/H_1}\right) - \sqrt{\left(1+\frac{3}{H/H_1}\right)^2 + 8}}{4}$$

$$N = \frac{1}{3 - 3H/H_1 + \sqrt{(H/H_1+3)^2 + 8(H/H_1)^2}}$$

Figure 5 shows the value of x_3 as a function of the inclination of the slope and figure 6, the critical slip lines in adimensional form.

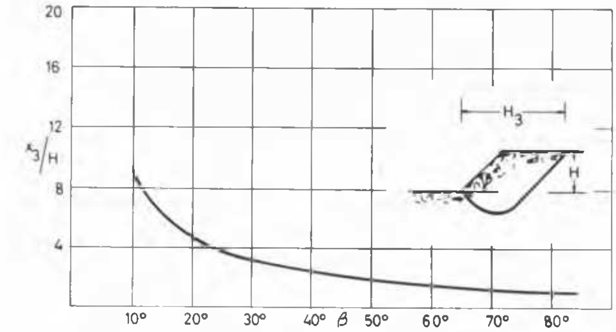

Fig.5.-Values of x_3 versus slope angle B

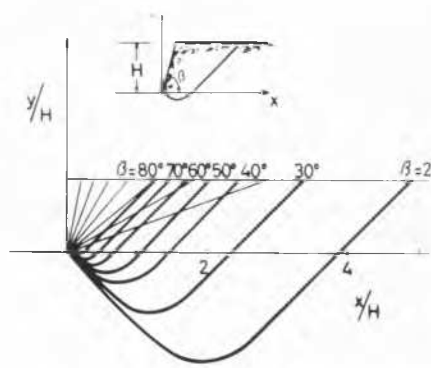

Fig.6.-Critical slip lines in adimensional form

3.1.2.- GENERAL SOIL

In this case the Euler equations are:

$$y_i'' = \frac{\gamma F\left[-A^2 y_i'^3 + A(2A^2-1)y_i'^2 + A\ P\ y_i' + Q\right]}{(2CA^2 - 2\gamma A^3 F f_i - 2\gamma A F f_i + 2C) + (2\gamma A^3 F + 2\gamma A F)y_i}$$

$(i = 1,2,3)$

where

$$A = \frac{tg\ \phi}{F}$$

$$P = 3A - A^2\ f_i' - f_i'$$

$$Q = A - A^2\ f_i' - f_i'$$

which must be integrated numerically. The results of this integration have been shown in figure 7 where the stability number N is given as a function of the inclination of the slope and the adimensional parameter M.

$$M = \frac{Htg\phi}{c}$$

If the point representing the slope is in zone I, the sliding line is a deep line, and, if in zone II, the sliding line passes through the toe of the slope

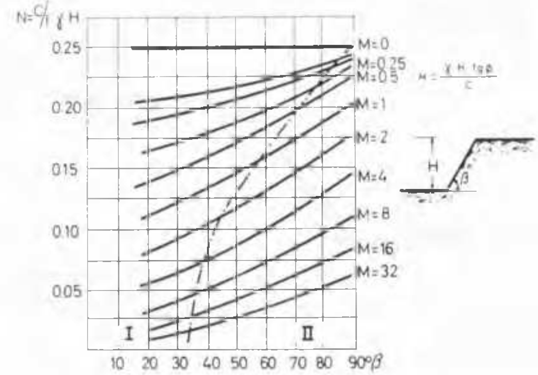

Fig.7.- Stability number versus slope angle

Figure 8 is a comparison with Taylor results. As it could be expected, the factors of safety obtained by the variational method are smaller, and in some cases the difference is large. Another important fact, for the case of ϕ=0, is that the slip lines do not pass through the toe of the slope if the slope angle is not 90°, which is in disagreement with Taylor's results.

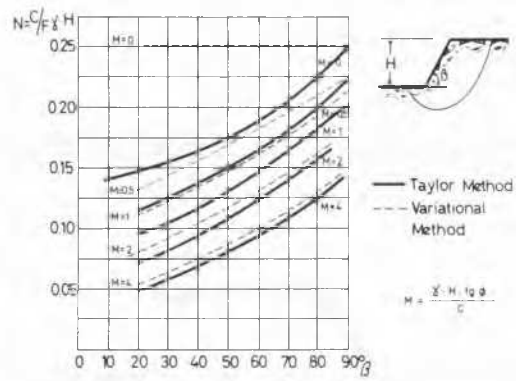

Fig.8.- Comparison with Taylor results

3.2.- SLOPE WITH A HARD STRATUM BELOW THE TOE

If there is a hard stratum below the toe of the slope the sliding line will be partially coincident with it (figure 9).

If the angle of internal friction of the soil is zero it always exists a solution of this type because the critical sliding line is deep. For this case the expresion of the safety factor becomes:

$$F = \frac{\int_{x_0}^{0} F_1 dx + \int_{0}^{x_1} F_2 dx + \int_{x_1}^{x_2} F_3 dx + \int_{x_2}^{-H_1} F_4 dx + \int_{-H_1}^{x_3} F_5 dx}{\int_{x_0}^{0} G_1 dx + \int_{0}^{x_1} G_2 dx + \int_{x_1}^{x_2} G_3 dx + \int_{x_2}^{-H_1} G_4 dx + \int_{-H_1}^{x_3} G_5 dx}$$

where

$$F_i = C(1+y_i'^2) \qquad (i=1,2,\ldots,5)$$
$$G_i = (y_i-f_i)\cdot y_i'$$

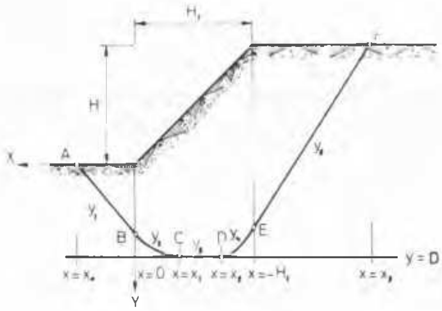

Fig.9.- Sliding line partially coincident - with the boundary of a hard stratum

Making use of the following equations:

a) Euler equations
b) $y_3(x) = D$
c) Transversality conditions in A and F
d) Natural boundary conditions in B and E
e) Continuity of $y_i(x)$ in B and E
f) $y'(x) = 0$ in C and D
g) $y(x) = D$ in C and D
h) The definition of F (equation (1))

The following results are obtained:

$$N = \frac{(2H+H_1+4D) - \sqrt{(2H+H_1+4D)^2 - \frac{16}{3}H_1\left(\frac{H}{2}+D\right)}}{\frac{8}{3}H_1}$$

which shows $N \rightarrow 1/4$ if $D \rightarrow \infty$ as it could be expected.

Figure 10 shows the stability number as a function of the slope angle and the relative depth of the hard stratum, and figure 11 - shows an example of sliding lines of this type.

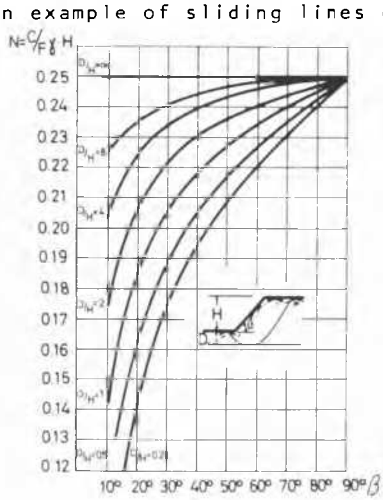

Fig.10.- Influence of a hard stratum in the stability number

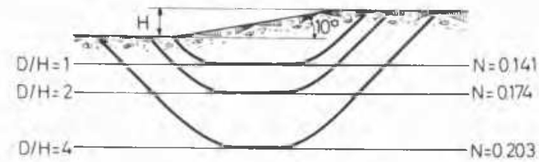

Fig.11.- Critical lines partially coincident with a hard stratum

3.3.- CASE OF SEVERAL STRATA

The same procedure can be followed for the case of several strata making use of the transversality conditions where there is a change of strata.

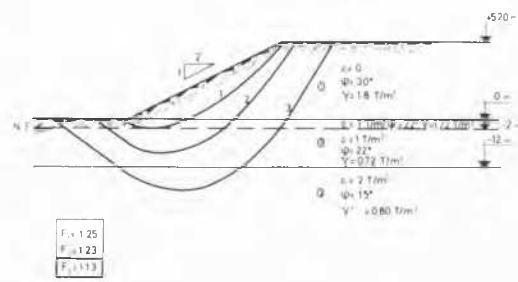

Fig 12.- Example 1

Figure 12 shows one example of horizontal - stratification. As it can be seen the two different kind of lines studied give relative minima, but line 3 gives the absolute mini-- mun for the safety factor F.

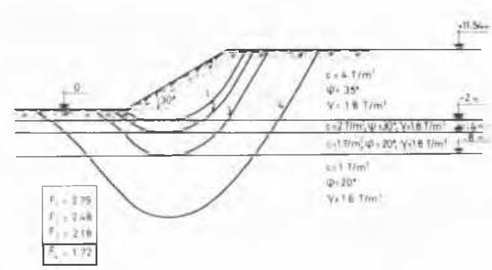

Fig 13.- Example 2

Figure 13 shows other example similar to the preceeding one but in this case the potency of intermediate strata is not sufficient to allow a free line to develope enterely betwe en strata. It is interesting to observe the concavity and convexity of the sliding lines on the right and left sides respectively and the changes in the first derivative at the points where a change of stratus exists. The concavity and convexity would be contrary if the soil would have been strength increasing instead of decreasing with depth.

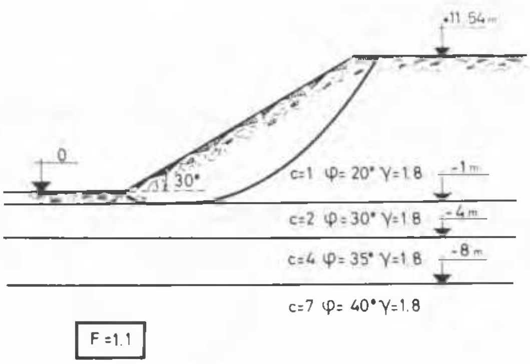

Fig. 14.- Example 3

Figure 14 shows one case of sthength increa sing with depth. This fact leads to a unique sliding line shown in the figure.

Finally figures 15 and 16 show the effect of a soft stratus in the stability of a slope. Figure 15 shows the critical line for an ho mogeneus soil having a safety factor of 1.04, and the figure 16 shows the same soil with a soft layer, In this case two lines give a relative minimun for the safety fac tor, line 1, with a value of 1.07 (greater than 1.04 as it must be expected) and line 2 with a safety factor 0.91 which is the factor of safety of the slope

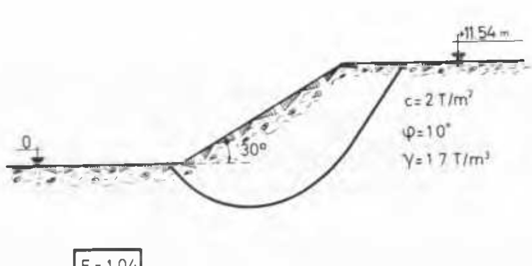

Fig. 15.- Example 4.

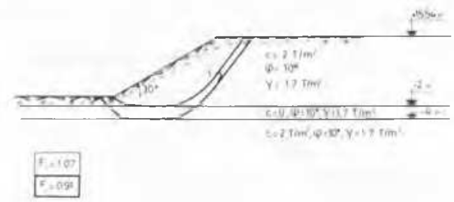

Fig. 16.- Example 4.

4.- CONCLUSIONS

1.- The mathematical theory of the calculus of variations allows a direct determination of the critical slip line of a given slope. Euler's equations together with the transversality, continuity and boundary conditions are used in order to solve the problem.

2.- The only assumption for the sliding line used in the new method is that of been continuous.

3.- In the case of a slope with several strata it is possible the existence of several relative minima. In some cases the sliding lines are partially coincident with the boundary between strata.

4.- Application of the calculus of variations to the Janbu's methods give excelents results as it has been shown in the text.

5.- AKNOWLEDGEMENTS

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6.- REFERENCES

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