Modelling Soil Behaviour under Cyclic Loading

Un Modèle du Comportement du Sol Soumis à la Charge Cyclique

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SYNOPSIS A comprehensive model has been developed to describe the stress-strain behaviour of overconsolidated soil, Pender (1977). This paper is concerned with the extension of the model to cover the response of soil to small strain cyclic loading. The model continues to be based on the critical state theory of soil behaviour and the idea that a constitutive relation for a work-hardening plastic material is appropriate for the calculation of the non-recoverable strains. Important objectives in the development of the model are (i) the minimization of the number of parameters needed to characterise a particular soil, and (ii) that values for these parameters should be obtainable from routine tests. It is concluded that the variation of apparent shear modulus and equivalent viscous damping ratio with cyclic strain amplitude can be represented most effectively. The model requires five parameters to characterise a given material.

INTRODUCTION

The behaviour of soil under cyclic loading is of great importance in many aspects of civil engineering, e.g. the response to small strain amplitude cyclic loading such as is of relevance to soil-structure interaction under earthquake loading.

Although mathematical models for the static stress-strain behaviour of soil are becoming increasingly common, as yet models for cyclic behaviour are not so well developed. The object of this paper is to investigate the possibility of extending a model initially developed for the static behaviour of overconsolidated soil, Pender (1973) and (1977), to cyclic loading.

A convenient starting point for the model for static behaviour was the idea that all distortion is irrecoverable. The model developed from this predicts that the apparent shear modulus at very small strain amplitudes is very large, tending to infinity as the strain amplitude tends to zero. There is a large body of experimental evidence showing that at very small strain amplitudes (less than 10^{-3}%) soil exhibits an elastic shear modulus, e.g. Richart (1975), Seed and Idriss (1970), and Taylor and Parton (1973). The magnitude of this shear modulus depends on the density of the soil, temperature, soil type, stress history etc. It is most conveniently determined by the measurement of the in situ shear wave velocity, Richart (1975). In this paper the stress-strain model is extended to include the small shear strain elastic behaviour. It is then found possible to model the behaviour of the apparent shear modulus and the equivalent viscous damping ratio over a wide range of strain amplitudes. The number of parameters needed to characterise a given material increases from four (static behaviour) to five. Values for four of these five parameters are determined from routine laboratory tests in such a way that the effect of sample disturbance is not critical.

The concept that the yield locus for the behaviour of overconsolidated materials is reduced to a straight line and follows a type of kinematic hardening (in that the current yield locus is always attached to the current stress point) provides a mechanism for modelling reversed and repeated loading in a simple yet realistic manner. Presented in this paper is a model for small strain amplitude cyclic loading developed from a model for large strain static behaviour. Thus it is possible to look at a wide range of soil behaviour from one viewpoint rather than adopt one model and set of material parameters for static behaviour and a different model and parameters for cyclic behaviour. Furthermore the obvious economy in the amount of data required to make these calculations has important implications for the practical application of the model.

OUTLINE OF THE STRESS-STRAIN MODEL FOR OVERCONSOLIDATED SOIL

The following four subsections give a very brief outline of the model as developed for static loading with the conventional triaxial apparatus; detailed discussion is given by Pender (1973) and (1977).

(a) The model is based on the critical state theory of soil behaviour. Critical state concepts are explained by Schofield and Wroth (1968). These concepts are assumed to provide a valid idealisation within which to interpret the failure and prefailure behaviour of soil. This idealisation assumes that when loaded, soil approaches a failure state at which the stress ratio reaches a constant value, whilst unlimited distortion occurs with no further change in effective stress or volume. The conditions at the critical state are given by:

$$q = Mp; \quad dq = dp = dv = 0; \quad d\varepsilon >> 0 \qquad \ldots\ldots\ldots\ldots(1)$$

where: q is the principal effective stress difference, $(\sigma_1' - \sigma_3')$
p is the mean principal effective stress, $(\sigma_1' + 2\sigma_3')/3$
dv is the volumetric strain increment, $(d\varepsilon_1 + 2d\varepsilon_3)$
$d\varepsilon$ is the distortion increment, $(d\varepsilon_1 - dv/3)$
M is the stress ratio, q/p, at the critical state.

The present model differs from the previous critical state models for soil behaviour, Cam-clay explained by Schofield and Wroth (1968), and modified Cam-clay Roscoe and Burland (1968). Those models provide realistic predictions of stress-strain behaviour only for stress paths on the state boundary surface (SBS). By contrast the present model gives predictions for the whole range of overconsolidated behaviour beneath the SBS.

(b) There is no recoverable shear strain. The only recoverable strain increment is volumetric and is given by:

$$dv^r = \frac{\kappa dp}{p\ (1+e)} \qquad \ldots\ldots\ldots\ldots (2)$$

where: κ is the slope of the line in the e, ln p plane for swelling under spherical stress conditions
e is the void ratio of the soil.

(c) The distinctive feature of the model is the recognition that overconsolidated soil can experience plastic strains - both shear and volumetric. A general form of constitutive relation for incremental plastic strain is given by Hill (1950):

$$d\varepsilon^p_{ij} = h \frac{\partial g}{\partial \sigma_{ij}} df \qquad \ldots\ldots\ldots\ldots (3)$$

where: $d\varepsilon^p_{ij}$ is the plastic strain increment tensor
σ_{ij} is the effective stress tensor
h is the hardening function
g is the plastic potential
and df is the differential of the function f which defines the yield locus.

Three hypotheses are introduced which enable the functions f, g and h in equation (3) to be determined. These are:

(i) Overconsolidated soil experiences plastic strain when, and only when, there is a change in the stress ratio q/p (denoted herein by η). Furthermore beneath the state boundary surface constant stress ratio lines are yield loci. Thus the function f specifying a particular constant stress ratio yield locus is given by:

$$f = q - \eta_i p = 0 \qquad \ldots\ldots\ldots\ldots (4)$$

where: η_i is the stress ratio defining a particular constant stress ratio yield locus, e.g. OA in Figure 1.

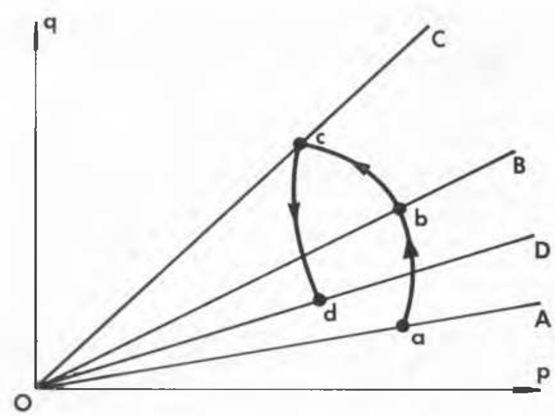

Fig. 1 Successive constant stress ratio yield loci OA, OB, OC and OD for the stress path abcd.

(ii) The undrained stress paths are parabolic in the q, p plane. The expression adopted is:

$$\left(\frac{\eta - \eta_o}{AM - \eta_o}\right)^2 = \frac{p_{cs}}{p}\left\{\frac{1 - p_o/p}{1 - p_o/p_{cs}}\right\} \ldots\ldots\ldots (5)$$

where: p_o is the value of p at the start of the undrained path
η_o is the stress ratio at the start of the undrained path
p_{cs} is the value of p on the critical state line corresponding to the current void ratio
A is +1 for loading in compression and -1 for loading in extension.

As is evident in Figure 2 equation (5) generates a family of undrained stress paths all of which are directed towards the critical state. It is assumed that the location of the critical state line in the stress space for extension is a mirror image of its location for compression.

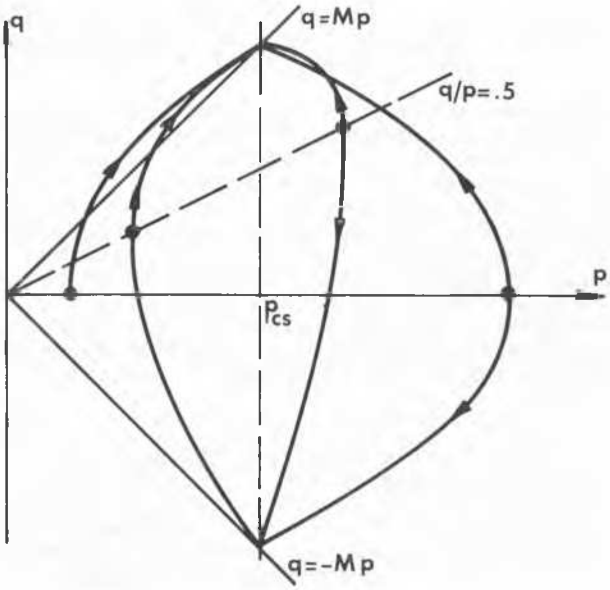

Fig. 2 Undrained stress paths for η_o = 0.0 and 0.5 calculated with equation (5).

(iii) An expression is adopted for the ratio of the plastic shear strain increment and plastic volumetric strain increment, i.e., $d\varepsilon^P/dv^P$. This is such that wet of critical the volumetric plastic strain is compressive and dry of critical it is dilatant. This expression for $d\varepsilon^P/dv^P$ enables the partial derivatives of the plastic potential to be determined, i.e. $\partial g/\partial p$ and $\partial g/\partial q$.

These three hypotheses provide all the information needed for the calculation of the plastic strain increments with equation (3). The first and third enable df, $\partial g/\partial q$ and $\partial g/\partial p$ to be determined. The hardening function is arrived at by considering two independent definitions of an undrained stress path. The first is that given by equation (5), and the second is that obtained by equating the recoverable volumetric strain increment (equation 2) with the plastic volumetric strain increment (equation 3). This equivalence leads to the hardening function, h. The details are given by Pender (1977).

(d) Substitution of the functions f, g and h into equation (3) gives the following equations for the plastic strain increments:

$$d\varepsilon^P = \frac{2\kappa(p/p_{cs})(\eta-\eta_o)d\eta}{M^2(1+e)(2p_o/p-1)\{(AM-\eta_o)-(\eta-\eta_o)p/p_{cs}\}} \qquad \text{............ (6)}$$

$$dv^P = \frac{2\kappa(p_o/p_{cs}-1)(p/p_{cs})(\eta-\eta_o)d\eta}{(AM-\eta_o)^2(1+e)(2p_o/p-1)} \qquad \text{............ (7)}$$

In general these equations for the plastic strain increments can be integrated only by a numerical process. However, there is one exception, a constant p path with $p=p_o=p_{cs}$; the integrated form of equation (6) is then:

$$\varepsilon^P_{p_o=p_{cs}} = \frac{2\kappa}{M^2(1+e)}\left\{(AM-\eta_o)\ln\left(\frac{AM-\eta_o}{AM-\eta}\right)-(\eta-\eta_o)\right\} + \varepsilon^P_o \qquad \text{............ (8)}$$

where: ε^P_o is the cumlative shear strain up to the start of the current loading cycle from η_o.

For such a path, equation (7) for the plastic volumetric strain increment reduces to zero and as p remains constant there is no recoverable volumetric strain.

The assumption that the current constant stress ratio yield locus always remains attached to the current stress point makes it possible to use equations (6) and (7) to calculate the response to reversed and cyclic loading in a simple manner. When the direction of loading changes from compression to extension, or vice versa, the sign of A is changed and η_o and p_o reset to new values. The kinematic behaviour of the yield locus means that there is no requirement to keep track of the position of the current yield locus.

The model described above enables the stress-strain behaviour, both drained and undrained, to be calculated for any state of overconsolidation. Though the presentation in this paper is for the stress conditions in the conventional triaxial apparatus extension to more general stress systems is possible. Four parameters are needed to characterise a given material: the stress ratio at the critical state (M), the location of the critical state line in the stress space (p_o/p_{cs} for a soil normally consolidated under spherical stress conditions), the slope of the line in the e, ln p plane for swelling under spherical stress conditions (κ), and the slope of the virgin compression line in the e, ln p plane (λ). The values of all four of these parameters are easily determined from routine laboratory tests. All calculated results presented herein are based on the following values: M = 0.90 (ϕ' = 23^0), p_o/p_{cs} = 1.90 for material normally consolidated under spherical stress conditions, κ = 0.04 (C_s = 0.09). For the undrained situations considered here the value of λ is not required. The void ratio must also be specified, in this case a value of 1.38 is used which is equivalent to a p_{cs} value of 220 kN/m^2.

APPARENT SHEAR MODULUS AND EQUIVALENT VISCOUS DAMPING RATIO FOR SMALL STRAIN AMPLITUDE CYCLIC LOADING

Figure 3a is a plot of the undrained response, calculated with equations (5) and (8), of an initially normally consolidated soil to a gradually increasing cyclic shear-stress. The decrease in apparent shear modulus and increase in plastic work for each cycle as the strain amplitude increases is evident.

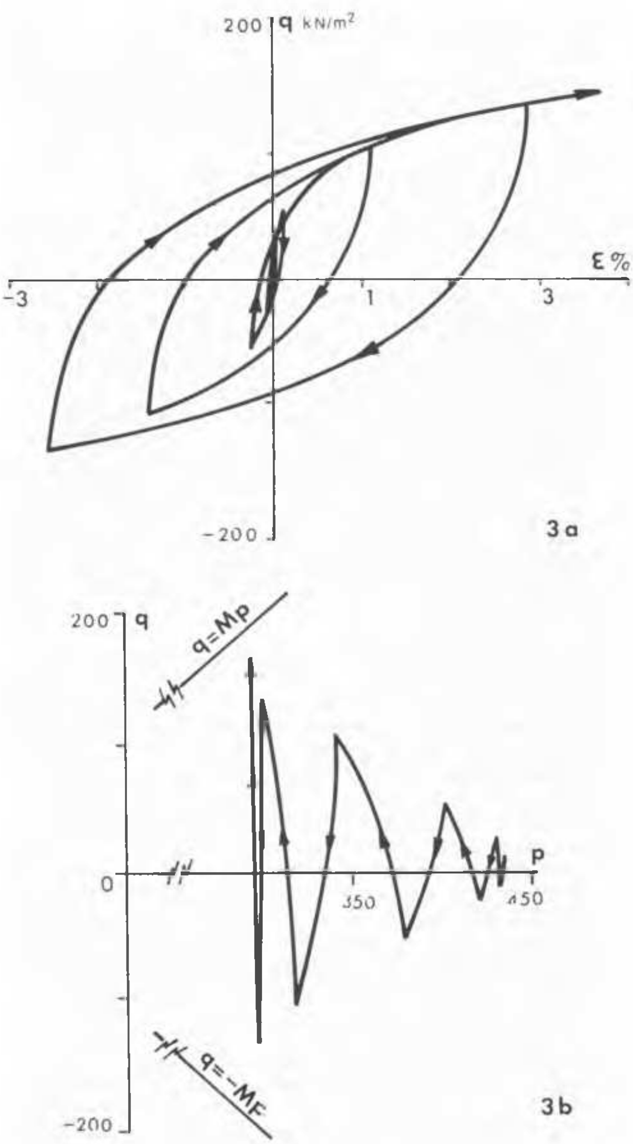

Fig. 3 Calculated response of normally consolidated clay to cyclic undrained shear.

Apparent shear modulus

G = slope of OA

Equivalent viscous damping ratio:

$$D = \frac{1}{4\pi} \times \frac{\text{area of loop}}{\text{area of } \Delta OAB}$$

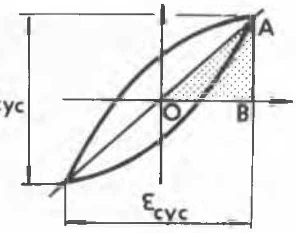

Fig. 4 Definition of apparent shear modulus (G) and equivalent viscous damping ratio (D).

A gradual increase in pore pressure is implied by the approach of the undrained stress path to the critical state value of p, Fig 3b. The apparent shear modulus (G) and equivalent viscous damping ratio (D) are defined in Fig. 4. Numerical evaluations of these are possible for any stress path by means of equation (6). However, equation (8) provides a more convenient method for the calculation of these parameters for lightly overconsolidated material. It will be used in majority of the calculations presented below.

In a closed cycle of stress change, $-\eta_j \rightarrow +\eta_j \rightarrow -\eta_j$, the stress-strain loop calculated with equation (8) is closed because there is no change in void ratio or p_{cs}. Thus equation (8) can be used to express the apparent shear modulus as a function of the cyclic stress ratio, η_j:

$$G = q_{cyc}/\varepsilon^p_{cyc}$$

$$= 2\eta_j p_{cs}/\varepsilon^p_{cyc}$$

On substitution for ε^p_{cyc} from equation (8), and with $\eta_o = -\eta_j$ and $\eta = +\eta_j$:

$$G = \frac{M(1+e)\eta_j p_{cs}}{\kappa\ \{(1+\frac{\eta_j}{M})\ \ln\ (\frac{M+\eta_j}{M-\eta_j}) - \frac{2\eta_j}{M}\}}$$

At the critical state $q_{cs} = Mp_{cs}$, thus:

$$G/q_{cs} = \frac{\eta_j(1+e)}{\kappa\ \{(1+\frac{\eta_j}{M})\ln(\frac{M+\eta_j}{M-\eta_j}) - \frac{2\eta_j}{M}\}} \qquad \text{............. (9)}$$

As η_j tends to zero the right hand side of equation (9) tends to infinity. The behaviour of the shear modulus with strain amplitude predicted with equation (8) is seen in Fig. 5.

Equation (8) can be used to evaluate the apparent shear modulus for stress cycles in which the mean stress ratio is not equal to zero. The effect on the apparent shear modulus of cyclic loading about a mean stress ratio other than zero shown in Fig. 6.

Examination of equations (6) and (8) reveals that when η_o is greater than zero the apparent shear modulus in compression is less than that in extension and when η_o is less than zero the apparent shear modulus in compression is greater than that in extension. Thus in Fig. 6 the plot of apparent shear modulus versus cyclic strain amplitude

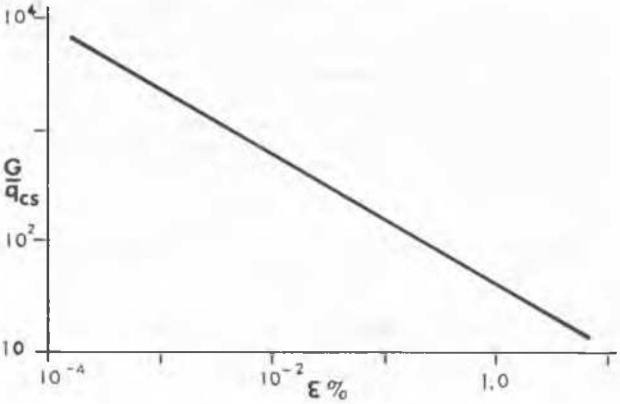

Fig. 5 Apparent shear modulus/strain amplitude relation for lightly overconsolidated clay.

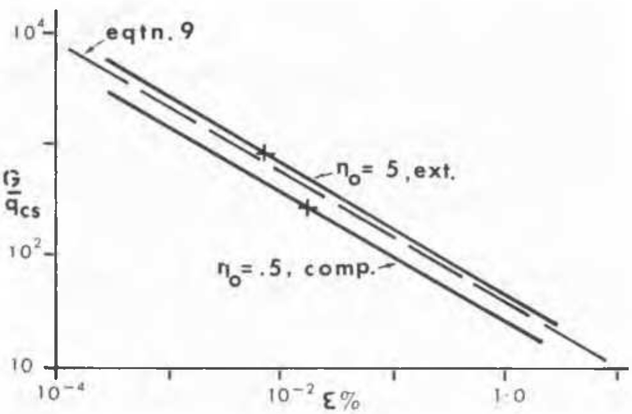

Fig. 6 Effect of initial stress ratio on apparent shear modulus.

calculated with equation (8) for loading in extension from η_o = 0.5 lies above the results calculated with equation (9) when the mean stress ratio for each cycle is zero. Similarly the plot for loading in compression from η_o = 0.5 lies beneath that calculated with equation (9). In contrast if η_o was less than zero, rather than 0.5, the relative position of the lines for compression and extension in Fig. 6 would be reversed. The figure shows that the initial value of η has little effect on the relation between apparent shear modulus and cyclic strain amplitude. However it does have a considerable effect on the values of the apparent shear modulus and shear strain for equal and opposite stress changes. This is shown by the two points marked '+' in Fig. 6 which are for equal and opposite changes in q from η_o = 0.5. Thus a closed cycle of stress change when the mean value of η is not zero will lead to a net build-up in strain.

The overconsolidation ratio when $p = p_{cs}$ for a soil with the properties specified above is 2.14. For the many natural soil deposits which exist in a lightly overconsolidated state, equation (9) provides a convenient means of predicting behaviour under cyclic loading although this equation is strictly valid only when the overconsolidation ratio is equal to p_{max}/p_{cs}. For the general case it is possible to calculate the apparent shear modulus and equivalent viscous damping ratio for any overconsolidation ratio by means of equation (6). Results of calculations for the apparent shear modulus for overconsolidation ratios 1, 5 and 20 are plotted against strain amplitude in Fig. 7. The q versus ε^p curve for each overconsolidation ratio was generated numerically and the value of G determined from this curve. The undrained nature of the paths is ensured by determining the incremental change in p, corresponding to each increment in q, from equation (5). All the stress cycles represented in the figure

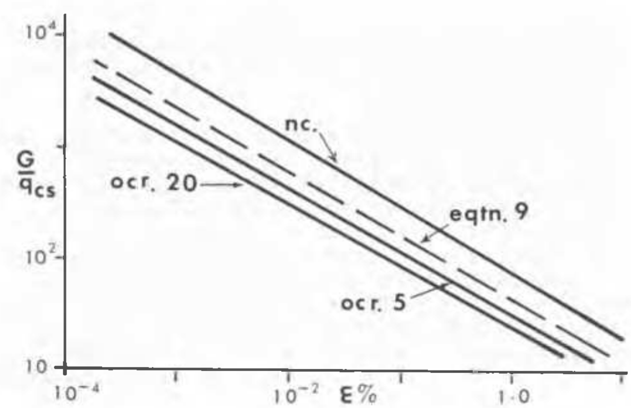

Fig. 7 Effect of overconsolidation ratio on apparent shear modulus/strain amplitude relation. (Mean stress ratio = 0.0 for all cycles).

have a mean stress ratio of zero. It is seen that increasing overconsolidation ratio leads to a decrease in the apparent shear modulus. A central concept of the model for overconsolidated soil is that during loading, the state of the soil is always moving towards the critical state value of p. Thus during a cyclic undrained test on normally consolidated soil there is a gradual increase in pore water pressure as the stress path moves towards p_{cs}. This is illustrated in Fig. 3. When the cyclic loading is applied to an overconsolidated material from an initial state dry of critical, negative pore water pressures are set up as the stress path moves towards p_{cs}. As the stress path approaches p_{cs} the apparent shear modulus/strain amplitude relation approaches that calculated with equation (9) and plotted in Fig. 5. Thus the apparent shear modulus of a normally consolidated material decreases slightly as the number of cycles increases. On the other hand for materials with an overconsolidation ratio substantially in excess of 2 the apparent shear modulus increases as the number of cycles increases. This phenomenon has been observed by Silver and Seed (1971) for medium dense sand. Thus equation (9) represents a limiting condition which the model predicts for a large number of loading cycles.

From the definition given in Fig. 4:

$$D = \frac{\int_A^B q\,d\varepsilon^P - 4 \times \text{Area } \Delta OBC}{4\pi \times \text{Area } \Delta OBC}$$

Substituting in equation (9) $\eta_o = -\eta_j$ and $\eta = +\eta_j$ this becomes:

$$D = \frac{(2/\pi)(1+\frac{\eta_j}{M})\{\ln(\frac{M+\eta_j}{M-\eta_j}) - 2(\frac{\eta_j}{M})\}}{(\frac{\eta_j}{M})\{(1+\frac{\eta_j}{M}) \times \ln(\frac{M+\eta_j}{M-\eta_j}) - 2(\frac{\eta_j}{M})\}} \quad \text{.....(10)}$$

As $\eta_j \to o$ it can be shown that the right hand side of equation (10) tends to a limiting value of 0.21. The relationship between damping ratio and strain amplitude is shown in Fig. 9.

It is emphasised that both equations (9) and (10) have been derived from the model for large strain static behaviour outlined earlier in the paper. No additional data or assumptions have been introduced. It is seen from Fig. 5 that the predictions of G are reasonable for strain amplitudes greater than 0.01% but for strain amplitudes less than this the predicted and observed behaviour diverge. This will now be remedied by the introduction of an elastic shear modulus.

INCLUSION OF ELASTIC SHEAR MODULUS

The hypothesis that soil has no recoverable shear strain may be realistic in many applications but at very small strains it is probably an unacceptable simplification. The predictions of equations (9) and (10) are modified below by the inclusion of an elastic shear modulus, the effect of which is most evident at very small strains. With the inclusion of this elastic shear modulus the apparent shear modulus is given by:

$$G = q_{cyc}/(\varepsilon^P_{cyc} + \varepsilon^e_{cyc})$$

$$= 2\eta_j p_{cs}/(\varepsilon^P_{cyc} + 2\eta_j p_{cs}/G_e)$$

Where: G_e is elastic shear modulus.
ε^e_{cyc} is the cyclic elastic distortion $= q_{cyc}/G_e$

On substitution for ε^P_{cyc} from equation (8) and rearranging for $\eta_o = -\eta_j$ and $\eta = +\eta_j$, the ratio of apparent shear modulus to q_{cs} becomes:

$$G/q_{cs} = \frac{\eta_j(1+e)}{\kappa\{(1+\frac{\eta_j}{M})\ln(\frac{M+\eta_j}{M-\eta_j}) - \frac{2\eta_j}{M}\} + \frac{\eta_j q_{cs}(1+e)}{G_e}} \quad \text{............ (11)}$$

As $\eta_j \to o$ the right hand side of equation (11) tends to G_e/q_{cs}. The apparent shear modulus, calculated with equation (11) using the same values for M, κ, e and p_{cs} given above and G_e = 51.0 Mpa, is plotted against strain amplitude in Fig. 8. This compares well with the pattern of experimental behaviour reported by Seed and Idriss (1970) and Taylor and Parton (1973).

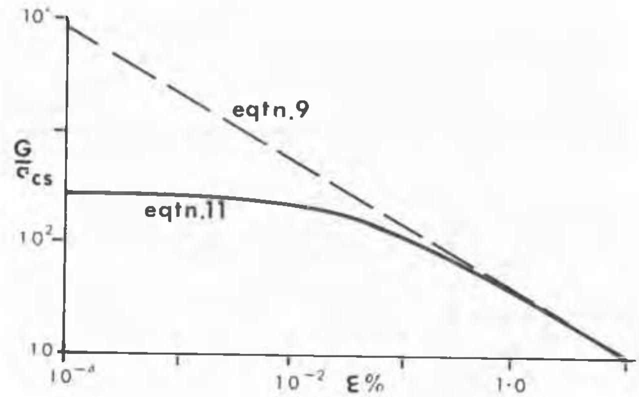

Fig. 8 Variation of apparent shear modulus with strain amplitude including the effect of a small strain elastic shear modulus.

With the inclusion of elastic shear strain the expression for the equivalent viscous damping ratio becomes:

$$D = \frac{(2/\pi)(1+\frac{\eta_j}{M})\{\ln(\frac{M+\eta_j}{M-\eta_j}) - 2(\frac{\eta_j}{M})\}}{(\frac{\eta_j}{M})\{(1+\frac{\eta_j}{M})\ln(\frac{M+\eta_j}{M-\eta_j}) - \frac{2\eta_j}{M}\} + \frac{(1+e)p_{cs}(\eta_j)^2}{\kappa G_e}} \quad \text{........... (12)}$$

It can be shown that the right hand side of equation (12) tends to zero as η_j tends to zero. The behaviour predicted with equation (12) is presented in Fig. 9. As with Fig. 8 the predictions in Fig. 9. compare well with the experimental behaviour reported by Seed and Idriss (1970) and Taylor and Parton (1973). Equation (12) is derived for stress cycles in which the mean stress ratio is zero. For cycles in which the mean stress ratio is not zero but p_o is still equal to p_{cs} equation (8) can be used to evaluate the equivalent viscous damping ratio. For the general case the equivalent viscous damping ratio may be evaluated using equation (6) for the plastic shear strain increment and dq/G_e for the elastic shear strain increment. In this way the shape of the stress-strain loop is generated, evaluation of the area of the loop then leads to the equivalent viscous damping ratio.

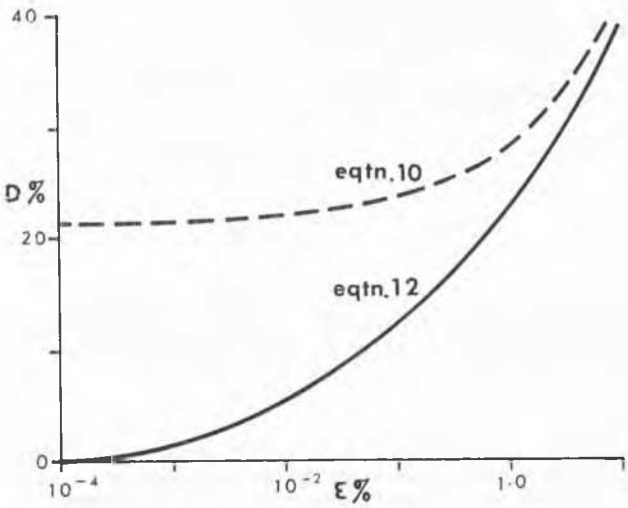

Fig. 9 Equivalent viscous damping ratio predicted with the inclusion of elastic shear behaviour.

CONCLUSIONS

This paper has shown how a stress-strain model for overconsolidated soil originally developed for large strain static behaviour, can be extended to give realistic predictions for small strain cyclic loading. One additional soil property, the elastic shear modulus, was introduced to make predictions at very small strain amplitudes realistic. Thus the number of properties required by the model to describe a given soil rises from four to five.

The model has the advantage of providing a highly unified treatment of stress-strain behaviour of soil. Thus one model is capable of treating a wide range of soil behaviour. Both the apparent shear modulus and the equivalent viscous damping ratio are predicted in a realistic manner. Four of the five parameters needed to characterise a given soil are determined from routine laboratory tests. The fifth parameter, the elastic shear modulus, is most conveniently determined from measurements of the in situ shear wave velocity. Thus assessment of the behaviour of a soil deposit under cyclic loading will be possible without any special laboratory testing to determine soil parameters.

At first sight equations (9) and (11) for the apparent shear modulus (G) and equations (10) and (12) for the apparent viscous damping ratio (D) might seem rather complex. However, the facilities available to the engineering profession today make computation straight forward and relatively inexpensive. In contrast the determination of values for properties to characterise a given soil by laboratory and/or field testing is often difficult, time consuming and costly. Two most important features of the present model are (i) the small number of parameters needed to characterise a given soil, and (ii) that values for these can be determined from routine tests.

Although the treatment in this paper is for the stress conditions in the conventional triaxial apparatus extension to more general stress system is possible.

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