SAND DEFORMATION IN THREE-DIMENSIONAL STRESS STATE

DEFORMATION DU SABLE SOUS UNE COMPRESSION TRIAXIALE

ДЕФОРМАЦИЯ ПЕСКА ПРИ ТРЕХМЕРНОМ НАПРЯЖЕННОМ СОСТОЯНИИ

M. ARNOLD, Senior Lecturer in Civil Engineering, University of Adelaide
P.W. MITCHELL, Postgraduate Research Student, University of Adelaide, Australia

SYNOPSIS. The stress-deformation behaviour of a uniform sand at a single initial density is studied under generalized three-dimensional conditions ($\sigma_a > \sigma_r \geqslant \sigma_t$) with the use of the hollow cylinder triaxial compression apparatus which allows independent variation of the principal stresses.

Because the displacement of sand under load is shown to be nearly irrecoverable, the concepts of classical plasticity theory may adequately be used to describe its deformation properties. The three parameters of the plasticity theory (yield surface, work-hardening relationship and plastic potential), together with the failure state are presented in a three-dimensional manner, suitable for the development of a generalized three-dimensional stress-strain relationship.

In this study the yielding is restricted to that region of stress within which the mass contracts, prior to the onset of dilatancy, a region within which most of the load-carrying capacity of the material occurs. The plastic potential is found to be a bullet-shaped surface in the entire stress-range up to failure.

INTRODUCTION

The hollow cylinder triaxial compression test, which allows independent variation of the inside and outside pressures and vertical load is used to study the deformation behaviour of a sand under three-dimensional systems ($\sigma_a > \sigma_r \geqslant \sigma_t$). Consideration of volume change is limited to that region of stress within which the sand contracts under increasing stress prior to the onset of dilatancy. This stress region has important engineering applications because most soil structures are designed to fall within it under normal operating conditions, and the stresses up to the onset of dilatancy are a large proportion of the failure stresses (Kirkpatrick 1961, Frydman 1969). Kirkpatrick showed that all granular soils would compress when subjected to a standard triaxial compression test until the principal stress ratio increased to approximately 2.8, corresponding to a developed angle of internal friction of 28°. Hence a study of the contractive region is of importance in the prediction of displacements in most practical soil structures.

It has been experimentally observed (e.g. Holubec 1968) that when a sand mass, not previously deformed is subjected to an external load only a part of the resulting deformation is recoverable or elastic, the remainder being irrecoverable or plastic. The material behaves as an elasto-plastic, work hardening substance. This has led to the belief that the concepts of the plasticity theory (Hill 1950) may accurately describe the response of a sand mass under an external agency. The determination of the relevant parameters of the plasticity theory by conventional soil tests, such as the standard triaxial compression test is limited because of their uncertainty regarding the true three-dimenional stress state. In this investigation, the results from the hollow cylinder test, which permits independent variation of the principal stresses are used to determine the basic parameters of the plasticity theory, enabling the development of a generalized stress-strain law for sand.

The three basic ingredients of any stress-strain law based on plasticity theory are a yield surface, a flow rule defining the plastic potential and a work-hardening law (Roscoe, 1970). These concepts hold only in the region of strain-hardening up to the peak stress condition, after which the material strain softens.

The yield surface is a surface in stress space representation such that if the sand is subjected to stresses within that surface, the sand deforms entirely elastically, if the stresses move outside the surface, then both elastic and plastic deformations occur, and the yield surface moves to a new location. From the results of simple shear tests, Roscoe concluded that the yield loci, when plotted on a shear stress vs normal stress plane, are a family of straight lines passing through the origin. This was in agreement with Poorooshasb et. al. (1966) using solid triaxial tests. Poorooshasb (1971) modified this simple yield locus relationship to include dependence on the mean normal stress. It is significant, however, that the determination of the exact yield point is susceptible to errors because of the steepness of the elastic stress-strain curve and the uncertainty in the precise location of the point of initiation of yielding (Havthorthwaite 1968). This is analogous to the uncertainty in the determination of the preconsolidation pressure for clays.

The work-hardening law relates the change of strains to the stresses as the yield locus of the sand changes as it hardens. For metals the work-hardening law relates an equivalent stress, σ_e, to the equivalent plastic strain, ε_p, through a hardening parameter H. This parameter is derived from a uniaxial tensile stress-strain curve by plotting σ_e against ε_p. Pending the development of a generalized stress-strain law, a similar procedure has been undertaken for soils. Roscoe (1970) suggested using the results of a simple shear test for the work hardening law, Smith & Kay (1971) used a plane strain test, whereas Poorooshasb et. al. used constant mean normal stress triaxial tests to determine the hardening parameter.

In a stress-space representation of the mechanical properties of a soil, the plastic potential is a surface that is everywhere perpendicular to the direction of the irrecoverable strain increment. Several attempts at applying the plastic potential concept to the yielding of cohesive soils have been made (e.g. Drucker et. al. 1957, Roscoe et. al. 1963). Poorooshasb et. al. showed that, for sands in conventional triaxial tests the potential is a function only of the principal stresses and the void ratio. On this basis they were able to obtain a family of plastic potential curves, the general equation for which may be incorporated in any stress-strain law.

The failure stress state is a region bounding the work-hardening behaviour and the applicability of the plasticity theory A clear distinction must be made between the concept of "yield" and the term "failure". In the sense of classical plasticity theory, "yield" indicates the onset of irrecoverable deformations, whereas the term "failure" is defined as the peak stress combination that a sand can support. With the use of various devices, numerous investigators (e.g. Kirkpatrick 1957, Sutherland & Mesdary 1969) have studied the failure state in a three-dimensional stress system. Generally, it has been found that the Mohr-Coulomb law for sand based on triaxial tests under-predicts the failure strength of the material in other stress conditions.

The parameters of any stress-strain law derived from solid cylinder compression tests are restricted in applicability because the intermediate principal stress is assumed equal to the minor principal stress. With the use of the hollow cylinder apparatus it has been found possible to extend the concepts of the plasticity theory for sands to a three-dimensional representation. From this, a generalised stress-strain theory for the sand under stress states of practical engineering interest could be formulated. The plastic potential, the work-hardening law limited to the contraction region, the failure and contraction - dilation reversion state are presented in terms of the three principal stresses, from which it is hoped that field problems in soil mechanics involving a generalized stress state could be accurately solved.

MATERIAL & APPARATUS USED

The tests were performed on a washed and oven dried, subrounded, quartzitic sand (Christies sand). This was artifically graded to the limits 0.42 mm $< d <$ 1.20 mm. The maximum and minimum void ratios, as determined by the conventional methods were 0.89 and 0.60 respectively.

A hollow cylinder apparatus, similar in design to that of Kirkpatrick (1957) was used with specimens of outer diameter 152 mm, inside dia. 102 mm, and height 142 mm. The essential feature of the apparatus was a complete seal between the bore and outside of the sample, allowing independent pressure control and volume measurements. Pressures were supplied by two Bishop-type, self-compensating mercury systems, and the volume changes were measured by the displacement of water from the bore and cell. These measurements were checked against a burette connected to the sample interior and almost complete agreement was observed. The vertical load and displacements were measured by an external proving ring and dial gauge respectively. Mineral oil was supplied to the piston to minimise friction and frictionless end platens were used to obtain uniform sample deformations.

The samples were prepared by pouring the dry sand from a constant height into the space between the inside membrane supported by a core, and the outside membrane supported by a mould. The sample was compacted to a constant void ratio (within the range 0.74 ± 0.01) by ramming with a wooden rod. To keep the sample firm when the moulds were removed it was subjected to a small negative pressure until the hydrostatic pressure was applied. The testing technique was similar to that of the conventional solid cylinder triaxial compression test, the major difference being the additional measurement of the bore pressure and volume change.

TESTING PROGRAMME

During any test, the sample was subjected to an axial stress, σ_a, in the vertical direction, a radial stress σ_r, and a tangential stress σ_t. The corresponding strains were ε_a, ε_r, ε_t.

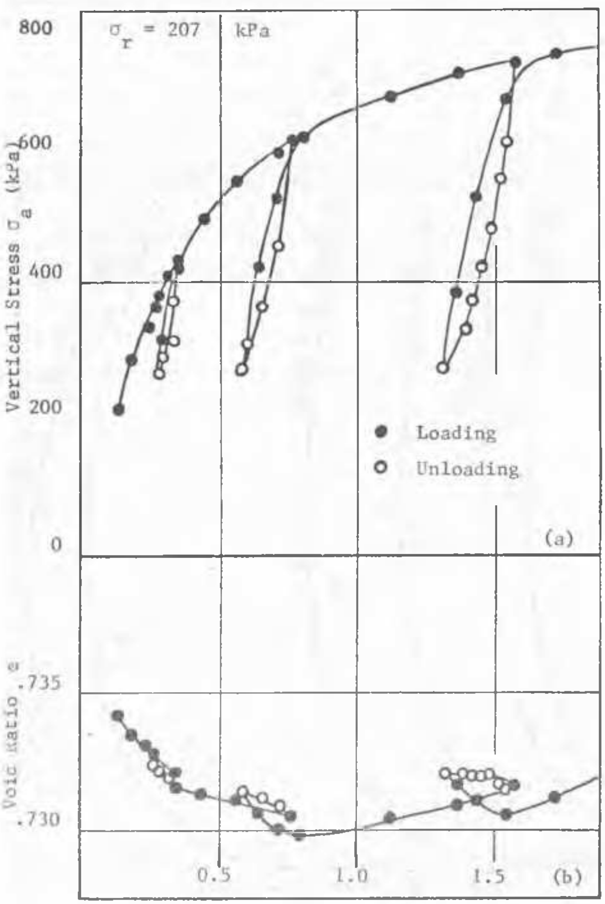
A total of 37 tests was performed, of which 14 were loaded to failure under the stress combination $\sigma_a > \sigma_r = \sigma_t$, (series A), 13 under the condition at failure of $\sigma_a > \sigma_r > \sigma_t$ (series B); and the remainder (series C^a) were similar to series A except that several unloading, reloading paths were followed. In all cases, σ_a was taken as the sum of the load applied and the pressure difference over the true specimen area.

The purpose of test series A, which involved a stress state equivalent to that of the standard triaxial test, was to allow comparison with the results from series B, involving an independent variation of the stresses.

The procedure for series A was to consolidate the sample initially to a certain stress level, keeping bore and cell pressures equal and then to load vertically to failure with the two lateral pressures equal and constant. During the consolidation stage, volume changes were recorded, with corrections being made for membrane penetration (following the procedure of Feda, 1969). For this series, the tangential and radial stresses were equal to the bore and the cell pressure, the radial and tangential strains were taken as half the difference between the volumetric strain and the vertical strain.

The procedure for series B was initially to consolidate the sample under equal bore and cell pressures, after which the inside pressure was increased by a certain amount over the outside pressure, which was kept constant. This raised σ_r, but decreased σ_a and σ_t. The specimen was then loaded axially to failure.

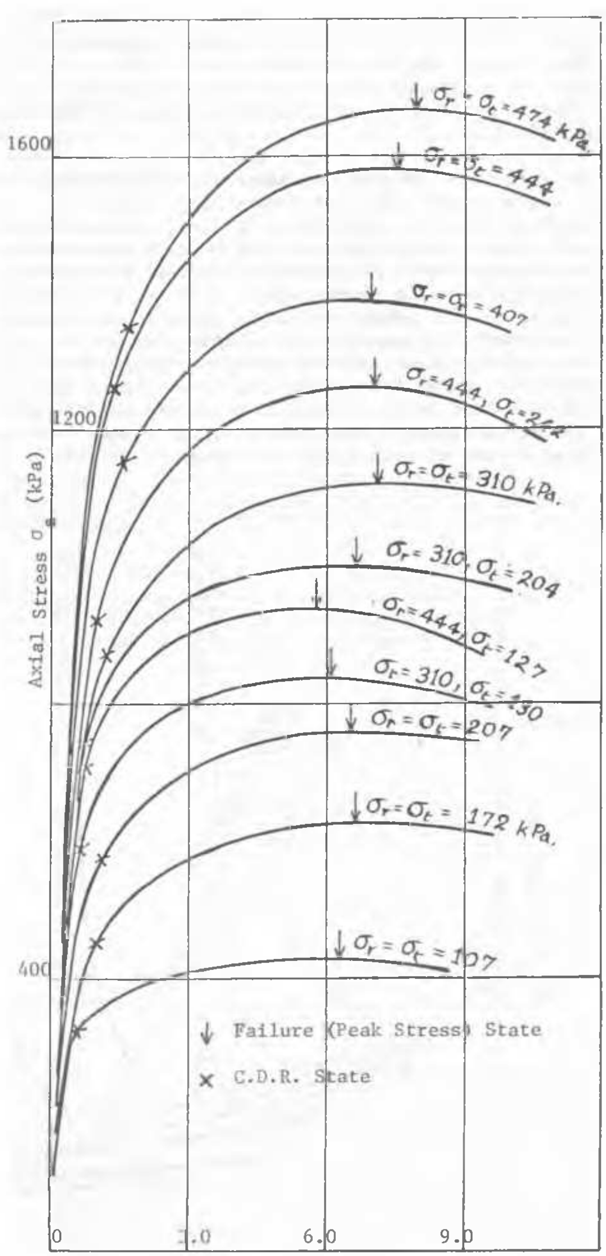
The difficulty with this type of test is that the analysis involves an assumed distribution of stress and strain. This is the major restriction of the hollow cylinder test, and the results from it must be accepted with this qualification. A linear distribution of stress and strain across the sample thickness was assumed with the radial stress as the mean of the two lateral pressures. The tangential stress and the strains could then be determined with the use of the equilibrium and compatibility equations respectively.



Vertical Strain ε_a %

Fig. 1. Unloading - Reloading Response.

The purpose of series C was to examine the proportions of irrecoverable and recoverable strains as the sample was deformed at equal radial and tangential stress. The testing technique was identical to that of series A, with the exception that the samples were not loaded progressively to failure, but were subjected to various cycles of loading and unloading at various stages of the shearing process.



Axial Strain ε_a %

Fig. 2. Vertical Stress - Strain Behaviour

ANALYSIS & DISCUSSION OF TEST RESULTS

1. Unloading - Reloading Behaviour (Series C)

Figure 1 is typical of the behaviour of a sample from series C. This indicates that as a sample of sand is loaded, both irrecoverable and recoverable deformations occur. However, the unloading response involves an increasing degree of hysteresis with consequent difficulties in the determination of the elastic components of strain. Fortunately, the irrecoverable strains are much larger than the elastic strains giving justification for regarding total strains as irrecoverable (Smith & Kay, 1971). This is in agreement with published results of triaxial tests on solid specimens. (e.g. Barden et. al. 1969).

The void ratio vs axial strain relationship (Fig. 1b) illustrates how a mass of sand will yield during loading, while on unloading only small volume changes will occur. Since the mass has been conditioned to take higher values of stress, no appreciable volume changes will occur on reloading until the void ratio returns to the value immediately prior to unloading; after which the material will continue to yield. This experimental evidence indicates that contours of equal void ratio are yield surfaces. When a sand specimen is normally loaded to a certain stress level (with corresponding void ratio) little change in axial strain or void ratio will occur on unloading. On reloading, large irrecoverable displacements will not be developed until the original stress level and hence original void ratio is reached. Thus curves of equal void ratio on a stress space representation are curves separating a region of recoverable strains within them and irrecoverable displacements outside them.

2. Stress - Strain Response

Typical results from series A and series B are shown in Figures 2 and 3. For series A ($\sigma_r = \sigma_t$), an increase in lateral stresses resulted in a linear increase in the vertical stress at failure (and corresponding vertical strain). The principal stress ratio, σ_a/σ_t, at failure was 3.6, giving a Mohr-Coulomb friction angle, $\phi = 34^o$.

For series B, ($\sigma_r \neq \sigma_t$), with σ_r constant, a decrease in σ_t resulted in a decrease in the vertical stress at failure (and a corresponding slight decrease in vertical strain). The stresses at failure are under-estimated by the Mohr-Coulomb relationship, as was also reported by Kirkpatrick and Sutherland & Mesdary. Figure 4 shows the relationship between the principal stress ratios

$$\frac{\sigma_a}{\sigma_r} \text{ vs } \frac{\sigma_t}{\sigma_r}$$

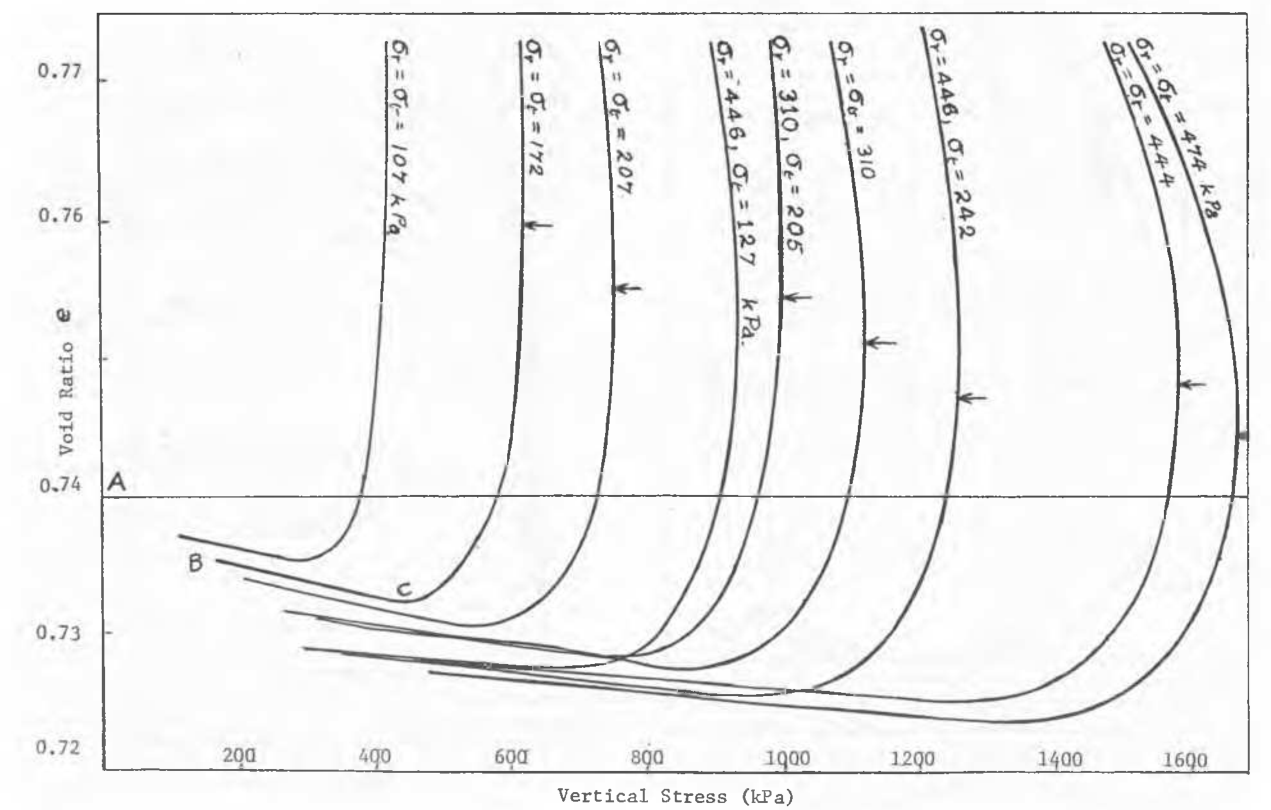


Fig. 3. Volume Change - Vertical Stress Behaviour.

This can be empirically expressed as:-

$$\frac{\sigma_a}{\sigma_r} = C - 2.24 \left(1 - \frac{\sigma_t}{\sigma_r}\right)^{1.55} \quad \ldots\ldots (1)$$

where C = 3.60 for the failure condition (Curve 1). This equation governs the permissible stresses that can act on a sand mass and reduces to the Mohr-Coulomb relationship for conventional triaxial compression tests involving $\sigma_r = \sigma_t$.

Figure 3 shows the two regions of volume change, that of contraction and that of dilation. The sample contracts during consolidation (for $\sigma_r = \sigma_t$ = 172 KPa this is represented by the path AB in Figure 3) and under subsequent shearing stress the contraction continues until a shear stress level is reached (Point C) beyond which the sample dilates. This shear stress level is a considerable proportion of the failure stress and occurs when the load deformation response has departed from initial linearity (Figure 2). The attainment of dilatancy creates a flattening of the stress-strain relationship, indicating the onset of failure. The full implication of this will be discussed later with reference to the work-hardening response.

For series A, as the stress $\sigma_r = \sigma_t$ is increased, the void ratio at failure and the amount of dilation is progressively reduced. The rate of change in void ratio with stress is a maximum at failure and a minimum at the contraction to dilation reversion stress level (CDR state). The principal stress ratio $\frac{\sigma_a}{\sigma_r}$, at this change-over point for series A is 2.78 corresponding to a developed friction angle of 28°.

For tests involving $\sigma_r \neq \sigma_t$, as σ_t is reduced at constant σ_r the void ratio at failure increases as does the ratio $\frac{\sigma_a}{\sigma_t}$ at the CDR state. The relationship $\frac{\sigma_a}{\sigma_r}$ vs $\frac{\sigma_t}{\sigma_r}$ at the CDR state is also shown in Figure 4 (Curve 2). It will be seen that this is similar to the relationship at failure and can be accurately expressed by equation (1) with C = 2.78.

3. Work-Hardening Behaviour

As was shown in the analysis of the unloading, reloading behaviour, contours of equal void ratio must form yield surfaces in stress space, illustrating the work-hardening behaviour of the sand. Figure 5 shows the data from series A plotted as void ratio contours on the stress space plane $\sigma_r = \sigma_t$. From this, two significant observations can be made. First, the void ratio contours in the contraction region are independent of the level of deviatoric shear, being only dependent on the distance along the hydrostatic axis. Second, in the dilation region, the void ratio contours form straight lines progressively aligning themselves parallel to the failure line. The contractive behaviour agrees with the work of Frydman (1969) who found that when sheared under a pure deviatoric stress, a triaxial sample of sand did not change in volume until dilation commenced. Hence the yielding of sand in the contraction region is completely defined by the volume changes occurring under a hydrostatic stress.

During all-round consolidation, the sand grains are capable of moving into more stable locations thus creating a slightly denser and stronger sample. During initial shearing, this movement continues,

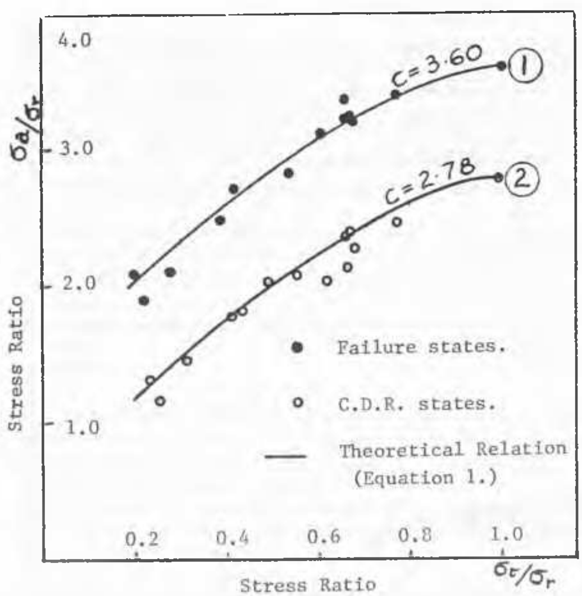


Fig. 4: Failure and C.D.R. stress-state ratios

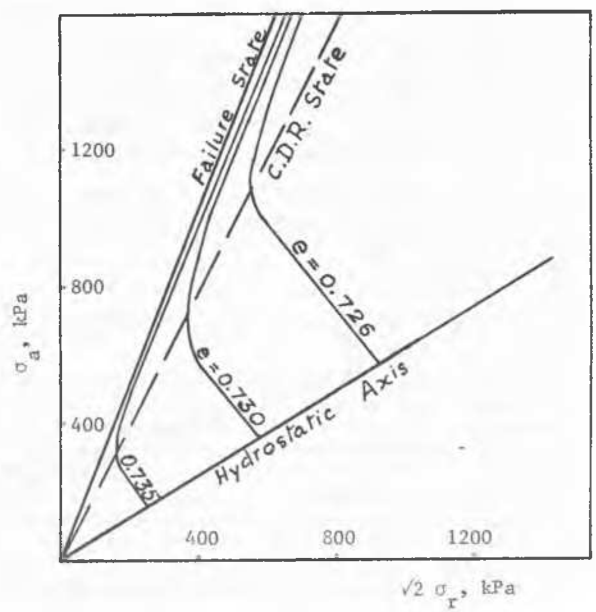


Fig. 5: Void Ratio Contours (Series A)

creating a still more dense and stable structure. It is this initial contraction that is responsible for a steep stress-strain response and which gives a sand mass its load-bearing capacity. Eventually a level of shear stress is reached when the grains must ride over each other and hence the mass dilates to accommodate further levels of shear. Thus the behaviour of sand in the contractive region under shear is merely an extension of the behaviour under all-round consolidation.

This important phenomenon is shown in Figure 6 which shows the volume change response as determined by the change in void ratio (e) with the distance along the hydrostatic axis $\left(p = \frac{\sigma_a + \sigma_r + \sigma_t}{\sqrt{3}}\right)$. The consolidation response is presented together with the contraction data from series A and B. As already mentioned, series B involved a very slight degree of unloading to obtain the pressure difference and the data points shown in Figure 6 are for the virgin loading conditions only. The three types of data in Figure 6 (consolidation, Series A and Series B) are almost indistinguishable, indicating a unique relationship, independent of the level of deviatoric shear, which can be expressed as:-

$$e = e_o - 0.0000794\, p^{0.76} \quad \ldots\ldots\ (2)$$

where e_o (= 0.741) is the initial void ratio.

This equation is the work-hardening law of the sand in the contraction region, which is governed by the stress range of equation (1) with C = 2.78.

4. Plastic Potential

The plastic potential is a function g (σ_{ij}) defining the increment of plastic strain through a non-negative parameter λ.

$$d\,\varepsilon_{ij} = \lambda \frac{\partial g}{\partial \sigma_{ij}} \quad \ldots\ldots\ldots\ldots\ (3)$$

The stresses acting on the sample and the strains produced are known at any point during a test. Because, as observed previously, considerable simplification and little loss of accuracy will result if total deformations are treated as irrecoverable, the total strain is taken as the plastic strain for the determination of the plastic potential.

The strain increments, plotted on the $\sigma_r = \sigma_t$ stress plane for series A are shown as vectors in figure 7, where the axis scale of stress and strain is coincident. Contours that are drawn everywhere perpendicular to these vectors are also shown, these curves are the intersection of the family of plastic potentials with the plane $\sigma_r = \sigma_t$.

The form of these curves suggests that the plastic potentials are bullet-shaped, the sharp end of the bullet corresponding to a point on the hydrostatic axis, in agreement with Roscoe et. al. (1963). This suggestion would be correct if the projection of the plastic potential surfaces onto a deviatoric plane perpendicular to the hydrostatic axis was a circle of the extended Mises type. A family of plastic potentials, as determined from the results of series B involving $\sigma_r \neq \sigma_t$ is shown projected onto a deviatoric plane at a hydrostatic axis distance of 715 k Pa (Figure 8).

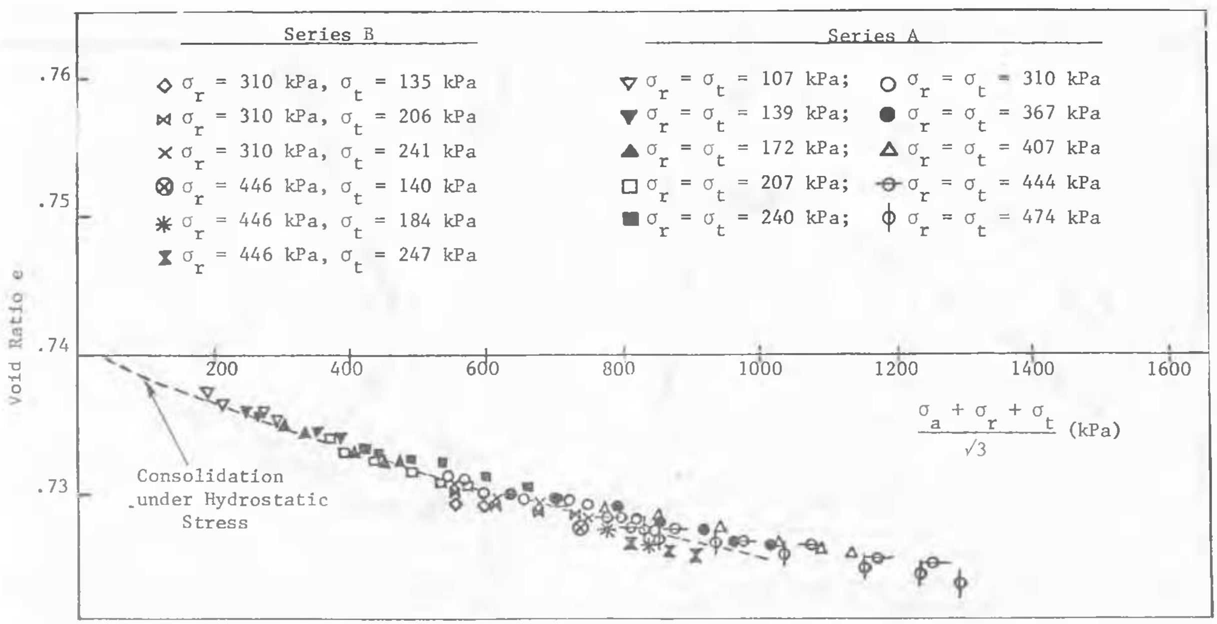


Fig. 6. Contraction Volume Change.

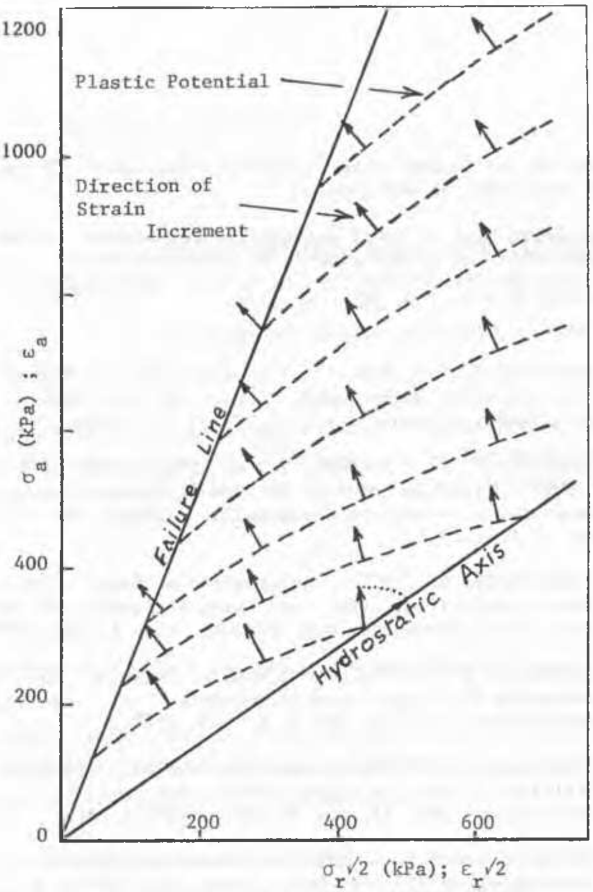


Fig. 7. Determination of Plastic Potentials from Strain Increments (Series A).

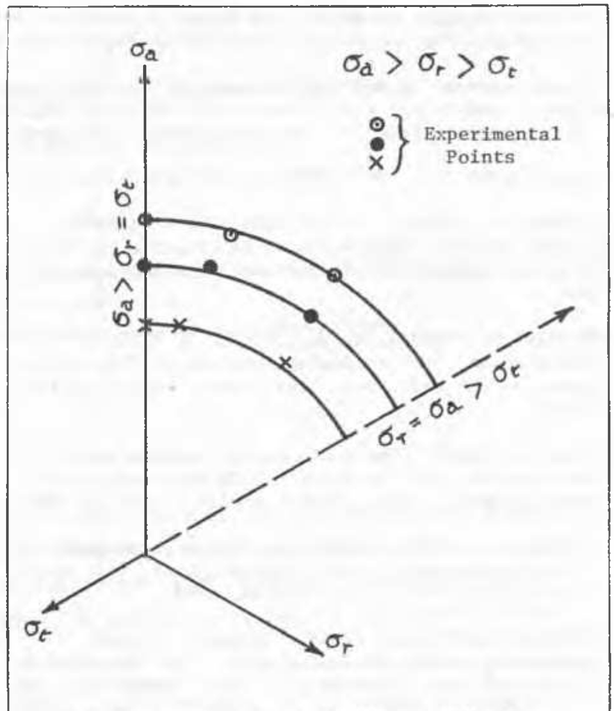


Fig. 8. Projection of Plastic Potentials on Deviatoric Plane.

The experimental results do not depart significantly from a circle whose radius is fixed on the $\sigma_a > \sigma_r = \sigma_t$ axis (as determined from series A), lending support to the Cambridge concept (Roscoe et. al.) of a bullet shaped plastic potential. This has ɔortant implications in that the plastic potential fully defined for generalized stress conditions, pplicable to field situations, by a longitudinal section on the $\sigma_r = \sigma_t$ plane, as determined by conventional triaxial tests.

The two equations incorporating the plastic potential (equation 3) can be coupled with the work-hardening law (equation 2) giving a solution for the three principal strains (ε_a, ε_r, ε_t) under imposed principal stress (σ_a, σ_r, σ_t).

CONCLUSIONS

The results of hollow cylinder triaxial compression tests are analysed to obtain the three dimensional stress-strain response of a sand of medium density using a generalized plasticity theory. This extends the results of conventional triaxial tests which are restricted by the uncertainty regarding the three-dimensional stress state.

Unloading tests indicate that recoverable deformations of the sand in this state are small, thus justifying the use of the plasticity theory in describing the total deformation behaviour of the material.

It was found that at least 75% of the failure stresses could be applied before the sand started to dilate. The concept of a contraction-dilation reversion surface (CDR) is used to define the boundary between regions in which contraction and dilation occurs during shear. This boundary is a unique surface similar in geometry to the failure surface.

In the contraction zone, deformation is characterized by a volume change response and a plastic potential surface. The volume change response can be represented by changes in void ratio, with planes of equal void ratio below the CDR state being directed normally to the hydrostatic axis in three-dimensional stress space.

The potential surfaces for both the contraction and dilation regions are bullet shaped surfaces of revolution about the hydrostatic axis, the points of which are directed away from the origin.

From these concepts, it is planned to formulate a generalized stress-strain response of the sand mass in the region of practical engineering interest, suitable for the solution of boundary value problems for cohesionless materials encountered in soil mechanics.

ACKNOWLEDGEMENTS

The investigation on which this paper is based was undertaken by the junior author in partial fulfilment of the requirements for the degree of Master of Engineering at the University of Adelaide.

Acknowledgement is made to Professor F. B. Bull, Head of the Department of Civil Engineering for support in the project, and to Mr. W.C.H. Hunter of the Civil Engineering Workshop for supervision of the manufacture of specialized equipment. The encouragement and advice from other members of the faculty is also appreciated.

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