MECHANICAL FOUNDATION OF THE DISPERSE SYSTEM"WATER–SOLID PHASE SKELETON" IN SOIL MECHANICS

MECANIQUE DE FONDATION DE SYSTEME DISPERSE "EAU–PHASE SOLIDE DU SQUELETTE" DANS LA MECANIQUE DES SOLS

МЕХАНИКА ОСНОВАНИЯ, СОСТОЯЩЕГО ИЗ СЫПУЧЕЙ СРЕДЫ ВИДА "ВОДА – ТВЕРДАЯ ФАЗА СКЕЛЕТА", В МЕХАНИКЕ ГРУНТОВ

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SYNOPSIS. The equilibrium of water-solid phase skeleton is disturbed by static or dynamic surface forces. The pore water movement and solid phase skeleton-motion resulting from these disturbances will be described. The following assumptions will be made: 1.) The pores of the solid phase skeleton situated under the water surface are completely filled with water (in the following called porewater); 2.) The material of the solid phase skeleton and the porewater are assumed to be incompressible; 3.) In view of the diffusivity the solid phase skeleton will be treated as laminated inhomogeneous and anisotropic layers; 4.) The capillary forces in the microflow are neglected; 5.) The molecular forces acting between the porewater and the solid phase skeleton are considered by a correction of the porevolume; 6.) The vaporation of water and chemical influences of any kind are neglected.

Nomenclature of the paper, as there are velocity of solid phase skeleton, filter-velocity, relative porevolume, grain to grain forces, pore water pressure and so on are determined in the sense of continuum mechanics. In deriving the equation of continuity we introduce the surface O_k fixed in space which encloses the volume V_k, where k is the number of the layer. Further we introduce the outer normal vector dO_k of the surface element, n_k, the relative effective porevolume (i.e. porevolume of the solid phase skeleton per unit spatial volume), $w_{k,K}$, the vectorial velocity of the solid phase skeleton and finally w_k denotes the filtervelocity of the liquid relative to the moving solid phase skeleton. The constituent of the solid are incompressible and deformable, the liquid phase is incompressible. The assumption is made that the pores which are benease the liquid surface are always completely filled with the liquid phase.

We also distinguish between a solid phase skeleton which contains pores filled by gas (air etc.) or liquid (water etc.). In the

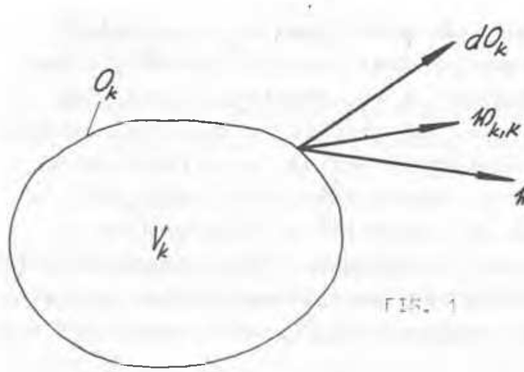

FIG. 1

following section we shall derive the fundamental equations of a solid phase skeleton completely filled with liquid. In this case the flux of solid phase volume through the surface O_k equals the time rate of the solid phase volume reduction in the interior of the closed surface O_k.

$$\int_{O_k}(1-n_k)w_{k,K}\, dO_k\, dt = -\frac{\partial}{\partial t}\int_{V_k}(1-n_k)\,dV_k\, dt$$

Using Gauss'formula and the following limiting procedure:

$$\lim_{V_k\to 0}\frac{1}{V_k}\int_{V_k}\left\{\nabla\cdot[(1-n_k)w_{k,K}]-\frac{\partial n_k}{\partial t}\right\}dV_k = 0$$

the following equation of continuity of the solid phase skeleton is established in differential form:

$$\nabla\cdot[(1-n_k)w_{k,K}]-\frac{\partial n_k}{\partial t}=0 \tag{1}$$

This equation holds for arbitrary volume Analogous the flux of liquid volume through the surface O_k equals the rate of liquid volume reduction within the surface:

$$\int(n_k w_{k,K}+w_k)\,dO_k\, dt = -\frac{\partial}{\partial t}\int_V n_k\, dV_k\, dt$$

Using Gaussian Theorem and taking the limit we get from

$$\lim_{V_k\to 0}\frac{1}{V_k}\int_{V_k}\left\{\nabla\cdot(n_k w_{k,K}+w_k)+\frac{\partial n_k}{\partial t}\right\}dV_k=0$$

the differential equation o continuity

$$\nabla\cdot(n_k w_{k,K}+w_k)+\frac{\partial n_k}{\partial t}=0 \tag{2}$$

Adding equation (1) and (2) it follows

$$\nabla\cdot(w_{k,K}+w_k)=0 \tag{3}$$

For sake of interpretation of the different meaning of a statistical cut and a well defined cut we consider a temporal constant solid body skeleton fixed in space under the assumption that for all makroscopic surface elements of the statistical cut, the ratio of the wet surface to the total surface of the element is defined as a function of the spatial coordinates. This holds only in the case of a statistical distribution of the solid body material. This assumption holds for a solid body skeleton made of a unique material component and is also true for a multicomponent material e.g. for the two constituent of concrete. For a makroscopic surface element the ratio of the wet surface to the total surface is denoted by λ_k This function λ_k takes on different values for different types of cuts (statistical cut or well defined cut.) In the following we show for statistical cuts in solid bodys which are anisotrop and inhomogeneous with respect to diffusivity, that λ_k equals the relative porevolume n_k . Now we consider a makroscopic volume element with corner lengths dx,dy,dz in an arbitrary oriented coordinate system (x,yz) where the surface of the element is formed by statistical cuts. The liquid contained in this element dx dy dz can be determined in three ways: Denoting $\lambda_{k,x}, \lambda_{k,y}, \lambda_{k,z}$ the λ_k values of the statistical surfaces normal to the corner directions x,y,z we have

$$n_k\, dx\,dy\,dz = \lambda_{k,x}\, dx\,dy\,dz = \lambda_{k,y}\, dy\,dx\,dz$$
$$\lambda_{k,z}\, dz\,dx\,dy$$

Hence, we have also in the case of anisotropic and inhomogeneous solid bodies (with respect to diffusivity) for statistical cuts the relations

$$\lambda_{k,x}=\lambda_{k,y}=\lambda_{k,z}=n_k \tag{4}$$

which are generalizations of the so called Delesse'law. [1]

For well defined cuts equation (4) is in general not valid. These cuts are of practical importance by the consideration of slope sliding of grained material since in this case we consider only cuts which do not cut the grains but follow the contact surfaces. Therefore, we consider in the following, to retain this possibility of well defined cuts λ_k as a as yet undefined value, which is assumed isotropic also in the case of anisotropic and inhomogeneous solid body with respect to diffusivity. We take a finite volume of the solidliquid mixture with the closed surface (well defined or statistical) and consider the forces acting on the solid and liquid phase and which enter the equations of motion. For this generalisation it is necessery to sum up all the wet parts of the external surface and all the internal liquid solid interfaces. This area surrounds the liquid body completely. Denotes $dO_{k,a,fl.}$ a wet vectorial surface element (orientes in the external direction) and denotes $dO_{k,i,fl.}$ a surface element of the internal liquid solid interface (oriented into the solid body), the resulting force of all the normal forces acting on these surfaces onto the liquid is given by $-\int p_k\, dO_{k,a,fl.} - \int p_k\, dO_{k,i,fl.}$, where p_k is the pressure in the liquid phase.

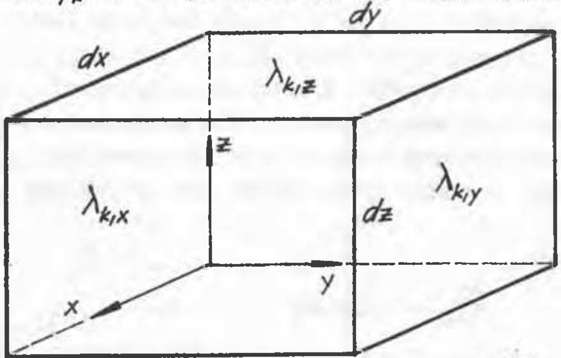

FIG. 2

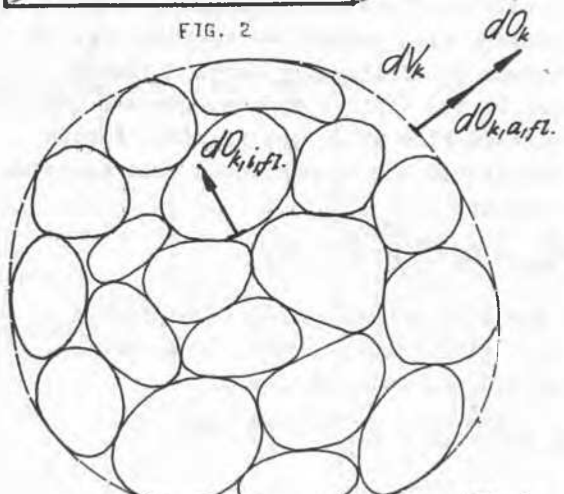

FIG. 3

Since this boundary surface surrounds completely the liquid body Gaussian law can be applied and renders

$$\int p_k\, dO_{k,a,fl.} + \int p_k\, dO_{k,i,fl.} = \int \nabla p_k\, dV_{k,fl.} = \int n_k \nabla p_k\, dV_k \tag{5}$$

where $$dV_{k,fl.} = n_k\, dV_k$$

denotes a volume element of the liquid body and dV_k denotes a volume element of the mixture. The quasivolume force resulting from skin friction between the solid grains and acting onto the liquid is given by $\int R_{k,fl.}\, dV_k$. Further the gravity force $-\int \frac{\gamma_{fl.}}{g} n_k \nabla U\, dV_k$ has to be considered.

In this makroscopic view we have devided the forces acting on the liquid body into surface forces, volume forces and quasivolume forces. U denotes the gravity potential, $\gamma_{fl.}$, the specific weight of the liquid and g denotes gravitational acceleration. The equation of motion of the liquid body therefore reads:

$$-\int p_k\, dO_{k,a,fl.} - \int p_k\, dO_{k,i,fl.} + \int R_{k,fl.}\, dV_k - \int \frac{\gamma_{fl.}}{g} n_k \nabla U\, dV_k = \int \frac{\gamma_{fl.}}{g} n_k \frac{D w_k^*}{dt}\, dV_k \tag{6}$$

Equation (5) together with (6) renders after the application of Gauss'theorem and using the following limit

$$\lim_{V_k \to 0} \frac{1}{V_k} \int_{V_k} \left[-n_k \nabla p_k + R_{k,fl.} - \frac{\gamma_{fl.}}{g} n_k \nabla U - \frac{\gamma_{fl.}}{g} n_k \frac{D w_k^*}{dt} \right] dV_k = 0$$

the differential form of the equation of motion of the liquid

$$\frac{\gamma_{fl.}}{g} n_k \frac{D w_k^*}{dt} = R_{k,fl.} - n_k \nabla p_k - \frac{\gamma_{fl.}}{g} n_k \nabla U \tag{6 a}$$

which holds for each volume. The frictional force follows from (6 a)

$$R_{k,fl.} = n_k \left[\nabla p_k + \frac{\gamma_{fl.}}{g} \left(\nabla U + \frac{D w_k^*}{dt} \right) \right] \tag{7}$$

This equation holds in general and does not use the assumption of a point contact between the grains ($\lambda_k = 1$). We denote the substancial derivative with respect to the liquid by

$$\frac{Da}{dt} = \frac{\partial a}{\partial t} + (w_k^* \cdot \nabla) \cdot a \tag{7 a}$$

and we use the absolute velocity of the liquid with respect to an inertial frame

$$w_k^* = w_{k,K} + \frac{w_k}{n_k} \tag{7 b}$$

We want to construct a generalized filter law, that means a general relation between the frictional force $R_{k,fl.}$ and the motion of the liquid. For this sake we introduce the filter velocity w_k .

For its definition we cut a small tetrahedron out of the flowed through body, where the axis x,y,z from a rectangular coordinate system.

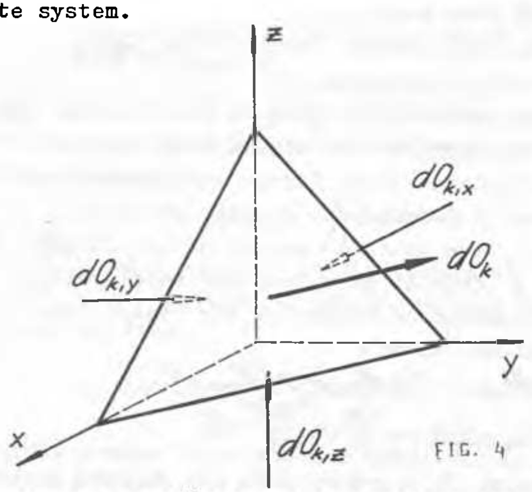

FIG. 4

We denote with dO_k the outer surface vector of the basis plane. The vectorial components in the coordinate directions $dO_{k,x}$, $dO_{k,y}$, $dO_{k,z}$ are identical to the inward normal vectors of the surface elements which are parallel to the planes x = const., y= const., z = const., respectively. We denote the liquid volume which flows through the surface element dO_k in the time interval dt by $dq_{k,0} \cdot dt$; analogous $dq_{k,x} \cdot dt$, $dq_{k,y} \cdot dt$, $dq_{k,z} \cdot dt$ denote the liquid volumina which enter in the time interval dt through the surface elements with normal vectors parallel x,y,z respectively. Under the assumption 2.) we have for the liquid

$$dq_{k,0} = dq_{k,x} + dq_{k,y} + dq_{k,z} \tag{8}$$

We denote the values of $dq_{k,x}$, $dq_{k,y}$, $dq_{k,z}$ per unit of the surface area by $v_{k,x}$, $v_{k,y}$, $v_{k,z}$ and it follows then

$$dq_{k,0} = |dO_{k,x}|\, v_{k,x} + |dO_{k,y}|\, v_{k,y} + |dO_{k,z}|\, v_{k,z} = \mathfrak{w}_k \cdot dO_k \tag{9}$$

there $\mathfrak{w}_k$ denotes the vector with components $v_{k,x}$, $v_{k,y}$, $v_{k,z}$
Since this vector behaves like a velocity vector it will be called filter velocity. Such a definition of the filter velocity is possible only for incompressible media. In general, the relation between the friction force $\mathcal{R}_{k,fl.}$ and the filter velocity vector $\mathfrak{w}_k$ is nonlinear:

$$\mathcal{R}_{k,fl.} = \varphi(\mathfrak{w}_k) \tag{10}$$

In contradistinction a linear relation between the friction force $\mathcal{R}_{k,fl.}$ and the filtervelocity $\mathfrak{w}_k$ exists in the case of a very slow moving liquid. The most general linear relation between these two vectorfields is constructed by the use of affinor $\mathfrak{A}_k$.
Whence

$$\mathcal{R}_{k,fl.} = -\mathfrak{A}_k \cdot \mathfrak{w}_k \tag{11}$$

This relation includes inhomogeneity and anisotropy with respect to diffusivity. If we assume a sufficiently large filtering action of the solid body that means a quasi-stationary flow where the inertial forces of the liquid can be neglected then equation (7) renders

$$\mathcal{R}_{k,fl} = n_k \nabla\left(\frac{\gamma_{fl.}}{g}\mathcal{U} + p_k\right) \tag{12}$$

The inertial forces cannot be neglected in a short time initial transitional motion. From (12) and (11) we have

$$n_k \nabla\left(\frac{\gamma_{fl.}}{g}\mathcal{U} + p_k\right) = -\mathfrak{A}_k \cdot \mathfrak{w}_k \tag{13}$$

or

$$\nabla\left(\frac{\mathcal{U}}{g} + \frac{p_k}{\gamma_{fl.}}\right) = -\mathfrak{W}_k \cdot \mathfrak{w}_k \tag{13 a}$$

with

$$\mathfrak{W}_k = \frac{\mathfrak{A}_k}{n_k \gamma_{fl.}} \tag{14}$$

$\mathfrak{W}_k$ is denoted drag affinor. A different form of (13 a) can be achieved by a left hand multiplication by the inverse affinor, called diffusivity affinor

$$\mathfrak{W}_k^{-1} = \mathfrak{K}_k \tag{15}$$

which is in general skew symmetric and a function of spatial coordinates. This renders

$$\mathfrak{v}_k = -\mathfrak{K}_k \nabla\left(\frac{U}{g} + \frac{p_k}{\gamma_{fl}}\right) \tag{16}$$

At a contact surface between two solid bodies the diffusivity affinor has a jump. $\mathfrak{K}_k$ is therefore a piecewice continuous function of spatial coordinates and allows the matrix representation

$$\mathfrak{K}_k = \begin{bmatrix} k_{k,xx} & k_{k,xy} & k_{k,xz} \\ k_{k,yx} & k_{k,yy} & k_{k,yz} \\ k_{k,zx} & k_{k,zy} & k_{k,zz} \end{bmatrix} \tag{17}$$

We set

$$\frac{U}{g} + \frac{p_k}{\gamma_{fl}} = h_k \tag{18}$$

and denote the static pressure by h_k .

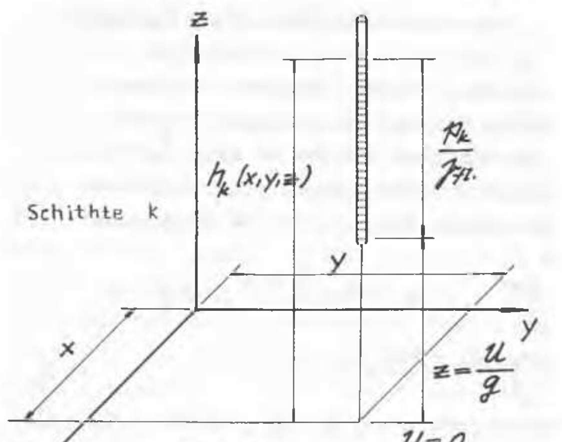

FIG. 5

From (16) and (18) we have

$$\mathfrak{v}_k = -\mathfrak{K}_k \cdot \nabla h_k \tag{19}$$

Introducing the gradient of the static pressure in the liquid

$$\vec{J}_k = -\nabla h_k \tag{19 a}$$

we find a new form of (19)

$$\mathfrak{v}_k = \mathfrak{K}_k \cdot \vec{J}_k \tag{19 b}$$

This is a generalisation of the wellknown Darcy filter law. If the diffusivity in the solid body is directional independent, then with $\mathfrak{K}_k = k_k \mathfrak{E}$ and from (19)

$$\mathfrak{v}_k = -k_k \nabla h_k = k_k \vec{J}_k \tag{19 c}$$

where k_k is the wellknown scalar diffusivity of a layer k . Denoting the absolute values of $\mathfrak{v}_k$ and $\vec{J}_k$ by v_k and J_k , respectively then the well known form of Darcy'law follows from (19 c)

$$v_k = k_k J_k \tag{19 d}$$

From (19 c) it can be seen that in the special case of constant diffusivity coefficients k_k the static pressure h_k playes the role of a velocity potential. In [2] this static pressure h_k is introduced into the problem of consolidation of clay layers following

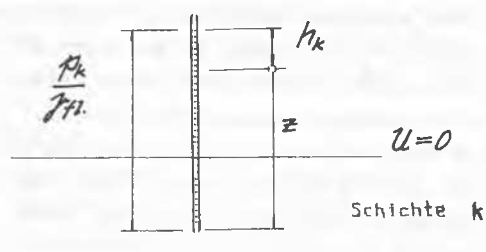

FIG. 6

Since $U=-gz$ it follows from (18) that

$$h_k = \frac{p_k}{\gamma_{fl}} - z \tag{20}$$

To find the equations of motion for the cutted part of the solid body skeleton we consider in a makroscopic view surface stresses acting on the solid portion of the outer surface everaged over the whole makroscopic surface element. The state of stress in such an element can be described by a symmetric stress tensor T_k . In the case of a grained solid body the danger of slope sliding is discussed using special (non statistical) cuts in such a way, that the grains are not cutted. Here, T_k is the stress tensor of the grain-to-grain contact forces. Using statistical cuts (in this case the cuts run through the solid body material), T_k represents the statistical mean of the stress tensor in the solid material. This expectation of the stress tensor is important in cases where the elastic or plastic property of the solid material has to be considered. The equation of motion for the cutted part of the solid body skeleton reads:

$$\int T_k \cdot dO_k + \int R_{k,fl}\, dV_k + \int p_k\, dO_{k,i,fl} - \int \frac{\gamma_{k,K}}{g}(1-n_k)\nabla U\, dV_k = \int \frac{\gamma_{k,K}}{g}(1-n_k)\frac{D_k v_{k,K}}{dt}\, dV_k \qquad (21)$$

where

$$\frac{D_k \square}{dt} = \frac{\partial \square}{\partial t} + (v_{k,K} \cdot \nabla) \cdot \square \qquad (22)$$

denotes the substantial derivative with respect to the skeleton motion and $v_{k,K}$ denotes the absolute velocity of the solid body skeleton with respect to an inertial system. Here dO_k denotes the outer normal vector of a surface element. The second and third term represents the reactions to the forces acting on the liquid body. The fourth integral represents the body force acting on the solid part of the body and the integral on the right hand side of equation (21) represents the inertial force. $\gamma_{k,K}$ is the specific weigth of the solid material. Considering the meaning of λ_k the following relation can be established

$$dO_{k,a,fl} = \lambda_k\, dO_k \qquad (23)$$

Using (5) and (23) and Gauss'theorem we have

$$\int p_k\, dO_{k,i,fl} = \int n_k \nabla p_k\, dV_k - \int p_k\, dO_{k,a,fl} \qquad (24)$$

$$= \int [n_k \nabla p_k - \nabla(\lambda_k p_k)]\, dV_k$$

Introducing (24) into (21) and using again Gauss'lemma and the limit $V_k \to 0$ we have

$$\lim_{V_k \to 0} \frac{1}{V_k}\int_{V_k} [\nabla\cdot T_k - R_{k,fl} - (1-n_k)\frac{\gamma_{k,K}}{g}\nabla U + n_k \nabla p_k - \nabla(\lambda_k p_k) - \frac{\gamma_{k,K}}{g}(1-n_k)\frac{D_k v_{k,K}}{dt}]\, dV_k = 0$$

and the differential form of the equation of motion in the following form

$$\frac{\gamma_{k,K}}{g}(1-n_k)\frac{D_k v_{k,K}}{dt} = \nabla\cdot T_k - R_{k,fl} + n_k \nabla p_k - (1-n_k)\frac{\gamma_{k,K}}{g}\nabla U - \nabla(\lambda_k p_k) \qquad (25)$$

which holds for every arbitrary volume. Together with (7) equation (25) renders

$$\nabla\cdot T_k - n_k\left[\nabla p_k + \frac{\gamma_{fl}}{g}\left(\nabla U + \frac{D v_k^*}{dt}\right)\right] + n_k \nabla p_k - (1-n_k)\frac{\gamma_{k,K}}{g}\nabla U - \nabla(\lambda_k p_k) - \frac{\gamma_{k,K}}{g}(1-n_k)\frac{D_k v_{k,K}}{dt} = 0 \qquad (26)$$

After rearranging and summing we have

$$n_k \frac{\gamma_{fl}}{g}\frac{D v_k^*}{dt} + (1-n_k)\frac{\gamma_{k,K}}{g}\frac{D_k v_{k,K}}{dt} = \nabla\cdot T_k - \frac{\gamma_{k,K} - n_k(\gamma_{k,K} - \gamma_{fl})}{g}\nabla U - \nabla(\lambda_k p_k) \qquad (26\ a)$$

$\nabla\cdot T_k$ represents the resulting surface forces in the solid body skeleton per unit of the volume. In the case of a grained solid body material and contact of the grains in singular points we have $\lambda_k = 1$, if we consider cuts through the points of contact between the grains. In this case we have

$$n_k \frac{\gamma_{fl}}{g}\frac{D v_k^*}{dt} + (1-n_k)\frac{\gamma_{k,K}}{g}\frac{D_k v_{k,K}}{dt} = \nabla\cdot T_k - \frac{\gamma_{k,K} - n_k(\gamma_{k,K} - \gamma_{fl})}{g}\nabla U - \nabla p_k \qquad (27)$$

For statistical cuts $\lambda_k = n_k$ equation (26 a) renders

$$n_k \frac{\gamma_{fl}}{g}\frac{D v_k^*}{dt} + (1-n_k)\frac{\gamma_{k,K}}{g}\frac{D_k v_{k,K}}{dt} = \nabla\cdot T_k - \frac{\gamma_{k,K} - n_k(\gamma_{k,K} - \gamma_{fl})}{g}\nabla U - \nabla(n_k p_k) \qquad (28)$$

Now we have given a clear answer to the question, if the pressure gradient ∇p_k in the full amount or the pressure p_k , by the action of the moving liquid on the solid body, has to be multiplied with the porevolume n_k in the construction of the gradient. This reduces the pressure gradient. The result now depends only on the form of the cut to be considered statistical or through the contact zones of the grains. In neglecting the acceleration terms of the liquid and solid equation (26 a) renders

$$\nabla \cdot \mathcal{T}_k = \frac{\gamma_{k,k} - n_k(\gamma_{k,k} - \gamma_{Fl.})}{g} \nabla U + \nabla(\lambda_k p_k) \tag{29}$$

Representing p_k by the pressure head h_k (18) gives

$$p_k = \gamma_{Fl.}(h_k - \frac{U}{g}) \tag{30}$$

and introducing this in (29) renders

$$\nabla \cdot \mathcal{T}_k = \lambda_k \gamma_{Fl.} \nabla h_k + \gamma_{Fl.}(h_k - \frac{U}{g}) \nabla \lambda_k + (\gamma_{k,k} - \frac{\lambda_k - n_k}{1 - n_k} \gamma_{Fl.}) \frac{1 - n_k}{g} \nabla U \tag{31}$$

from (19) we have

$$\nabla h_k = - K_k^{-1} \cdot w_k = W_k \cdot w_k \tag{32}$$

Looking at (31) and (32) we see that the resulting surface force of the solid body skeleton per unit of volume, of the mixture is a sum of three parts: The first term stems from the motion of the liquid, the second term is the product of the pressure in the liquid multiplied by the gradient of the wet to total surface ratio and the third represents the weight of the solid body skeleton reduced, in general, by the lifting force. The second term is zero everywhere where this ratio is independent of the spatial coordinates. In the case of point contact and for special cuts through the points of contact ($\lambda_k = 1$) the total lifting force is acting. An application is, for instance, the stability consideration of a flow through dam with point contact of the grains ($\lambda_k = 1$), ete. For statistical cuts ($\lambda_k = n_k$) the lifting force vanishes in the case of point contact and area contact. For special cuts where $\lambda_k < n_k$ the lifting force alternates its sign. In all the cases where the statistical mean of the stresses in the solid body skeleton has to be considered, e.g., if the stress strain relation is considered for the elastic or plastic behavior of the solid, a non zero lifting force cannot be expected, since the solid body has to be cutted in a statistical way, there fore $\lambda_k = n_k$. In continuum theory the deformations and deformation rates are related to the surface stresses and are not related in a direct way to the additional forces as, e.g. pressure forces acting from the liquid to the skeleton. In addition it is required that the relation between the deformations and their rates and also their higher derivatives with respect to time t, to the surface stresses and their time derivatives is all with the same in the dry and wet skeleton. This is necessary since the material constants of the dry state should be used in both cases. The cuts have to be constructed in such a way that in the liquid filled skeleton no additional surface stresses which alter the lifting forces come into consideration. This condition is full filled only from the statistical cut.

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[2] G. Heinrich und K. Desoyer, Theorie dreidimensionaler Setzungsvorgänge in Tonschichten, Ing. Arch. 30 (1961), S.225.

A detailed report with examples is given in the book "Soil Mechanics I " (Consolidation of buildings and Groundwatermovements) edited by J. Schimmerl. Springer Verlag Wien (Oct. 1973)