THE DERIVATION OF COMPLEX STRESS-STRAIN RELATIONS

LES RELATIONS CONTRAINTES-DEFORMATIONS NON-LINEAIRES

ВЫВОД ОБОБЩЕННЫХ ЗАВИСИМОСТЕЙ МЕЖДУ НАПРЯЖЕНИЕМ И ДЕФОРМАЦИЕЙ

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SYNOPSIS - A method for obtaining empirically based, non-linear, multi-dimensional stress-strain relations for soils is described. The method permits the relationship between any two, or more, stress and strain components to be closely approximated. The technique can be applied to data obtained by a variety of laboratory procedures.

An experimental examination of the forms that rigorous stress-strain relationships may assume for a sand-clay is then reported. These relationships are derived in terms of invariant parameters, the octahedral stresses and strains, and are believed to represent an advance on earlier stress-strain models. The relative significance of the two octahedral stresses in controlling the deformation response of the soil is assessed. The stress-strain models are then extended to include the effects of stress history. Specifically, the effects of cycling the applied stresses along stress paths similar to those encountered in pavements, and beneath machine bases, are considered.

INTRODUCTION

The study of stress-strain relationships is a key problem in Soil Mechanics. However, despite an extensive literature, reviewed elsewhere (Scott and Ko, 1969) no satisfactory constitutive relations are generally available for soils. Yet in many instances, including, for example, the design of shallow machine foundations and the analysis of pavements, the accurate prediction of stresses and deflections is of greater importance than considerations of bearing capacity and strength.

In this paper an original technique for determining three-dimensional stress-strain relationships is described and typical experimental results are reported. The technique is based on a series of experimental observations that can be made using conventional triaxial compression tests. However, in the experimental work reported here, special cyclic loading triaxial equipment was employed and stress-strain relationships were determined not merely for the first load application, but also for subsequent cycles. Thus, whilst providing no information on failure conditions, the cyclic loading tests supplied most of the data that could be obtained from a conventional triaxial test. In addition, the effects of a stress history, similar to that encountered in many types of dynamically loaded foundation, were assessed.

NOTATION

E_s = secant modulus
E_t = tangent modulus
M_r = resilient modulus
N = number of load applications
S_r = degree of saturation
ε_{ij} = strain tensor
ε_a = axial strain amplitude
ε_f = cumulative non-recoverable axial strain
ε_p = plastic axial strain
ε_r = resilient axial strain
ε_{oct} = octahedral direct strain
γ_{oct} = octahedral detrusion
γ_d = dry density
σ_{ij} = stress tensor
σ_{oct} = octahedral normal stress
τ_{oct} = octahedral shear stress
ν = Poissons ratio

A SURVEY OF STRESS-STRAIN RELATIONSHIPS

Three approaches to the problem of obtaining constitutive relations coupling stresses and strains can be identified (Lazan, 1968). These can be classified under the headings; 1) micromechanistic; 2) phenomenological; 3) ad hoc testing or simulation service evaluation. Most work in Soil Mechanics has concentrated on the phenomenological and ad hoc testing approaches.

Following the phenomenological approach, the stresses and strains in a continuum can be conveniently represented by the second order cartesian tensors, $_{ij}$ and $_{ij}$ which are independent of any assumed properties of the material. The stress-strain relationships of the material can then be defined as

$$\varepsilon_{pq} = B_{pqrs}\ \sigma_{rs} \tag{1}$$

In the most general case, 36 coefficients B_{pqrs} exist and must be evaluated before consummate stress-strain relationships can be formulated. For soils, this cannot be accomplished using existing techniques. Not only are present soil testing procedures incapable of generating a completely generalised stress system but the stresses set up are, with the possible exception of the confining stress in the triaxial test, almost invariably indeterminate. Thus, where the phenomenological approach is adopted, it becomes necessary to make several assumptions about both the test conditions and also the mechanical properties of the soil. By contrast, the simulated service approach to obtaining stress-strain relationships requires fewer prior assumptions about the soil behaviour. This approach has been adopted in the work reported here.

If the simulated service approach is to be successfully applied it is necessary to include as many of the elements of the stress and strain tensors as possible both in the experimental observations and in the subsequent analysis. Moreover, the generality of any empirical stress-strain relationship is improved if it can be expressed in terms of stress and strain invariants. For the simple system of triaxial compressive stress this can be conveniently achieved by expressing the relationship in terms of the octahedral stresses, σ_{oct} and τ_{oct} and the octahedral strains, ε_{oct} and γ_{oct}.

As a first approximation, the general form of a non-linear stress-strain relationship for cohesive soils might be defined by;

$$\varepsilon_{oct} = f_1\ (\sigma_{oct}) + f_2\ (\tau_{oct}) \tag{2}$$

$$\gamma_{oct} = f_3\ (\sigma_{oct}) + f_4\ (\tau_{oct}) \tag{3}$$

where f_1 to f_4 are arbitrary functions. However the generality of these relationships may be restricted by the fact that the octahedral stresses and strains are related to only the first and second stress and strain invariants. It may, therefore, be desirable, when fully stating stress-strain relationships, to include parameters which are functions of the third invariants of the stress and strain tensors (Newmark, 1960). However, the work reported here has been based on the simplified approach represented by eqns. 2 and 3.

Whilst some previous studies of stress-strain relationships have made use of the two invariant stress parameters, σ_{oct} and τ_{oct}, most investigations have been reported in terms of only one non-invariant strain measurement; usually the axial strain. Here, the relationships have been defined in terms of either the tangent or secant moduli, E_t or E_s, or, in the case of cyclically loaded soils, in terms of the resilient modulus M_r; the ratio of the cyclic deviator stress to the recoverable axial strain.

Almost without exception, prior investigations of the stress-strain response of cyclically stressed soils have been reported solely in terms of either the confining stress, σ_3, (e.g. Biarez, 1962; Mitry, 1965) or, alternatively, the octahedral normal stress, σ_{oct}, (e.g. Mitry, 1965). The general forms of these relations could be written as

$$M_r = K_1\ \sigma_{3m}^{\ n} \tag{4}$$

$$M_r = K_2\ \sigma_{oct}^{\ m} \tag{5}$$

where K_1, K_2, n and m are constants. These relationships were usually stated to be independent of the deviatoric stress component (e.g. Mitry 1965) and to apply solely to granular (noncohesive) soils.

However, a better understanding of the relative contributions of the spherical and deviatoric components of stress to the deformation of granular soils has recently begun to emerge. Thus it has been quantitatively demonstrated (Holden 1967; Gerrard, 1967) that for single loadings and for simple stress systems, the modulus can be related to the stress components by relationships of the form;

$$E_t = a + b\ \sigma_{oct} - c\ \tau_{oct} \tag{6}$$

where a, b, and c are positive empirical constants.

Subsequently, the author has shown (Shackel, 1971, 1972) that, for a cyclically stressed sand-clay, the axial strain amplitudes, ε_a, the cumulative, non-recoverable axial strains, ε_f, and the resilient axial strains, ε_r, can be related to the applied stresses by relationships of the general form

$$\varepsilon = K\ \frac{\tau_{oct}^{\ v}}{\sigma_{oct}^{\ u}} \tag{7}$$

where u and v are empirical constants which depend on the number of stress cycles. Thus the resilient moduli are related to the octahedral stresses by the expression

$$M_r = K \frac{\sigma_{oct}^{\,u}}{\tau_{oct}^{\,v-1}} \qquad (8)$$

It has been determined that the model represented by eqn. 8 can be used to accurately characterise data published earlier for seven cyclically stressed soils, ranging from silty clays (Kallas and Riley, 1967) to sands and gravels (Kallas and Riley, 1967; Mitry, 1965 Tanimoto and Nishi, 1970). Moreover, in most cases, statistical analyses of the data have shown that, of the two stress components, the octahedral shear stress, τ_{oct}, was the more significant in controlling the soil deformation.

The relationships expressed in equations 7 and 8 completely define the stress state investigated and therefore represent an advance on the simpler models described by eqns. 4 and 5. However, they do not provide complete descriptions of soil behaviour since they are based on measurements of axial strain alone. It was decided, therefore, to investigate whether the response of the soil could be characterised by more rigorous relationships based not just on the octahedral stresses but also on the octahedral strains ε_{oct} and γ_{oct}.

A TECHNIQUE FOR DEFINING EMPIRICAL STRESS-STRAIN RELATIONS

The author has found that, for the triaxial test, the relationship between some arbitrary direct strain, ε, and the stresses, σ_{oct} and τ_{oct}, can be represented by a unique surface inclined in three dimensional, ε, τ_{oct}, σ_{oct} space. Such a surface is shown in Fig. 1. for a soil exhibiting both internal friction and cohesion. The surface is bounded by the conditions of unconfined compression (OABD in Fig. 1.), isotropic compression (OUVW) and failure in unconfined and triaxial compression (ANPQ). The general shape of the surface shown in Fig. 1. is based on experimental data for axial strain obtained by the author for a sand-clay (Shackel, 1972) and is consistent with that published for other soils (e.g. Gibbs et al, 1960).

The effect of cycling the applied stresses is shown in Fig.2. If the soil is loaded and unloaded along a stress path chosen so that the principal stress ratio, σ_1/σ_3, remains constant, the deformation follows a path such as abc in Fig.2. At peak stress the total strain will comprise the sum of a plastic component, ε_p, and an elastic or recoverable deformation, ε_r. During the second stress application, the deformation follows path cde in Fig.2. Thus by conducting a series of cyclic loading tests over a range of principal stress ratios, σ_1/σ_3, or confining stresses, σ_3, it is possible to define the stress-strain response of the soil in terms of a series of surfaces in ε, σ_{oct}, τ_{oct}, space each corresponding to a cycle of either loading or unloading.

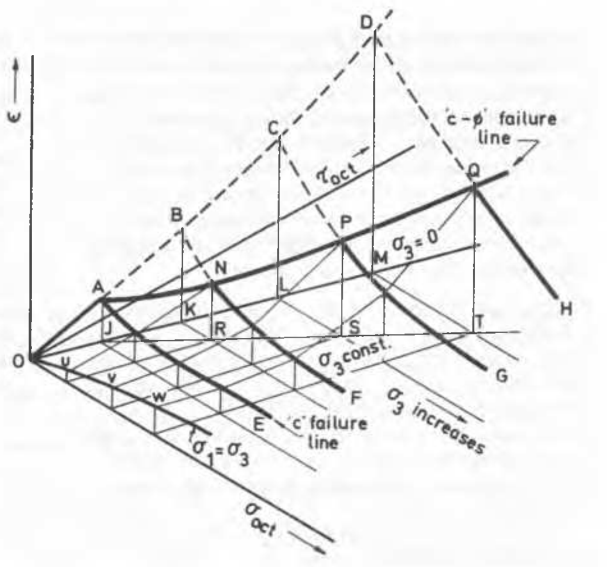

Fig.1. TYPICAL STRAIN VS. STRESS SURFACE

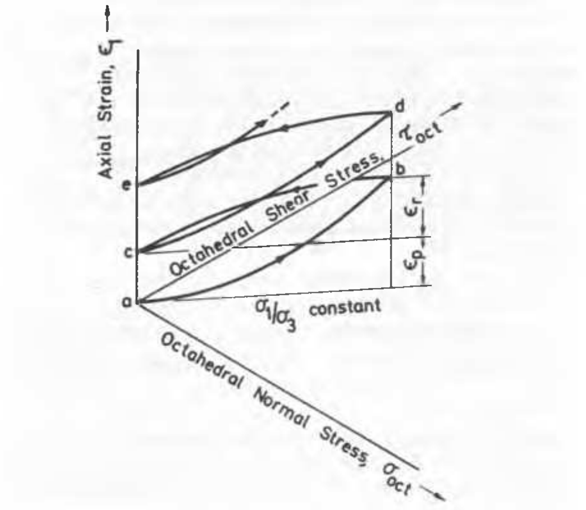

Fig.2. THE EFFECT OF CYCLING THE APPLIED STRESSES

In order to obtain stress-strain relationships in a form suitable for engineering application it becomes necessary to describe these surfaces mathematically. This can be done in two ways. The first approach is based on the use of a series of bicubic spline functions to provide a 'best fit' to the experimental stress-strain data for the soil (Desai, 1971). The second alternative method employs the statistical techniques of multiple regression analysis to approximate the experimentally defined surfaces.

In contrast to the use of spline functions, regression analyses do not necessarily provide the best available fit to the data. Moreover, the range of data that can be represented by a regression model is often limited. Nevertheless, regression models can provide excellent engineering predictions of the response of a soil to changes in the stress regime, provided that such changes do not fall outside the domain of the experimental observations upon which the models are based.

The technique adopted in the experimental work was to determine the octahedral strains, ε_{oct} and γ_{oct}, corresponding to various combinations of the octahedral stresses σ_{oct} and τ_{oct}. The stress-strain relationships were then defined by fitting multiple regression models to the experimental data. The general forms of these models were based on previously published relationships such as those given in eqns. 5, 6 and 7.

THE EXPERIMENTAL WORK

(a) The Soil Characteristics

Most reported investigations of constitutive relations for soils have been restricted to either purely granular or purely cohesive soils and little work has been done on soils combining both friction and cohesion. It was therefore decided to examine a sand-clay which combined both frictional and cohesive components of shear strength in roughly equal proportions. The soil comprised 60% sand and 40% kaolinite by weight. The properties of the soil are given in Table 1. The soil was mechanically mixed and then compacted to a dry density, γ_d, of 115.8 lb/ft^3 at a saturation, S_r, of 80% by floating mould compaction; a variant of static compaction giving specimens of exceptional uniformity (Shackel, 1970a).

Table 1

Soil Properties

Liquid Limit	26.0%
Plastic Limit	17.6%
Shrinkage Limit	14.9%
Maximum Proctor Dry Density	112.0 lb/ft^3
Maximum Mod.AASHO Dry Density	119.5 lb/ft^3
Optimum Saturation, S_r,	86.5%

(b) The Test Conditions

After curing for 3 days at 20°C the specimens were subjected to cyclic triaxial loading using special triaxial equipment described in detail elsewhere. (Shackel, 1970b). The triaxial tests were essentially unconsolidated and undrained but the pore air pressures were allowed to equalise with atmospheric pressure. Each test comprised 10,000 applications of axial stress of controlled amplitude applied at fixed principal stress ratios, σ_1/σ_3, along stress paths in which both the axial and confining stresses increased simultaneously and in phase.

The stress conditions and stress paths examined are shown diagrammatically in Fig.3. These formed an incomplete fully randomised factorial experiment comprising 17 tests. Four levels of τ_{oct} from 5.66 to 22.62 lb/in^2 and five levels of σ_1/σ_3 from 3.25 to 10.27 were examined. Thus each test was conducted at a different value of the octahedral normal stress, σ_{oct}. The magnitudes of the stresses and the range of the principal stress ratios employed were chosen as being typical of those encountered in road subgrades and many shallow foundations.

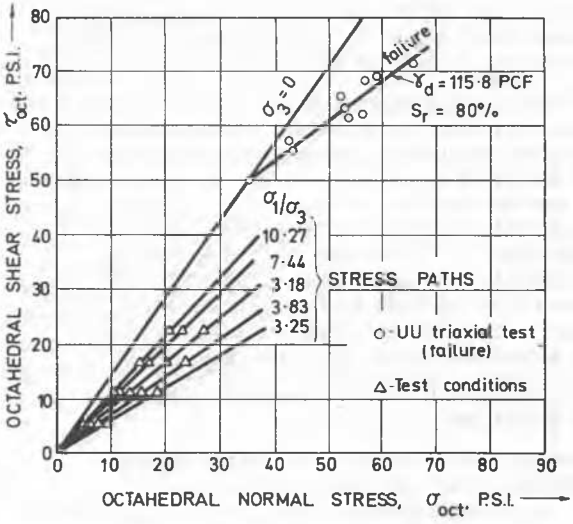

Fig.3. THE STRESS PATHS AND STRESS LEVELS USED IN THE EXPERIMENTAL WORK SHOWN IN RELATION TO THE FAILURE ENVELOPE

Throughout the cyclic loading tests the frequency of stress application was maintained at 2 cpm. The duration of each test was then approximately 84 hours. Observations of the applied axial and confining stresses and of the axial and diametral strains were recorded automatically at selected intervals during the tests. In all cases, complete stress-strain hysteresis loops were recorded for the selected stress cycles.

(c) Results

It was found that, for the stress paths and principal stress ratios used in the experimental work, the diametral strains were extremely small. During the tests the soil was, therefore, close to the condition of earth pressure at rest; the so called K_o condition. Values of the apparent Poisson's Ratio, ν, were generally within the range 0 to 0.2 with the majority of values being less than 0.05. These values of ν are of similar magnitude to those reported earlier for a repetitively stressed, cement-stabilised sand (Morgan and Williams, 1970). Thus in the work reported here, the axial strain, ε_1, formed the major component of the octahedral strains ε_{oct} and γ_{oct}

In order to determine the form of the octahedral stress-octahedral strain relationships the experimental results were analysed using a digital computer. First the data were grouped according to the numbers of stress applications at which observations had been made. Each data group then comprised a series of strain observations for a range of applied stresses, thereby defining a series of surfaces in three dimensional, ε, σ_{oct}, τ_{oct} space.

Using the techniques of multiple linear regression it was established that, irrespective of the number of stress applications considered, the surfaces relating the invariant strains, ε_{oct} and γ_{oct}, and the octahedral stresses, σ_{oct} and τ_{oct}, could be defined by a comparatively simple mathematical model. Thus, following logarithmic transformation, the data for each strain parameter lay in a plane inclined in ε, σ_{oct}, τ_{oct} space. This is illustrated in Fig.4.

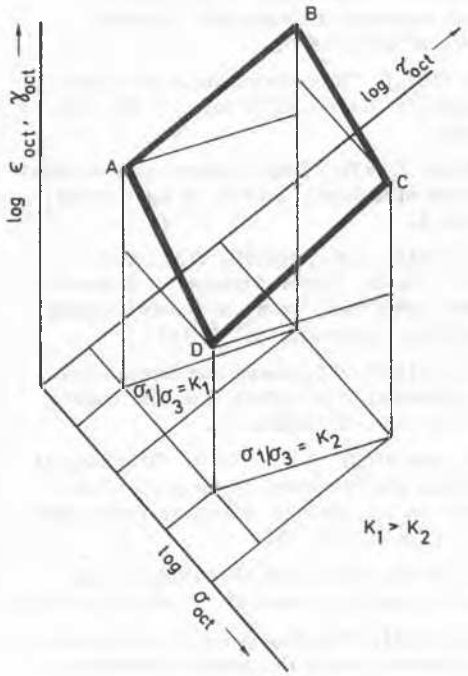

Fig.4. THE REGRESSION PLANE USED TO FIT THE EXPERIMENTAL STRESS-STRAIN DATA

Following normal statistical procedures the linearity of the regression models was tested and found to be significant at levels of significance, α, better than 0.025.

For a designated number of load applications, the regression models can be represented as follows:

$$\varepsilon_{oct} = K_\varepsilon \frac{\tau_{oct}^{\,b}}{\sigma_{oct}^{\,a}} \qquad (9)$$

$$\gamma_{oct} = K_\gamma \frac{\tau_{oct}^{\,d}}{\sigma_{oct}^{\,c}} \qquad (10)$$

where K_ε and K_γ, a, b, c, and d are empirical constants. In all cases, the regression analysis revealed that the octahedral shear stress, τ_{oct}, had a greater influence on the stress-strain behaviour of the soil than the octahedral normal stress, σ_{oct}.

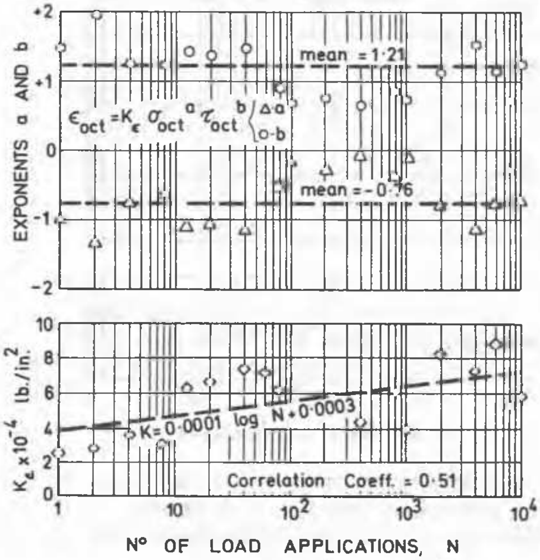

Fig. 5. THE RELATIONSHIP BETWEEN THE OCTAHEDRAL NORMAL STRAIN AND THE OCTAHEDRAL STRESSES AS A FUNCTION OF THE NUMBER OF LOAD CYCLES

The effects of load repetition on the octahedral stress-octahedral strain relationships are shown diagrammatically in Figs. 5 and 6. It was found that the exponents a, b, c and d in eqns. 9 and 10 were all substantially independent of the number of load applications. However, the constants, K_ε and K_γ showed a weak but significant correlation with the logarithms of the number of load applications, tending to gradually increase with increasing numbers of cycles.

By including the number of load applications, N, as a variable in the multiple regression analyses, the following simple stress-strain-stress history relationships were obtained.

$$\varepsilon_{oct} = 0.38 \frac{\tau_{oct}^{1.24} N^{0.009}}{\sigma_{oct}^{0.82}} \qquad (11) \text{ (microstrain)}$$

$$\gamma_{oct} = 0.98 \frac{\tau_{oct}^{1.58} N^{0.08}}{\sigma_{oct}^{1.14}} \qquad (12) \text{ (microstrain)}$$

Here it was found that the regression models were significantly linear at a level of significance, α of 0.005 or better.

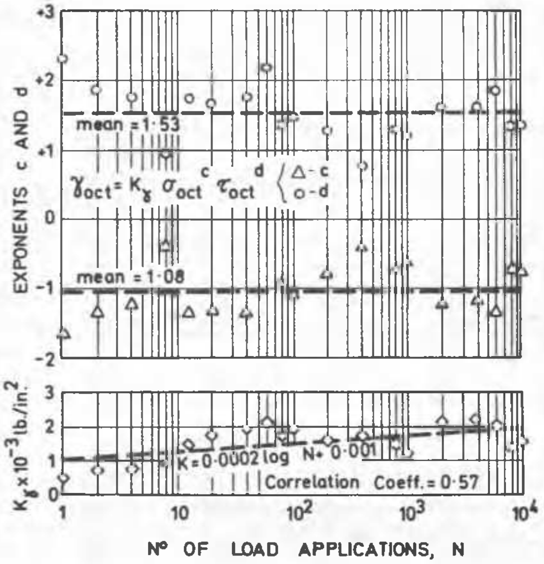

Fig.6. THE RELATIONSHIP BETWEEN THE OCTAHEDRAL DETRUSION AND THE OCTAHEDRAL STRESSES AS A FUNCTION OF THE NUMBER OF LOAD APPLICATIONS

SUMMARY AND CONCLUSIONS

A method for deriving empirically based constitutive relations for soils has been described. The method employs the well established statistical techniques of multiple linear regression. Using these techniques it was found possible to formulate stress-strain relationships for a $c - \phi$ soil in more general terms than those reported hitherto.

In the experimental work the techniques of cyclic triaxial loading were employed to delineate both the stress-strain relationships and the influence of stress history. However, the method described here is equally applicable to the analysis of data from conventional triaxial tests.

The method has the advantage that it enables stress-strain relationships to be defined in terms of all the components of stress and strain using invariant stress and strain parameters. Moreover, it is possible both to include the effects of stress history in the analyses and to statistically assess the relative significance of the various stress and strain components in controlling the soil response.

It is not suggested that stress-strain relations obtained using the techniques described in this paper are fundamentally correct. However, the multiple regression analyses provide statistical measures of the significance of the regression as a whole. Thus, if it can be shown that there is no significant lack of fit between the regression models and the empirical data, the models can be regarded as being useful engineering approximations to the true stress-strain relationships. The models may then be used, with confidence, to predict the strains resulting from stated stress changes, provided that these changes do not fall outside the domain of the experimental observations.

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