CONSOLIDATION OF A SEMI-INFINITE POROUS MEDIUM SUBJECTED TO SURFACE TANGENTIAL LOADS

CONSOLIDATION DU MILIEU POREUX SEMI-INFINI MIS A CHARGE TANGENTIEL SUPERFICIEL.

КОНСОЛИДАЦИЯ ПОЛУБЕСКОНЕЧНОЙ ПОРИСТОЙ СРЕДЫ ПРИ РАСПРЕДЕЛЕННОЙ ПО ПОВЕРХНОСТИ КАСАТЕЛЬНОЙ НАГРУЗКЕ

N. BABU SHANKER, M. Tech., Lecturer in Civil Engineering
K.S. SARMA, Ph.D. Lecturer in Mathematics
M.V. RATNAM, Ph.D. Professor of Civil Engineering (India)

SUMMARY. This paper presents and analytical investigation of the consolidation of a semi-infinite clay layer subjected to a uniform tangential load applied over a long strip at its surface. Biot's theory (1941) is made use of along with McNamee and Gibson's (1960) displacement functions. Application of Fourier Sine and Laplace Transforms leads to a solution of the problem. The excess pore-pressures and vertical settlements are evaluated for two specific cases: firstly, when the top surface is free-draining and the secondly, when it is completely impervious.

NOTATION

G	Shear modulus
$c = 2Gn\,k$	Coefficient of consolidation
$k=$	Coefficient of permeability
$n=(1-\nu)/(1-2\nu)$	Auxiliary elastic constant
t	time coordinate
u	displacement in x-direction (horizontal)
w	displacement in z-direction (vertical)
x	space coordinate (horizontal)
z	space coordinate (vertical)
ν	Poisson's ratio
σ	excess pore pressure
σ_{xx}, σ_{zz}	total compressive stresses
τ_{xz}	shear stress
$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$	Two dimensional Laplacian operator

INTRODUCTION

A self-consistent and general theory of three-dimensional consolidation (Poro-elasticity) was developed by Biot (1941). The theory successfully explains the consolidation behaviour of a solid sphere (Gibson et al, 1963) and practical phenomenon like Noordbergum-effect (Verrujt, 1969). The classical one-dimensional consolidation theory (Terzaghi,1943) can be treated as a special case of Biot's theory.

Problems of poro-elasticity (like elasticity can easily be solved by introducing one or more stress or displacement functions (Biot,1956; Josselin de Jong, 1957;McNamee and Gibson, 1960; Schiffman and Fungaroli, 1965). Few problems, wherein the normal loads are applied on the boundary of a consolidating semi-infinite body, have been solved. However, in Foundation Engineering Practice, the footings may be subjected to inclined loads, which can be resolved into normal and tangential components. Moreover, the friction between the rough footing and foundation soil may introduce some shear at the foundation soil may introduce some shear at the foundation level. Hence an attempt is made in this paper to study the consolidation behaviour of a long flexible strip footing (plane-strain case) resting on a very thick clay layer, with shear load uniformly distributed along the width of the strip.

GOVERNING EQUATIONS

Biot's theory in its simplest form will be used,wherein the clay is assumed to be a homogeneous, isotropic linearly elastic and saturated (with incompressible water) medium. In terms of the displacement functions E and S (McNamee and Gibson,1960) the governing equations are

$$\nabla^4 E = c \frac{\partial}{\partial t} \nabla^2 E$$

$$\nabla^2 S = 0 \qquad (1)$$

The boundary conditions of the problem are (for $t > 0$)

$$\sigma_{zz} = 0, \quad z = 0, \quad |x| < \infty$$

$$\left.\begin{aligned}\tau_{xz} &= f, \qquad z = 0, \qquad |x| < b \\ &= 0, \qquad z = 0, \qquad |x| > b\end{aligned}\right\} \tag{2}$$

$$\sigma = 0, \qquad z = 0, \qquad |x| < \infty$$

(for pervious case)

$$\frac{\partial \sigma}{\partial z} = 0, \qquad z = 0, \qquad |x| < \infty$$

(for impervious case)

The governing equations and the boundary conditions can be non-dimensionalised by dividing all the stresses by "f", all lengths by "b" and all times by "b^2/c". For simplicity, hereafter only the new non-dimensional variables will be used with old notation.

The various displacements and total stresses are given by

$$\begin{aligned} u &= -\frac{\partial E}{\partial x} + z\,\frac{\partial S}{\partial x} \\ w &= -\frac{\partial E}{\partial z} - S + z\,\frac{\partial S}{\partial z} \end{aligned} \tag{3}$$

$$\frac{\sigma_{zz}}{2G} = -\frac{\partial^2 E}{\partial x^2} - z\,\frac{\partial^2 S}{\partial z^2} + \frac{\partial S}{\partial z}$$

$$\frac{\tau_{xz}}{2G} = \frac{\partial^2 E}{\partial x\,\partial z} - z\,\frac{\partial^2 S}{\partial x\,\partial z}$$

$$\frac{\sigma}{2G} = \frac{\partial S}{\partial z} - n\,\nabla^2 E$$

METHOD OF SOLUTION

Successive application of Fourier Sine Transform with respect to "x" and Laplace Transform with respect to "t" on the variable E, yields its transformed value $\bar{E}$:

$$\bar{E}(\alpha, z, p) = \int_0^\infty\!\!\int_0^\infty E\,(x, z, t) \times \sin \alpha x \,.\, e^{-pt}\, dx\, dt \tag{4}$$

and the inverse relation is:

$$E(x, z, t) = \frac{1}{\pi^2 i} \int_0^\infty \int_{Bp} \bar{E}(\alpha, z, p) \times \sin \alpha x \,.\, e^{pt}\, d\alpha\, dp \tag{5}$$

Similar transformations may be defined for other variables like S etc.

The solutions of eqs. (1) using the conditions in eq.(2) will yield

$$\begin{aligned} \bar{E} &= A_1\, e^{-\alpha z} + A_2\, e^{-\alpha z (1+s)^{1/2}} \\ \bar{S} &= B\, e^{-\alpha z} \end{aligned} \tag{6}$$

where for Pervious Bundary

$$2GA_1 = -\frac{\sin \alpha\,(1+ns)}{\alpha^5\, s\, P}$$

$$2GA_2 = \frac{\sin \alpha}{\alpha^5\, s\, P}$$

$$2GB = -\frac{n \sin \alpha}{\alpha^4\, P}$$

$$P = 1 + ns - (1+s)^{1/2}; \qquad p = \alpha^2 s$$

and for impervious case:

$$2GA_1 = -\frac{\sin \alpha\,[1 + ns\,(1+s)^{1/2}]}{\alpha^5\, s\, Q}$$

$$2GA_2 = \frac{\sin \alpha}{\alpha^5\, s\, Q}$$

$$2GB = \frac{-n \sin \alpha \,.\, (1+s)^{1/2}}{\alpha^4\, Q}$$

$$Q = 1 + (ns - 1)(1+s)^{1/2}$$

DISCUSSION

a) PERVIOUS CASE

From eqs (4,5 and 6), one obtains after effecting inversion with respect to "t":

$$2GE = -\frac{2}{\pi} \int_0^\infty \frac{\sin \alpha \,\sin \alpha x}{\alpha^3}\, d\alpha \left[\frac{2n + \alpha z}{2n-1}\, e^{-\alpha z} - e^{-\alpha^2 t}\left(I_4 + e^{-\alpha z} I_5\right)\right]$$

and similarly:

$$2GS = -\frac{2n}{\pi} \int_0^\infty \frac{\sin \alpha \,\sin \alpha x}{\alpha^2}\, e^{-\alpha z}\, d\alpha \left[\frac{2}{2n-1} - e^{-\alpha^2 t}\, I_6\right]$$

where the various integrals I_4, I_5, I_6 etc. are tabulated under appendix.

The surface settlement (w at z=0) is given by:

$$2GW_{z=0} = \frac{2}{\pi} \int_0^\infty \frac{\sin \alpha \,\sin \alpha x}{\alpha^2}\, d\alpha \left[\frac{1}{2n-1} - n e^{-\alpha^2 t}\, I_6\right]$$

The settlement of the centre of the strip (x=0) is zero at all times, and the edge settlement (x=1) is given by:

$$2GW_{\substack{z=0\\x=1}} = \frac{2}{\pi} \int_0^\infty \frac{\sin^2 \alpha}{\alpha^2}\, d\alpha \left[\frac{1}{2n-1} - n e^{-\alpha^2 t}\, I_6\right] \tag{7}$$

the ultimate edge settlement ($t \to \infty$) is given by:

$$2G\, w_{\substack{z=0\\x=1\\t=\infty}} = \frac{1}{2n-1} \tag{8}$$

Thus the ultimate settlement is a function of n and G (or ν and G), which is a fact from elasticity, It is of interest to note that the immediate settlement (i.e., $w_{\substack{z=0\\x=1\\t=0}}$) turns out to be zero for all values of n. This fact can also be verified from eq.(8), by letting $n \to \infty$ (or $\nu = 0{,}5$). Physically this can be explained by the fact that at t=0 i.e., immediately after the application of the load, there is no chance for the clay to dilate and it behaves as an incompressible material.

The time-dependent consolidation settlement is defined as:

$$2GW_c = 2GW_{t=t} - 2GW_{t=0}$$

The progress of consolidation is measured by the ratio, called the degree of settlement U_s as:

$$U_s = \frac{2GW_{t=t} - 2GW_{t=0}}{2GW_{t=\infty} - 2GW_{t=0}} \tag{9}$$

The excess pore pressure turns out to be

$$\sigma = \frac{2n}{\pi}\int_0^\infty \frac{\sin\alpha \sin\alpha x}{\alpha} e^{-\alpha^2 t} d\alpha \left[I_4 - \frac{1}{\alpha^2}\frac{d^2 I_4}{dz^2} - e^{-\alpha z} I_6\right]$$

At the surface (z=0), the pore pressure is zero for all time since a pervious boundary is assumed. However, at any other depth also the pore pressure is zero under the centre of the strip (x=0). The pore pressure under the edge of the strip (x=1 and at z=1) is given by

$$\sigma_{\substack{x=1\\z=1}} = \frac{2n}{\pi}\int_0^\infty \frac{\sin^2\alpha}{\alpha^2} e^{-\alpha^2 t} d\alpha \left[I_{4\,z=1} + I_{4m} - e^{-\alpha} I_6\right] \tag{10}$$

b) IMPERVIOUS CASE

The various quantities for this case turn out to be:

$$2GE = -\frac{2}{\pi}\int_0^\infty \frac{\sin\alpha \sin\alpha x}{\alpha^3} d\alpha \times \left[\frac{2n+\alpha z}{2n-1} e^{-\alpha z} - e^{-\alpha^2 t}\left(I_1 + e^{-\alpha z} I_2\right) - \frac{2e^{-(1-y)\alpha^2 t}\left\{e^{-\alpha z y^{1/2}} - y^{1/2} e^{-\alpha z}\right\}}{(1-y)\left\{(n+1)^2 - (2-ny)^2\right\}}\right]$$

$$2GS = \frac{2n}{\pi}\int_0^\infty \frac{\sin\alpha \sin\alpha x}{\alpha^2} e^{-\alpha z} d\alpha \times \left[\frac{2}{2n-1} - e^{-\alpha^2 t} I_3 - \frac{2y^{1/2} e^{-(1-y)\alpha^2 t}}{(n+1)^2 - (2-ny)^2}\right]$$

where $y = 1 - \frac{(n^2+4n)^{1/2} - (2-n)}{2n}$

$$2GW_{z=0} = \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha \sin\alpha x}{\alpha^2} d\alpha \left[\frac{1}{2n-1} - ne^{-\alpha^2 t} I_3 - \frac{2ny^{1/2} e^{-(1-y)\alpha^2 t}}{(n+1)^2 - (2-ny)^2}\right]$$

Again it is observed that the centre of the strip does not settle at all (as was in the perivous case).

The edge settlements are given by

$$2GW_{\substack{z=0\\x=1}} = \frac{2}{\pi}\int_0^\infty \frac{\sin^2\alpha}{\alpha^2} d\alpha \left[\frac{1}{2n-1} - ne^{-\alpha^2 t} I_3 - \frac{2ny^{1/2} e^{-(1-y)\alpha^2 t}}{(n+1)^2 - (2-ny)^2}\right] \tag{11}$$

The ultimate edge settlement is again given by:

$$2GW_{\substack{z=0\\x=1\\t=\infty}} = \frac{1}{2n-1}$$

which tallies with the result of eq.(8). This coincidence must be obvious from the physical point of view. It is again to be noted that immediate settlements are zero even in this case.

Similarly the pore pressures are observed to be zero under the centre of the ship at all times, irrespective of the surface conditions. The pore pressure for the impervious case is given by

$$\sigma_{x=z=1} = -\frac{2n}{\pi}\int_0^\infty \frac{\sin^2\alpha}{\alpha^2} e^{-\alpha^2 t} d\alpha \times \left[e^{-\alpha} I_3 - I_{1\,z=1} + I_{1m} - \frac{2e^{y\alpha^2 t}\left\{e^{-\alpha y^{1/2}} - y^{1/2} e^{-\alpha}\right\}}{(n+1)^2 - (2-ny)^2}\right] \tag{12}$$

c) STEADY STATE SOLUTIONS

The steady state (the state when the dissipation of pore pressure is complete) solutions of poro-elasticity reduce to the solutions of corresponding problems of the classical theory of elasticity. It is observed that the utlimate values of E and S (as t are the same for both the surface drained conditions. If these values are substituted in eq. (3), one gets the steady state value of σ_{zz} as:

$$\sigma_{\substack{zz\\t\to\infty}} = \frac{2z}{\pi}\int_0^\infty \sin\alpha \sin\alpha x\, e^{-\alpha z} d\alpha = \frac{4xz^2}{\pi\left[(z^2+x^2-1)^2 + 4z^2\right]}$$

which coincides with that obtained independently from theory of elasticity (Harr,1966). Incidentally, this value is identical with the steady state value of σ_{xz} for the case strip, which is loaded normally.

d) NUMERICAL RESULTS

The numerical results based on eqs.(9,10,12) and evaluated by computer are shown graphically in Figs.1 and 2. The results show that the effect of ν is more pronounced in the pervious case. Fig.2 shows that pore pressures obtained from Biot's theory temporarily increase and then decrease with time, unlike the gradually decreasing pore pressures as predicted by Terzaghi's theory.

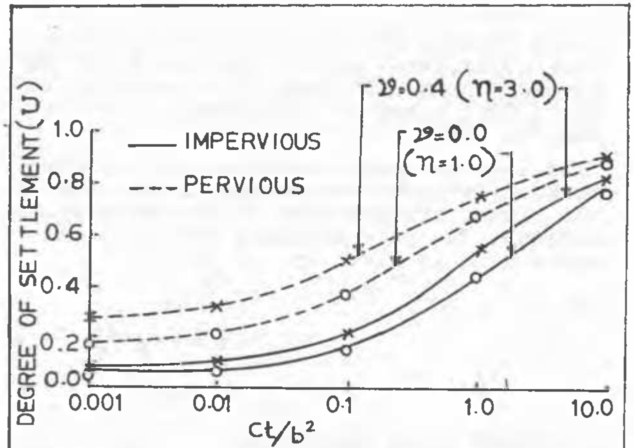

FIG.1.SURFACE SETTLEMENTS UNDER THE EDGE OF STRIP (X = b ; Z = 0)

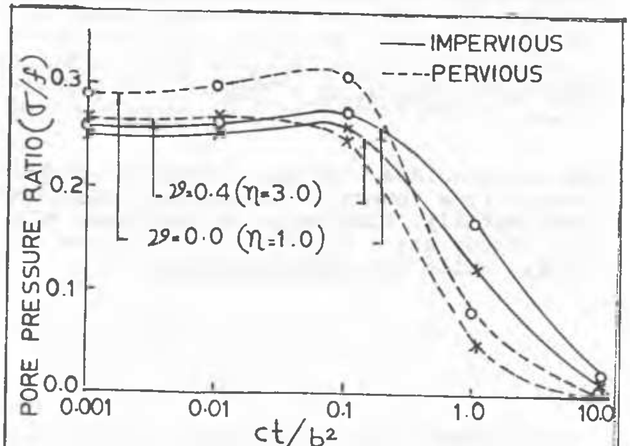

FIG.2.PORE-PRESSURE UNDER THE EDGE OF THE STRIP (X = b = Z)

APPENDIX

$$I_1 = J\int_0^\infty MR\left(NV^2 \cos V\alpha Z - V \sin V\alpha Z\right) dV$$

$$I_{1m} = J\int_0^\infty MR\left(V^3 \sin \alpha V - NV \cos \alpha V\right) dV$$

$$I_2 = J\int_0^\infty MR\left(V^2\right) dV; \quad I_3 = J\int_0^\infty MR\left(V^4 + V^2\right) dV$$

$$I_4 = J\int_0^\infty MT\left(V^2 \cos \alpha VZ + (N-2)\, V \sin \alpha VZ\right) dV$$

$$I_{4m} = J\int_0^\infty MT\left(V^4 \cos \alpha V + (N-2)\, V^3 \sin \alpha V\right) dV$$

$$I_5 = J\int_0^\infty MT\,(N-2)\, V^2 dV;$$

$$I_6 = J\int_0^\infty MT\left(V^4 + V^2\right) dV$$

where $(R)^{-1} = (V^2+1)^2 (V^2+y)\left[V^2 + (1/n^2 y)\right]$

$$(T)^{-1} = (V^2+1)^2\left[V^2 + \left(n - \frac{1}{n}\right)^2\right]$$

$$J = 2(\pi n^2)^{-1}; \quad M = e^{-\alpha^2 t V^2};$$

$$N = n(V^2+1) + 1$$

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