APPLICATION OF LAYERED ELASTIC THEORY TO PRACTICAL PROBLEMS

APPLICATION DE LA THEORIE ELASTIQUE DES COUCHES AUX PROBLEMES PRATIQUES

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SYNOPSIS An extensive tabulation of stresses, strains and displacements in a two layer elastic solid subjected to various surface loadings has recently been produced by Gerrard (1968). The loading systems include vertical stress, horizontal unidirectional shear stress and inwardly acting shear stresses on a circular loaded area. A wide range of surface layer depths, modular ratios and Poisson's ratios of each layer has been included and values have been given for surface, centreline and interface locations. This paper describes the use of the tables to solve directly problems such as: (i) stresses and surface displacements resulting from loads applied to single or group footings, (ii) stresses and deflections in roads and runways due to various wheel load configurations and typical tyre loadings. Thus equivalent wheel loads may be estimated based on various criteria. In addition, by superposition of results, cases such as the following may be analysed: (i) stresses and displacements of footings subjected to vertical and horizontal loads, (ii) stresses and deflections in pavements due to traction forces arising from braking and acceleration.

INTRODUCTION

The use of elastic theory in the analysis and design of soil engineering structures is becoming more prevalent. This has resulted from the availability of a large range of elastic solutions to the response of the semi-infinite medium subjected to external load; and the recognition that meaningful elastic parameters can be assigned to real soils and similar materials. The efficacy of elastic theory is seen in the approach to calculation of three-dimensional consolidation settlements developed by Davis and Poulos (1968) and the more general 'stress-path method' for calculating soil deformations suggested by Lambe (1967). In these circumstances large plastic strain components exist, but these do not influence the stress distribution, and are taken into account in the deformation calculation by the use of the appropriate stress path. In flexible pavements, where plastic strains are usually insignificant compared with elastic, theory has been successfully applied in the development of the rational design method (Dorman and Edwards, 1967).

Solutions for deflections in a two-layer elastic system were first published by Burmister (1943). Since that time many other solutions have been published, extending to three and more layers. However the solutions have usually been presented for a limited range of variables, parameters or loading situations because of the large number of possible combinations. Therefore the behaviour on the surface (other than at the load axis) has been ignored, and the influence of the Poisson's ratios of the layers has received scant treatment. The value of Poisson's ratio = 0.5 often assumed is a convenient upper limit since it implies an incompressible material. To simulate a much more compressible material a minimum value of 0.2 has been used in the compilation of the tables. There is considerable evidence (Morgan and Scala, 1968) that such values more closely reflect the behaviour of granular materials, such as sands and fine crushed rock.

The tables (Gerrard, 1968) on which this paper is based have been calculated by applying integral transform methods to the classical elastic equations expressed in cylindrical co-ordinates (Gerrard and Mulholland [1968]). The integral solutions were obtained numerically by electronic computer. A layered elastic system in which the top layer has a stiffness (Young's modulus) equal to or greater than the bottom layer has been considered and involves the following variables:

Modular ratio $\frac{E_1}{E_2}$ 1, 2, 5, 10

Poisson's ratios ν_1, ν_2, 0.2, 0.35, 0.5

layer depth $\frac{h}{a}$ 1/2, 1, 2, 4

where E_1 = Young's modulus of top layer
E_2 = " " " bottom layer
ν_1 = Poisson's ratio of top layer
ν_2 = " " " bottom layer
h = depth of top layer
a = radius of loaded area

The following notation has been used in the figures:

w = vertical deflection
ϵ_{rr} = radial strain
ϵ_{zz} = vertical strain
zz = vertical stress

p = uniform vertical contact stress
r = radius of loaded area
d = centre to centre spacing of dual loads

Vertical displacements in a downward direction are taken as positive, as too are compressive stresses and strains.

Stresses, strains and deflections have been calculated on the surface and along the interface to a distance of ten radii (10a) and on the centreline to a depth of ten radii. For foundation problems surface deflections and vertical stresses are of most significance, whereas for flexible pavement design these values, as well as radial strains in the top layer and vertical strains in the bottom layer, are of interest. In the rational design procedure for flexible pavements the design criteria assumed are the tensile strain in the asphaltic layers and vertical stress or strain at the top of the subgrade.

The loadings considered are all assumed to act over a circular area. The 'vertical' load is uniformly distributed over the area; the 'traction' load applies a uniform unidirectional shear stress, and the 'inward shear' applies a shear stress directed everywhere to the centre of the loaded area and increasing linearly with radius from zero at the centre to a maximum on the circumference. The combination of the vertical and traction load enables solutions for foundations subject to combined vertical and horizontal loads to be obtained, or for pavements under braking or acceleration forces. The inward shear case combined with the vertical load gives an indication of the behaviour beneath a rough foundation where shear stresses are induced or for a pavement under the usual tyre contact stresses.

SINGLE VERTICAL LOAD

In order to identify the significant variables influencing the various displacements, stresses and strains, the values have been plotted. The plots are not reproduced here because of the great detail contained therein but instead the main variables influencing the behaviour are listed. "Major" factors are arbitrarily considered as those causing a range of values of more than 30% of the average value, while minor factors give a range of more than 10%. The levels of 30% and 10% must be considered in relation to the previously stated ranges of the controlling factors; $\frac{E_1}{E_2}$, $\frac{h}{a}$, ν_1, and ν_2. The sign after the factor indicates the sign of the change in the quantity for an increase in the factor (variable) listed.

(a) Displacements. These are expressed in the form $\left(\frac{w.E_2}{p.a}\right)$ so in the comparison it is assumed that the other factors p, a, E_2 remain unchanged

(i) Vertical surface displacement on the centreline

$(\frac{r}{a} = 0)$

major factors $\frac{E_1}{E_2}$ (-), $\frac{h}{a}$ $(\frac{E_1}{E_2} \geq 2)$ (-),
ν_1 $(\frac{E_1}{E_2} = 1)$ (-)

minor factors ν_1 $(\frac{E_1}{E_2} \geq 2)$ (-)

(ii) Vertical surface displacement at $\frac{r}{a} = 2.5$

major factors $\frac{E_1}{E_2}$ $(\frac{h}{a} \geq 2)$ (-), $\frac{h}{a}$ $(\frac{E_1}{E_2} \geq 5)$ (-)
ν_1 $(\frac{E_1}{E_2} = 1$ and $\frac{h}{a} = 4)$ (-)

minor factors ν_2 $(\frac{h}{a} \leq 1)$ (-),
ν_1 $(\frac{E_1}{E_2} = 2, \frac{h}{a} = 4; \frac{E_1}{E_2} \leq 2, \frac{h}{a} = 2)$ (-)

(iii) Vertical surface displacement at $\frac{r}{a} = 7$

minor factors ν_2 $(\frac{h}{a} < 2; \frac{E_1}{E_2} = 1, \frac{h}{a} = 4)$ (-),
ν_1 $(\frac{E_1}{E_2} = 1, \frac{h}{a} \geq 2; \frac{E_1}{E_2} = 2, \frac{h}{a} = 4)$ (-)

(b) Stresses. These are expressed as a fraction of the surface contact pressure

(i) Vertical stress at interface on centreline

major factors $(\frac{E_1}{E_2}$ (-), $\frac{h}{a}$ (-)

minor factors ν_1 $(\frac{h}{a} \leq 1)$ (-)

(ii) Radial stress at surface on centreline

major factors $\frac{E_1}{E_2}$ $(\frac{h}{a} \leq 2)$ (+), ν_1 (+),
$\frac{h}{a}$ $(\frac{E_1}{E_2} > 2)$ (-)

minor factors $\frac{h}{a}$ $(\frac{E_1}{E_2} = 2)$ (-), $\frac{E_1}{E_2}(\frac{h}{a} = 4)$ (+)

(iii) Radial stress at interface (top layer) on centreline

major factors $\frac{E_1}{E_2}$ (-), $\frac{h}{a}$ $(\frac{E_1}{E_2} > 2)$ (+),
$\frac{h}{a}$ $(\frac{E_1}{E_2} = 1, \nu_1 > .2)$ (-),
ν_1 $(\frac{E_1}{E_2} = 1)$ (+), ν_1 $(\frac{E_1}{E_2} \geq 5)$ (-)

(Note: maximum negative values occur for high $\frac{E_1}{E_2}$)

minor factor ν_2 $(\frac{E_1}{E_2} \geq 2, \frac{h}{a} \leq 1)$ (-)

(c) Strains. These results are expressed in the form $\epsilon\frac{E}{p}$ where the modulus is taken as that of the layer in which the strain is calculated

(i) Vertical strain at interface (bottom layer) on centreline

major factors $\frac{h}{a}$ (-), $\frac{E_1}{E_2}$ $(\frac{h}{a} \geq 1)$ (-),
ν_2 $(\frac{h}{a} = 1/2)$ (-)

minor factors $\frac{E_1}{E_2}$ $(\frac{h}{a} = 1/2)$ (-), ν_2 $(\frac{h}{a} \geq 1)$(-)

(ii) Radial strain at interface (top layer) on centreline

major factors $\frac{h}{a}$ ($\frac{E_1}{E_2} > 2$) (+), $\frac{E_1}{E_2}$ (-)

minor factors ν_1 ($\frac{h}{a} < 2$ $\frac{E_1}{E_2} < 5$),

ν_2 ($\frac{h}{a} < 1$) (-)

(iii) Radial strain on surface, maximum tensile value

major factors ν_1 (+), $\frac{h}{a}$ ($\frac{E_1}{E_2} > 5$) (+)

Increase in these factors reduces the maximum negative value.

(Other factors interact in a complex fashion.)

COMBINATION OF LOADING TYPES

The traction and the inward shear loadings have been separately combined with the vertical load to give the solutions shown in Figures 1, 2 and 3. In each case the maximum value of the shear force has been made equal to the vertical load. The vertical load-inward shear combination gives results symmetrical about the load centreline, whereas the traction load case gives values which are added to the vertical load values on one side of the centreline and subtracted on the other. Only the additive case is shown in the figures, and it will be seen also that the traction load has no influence on the centreline. The figures show only the cases for $\frac{h}{a} = 1$ and 4, and $\frac{E_1}{E_2} = 2$ and 10. On each figure values for $\nu = 0.5$ in both layers and $\nu = 0.2$ in both layers are given. In general the Poisson's ratio in one of the layers has a substantially greater effect on the quantity concerned than the value in the other layer, hence the values chosen above approximately cover the full range.

SURFACE DEFLECTION (Fig.1)

Most notable is the fact that, for $\nu = 0.5$ in both layers, the shear loadings have negligible influence on vertical deflection. However, when Poisson's ratio is 0.2 it is noted that

- on the centreline the deflections are significantly higher than for $\nu = 0.5$
- traction or horizontal load causes a marked change in deflection away from the centre line, whereas on the centreline the inward shear only has a marked effect.

This latter result is characteristic of the more rapid diminution with distance from centreline of the effect of the inward shear load than the traction load, which is seen in many of the other results. This characteristic gives the deflection bowls for $\nu = 0.2$ and inward shear a very 'peaked' shape compared with the other loadings. The deflection bowl for the vertical and horizontal load combination is asymmetrical. The shape for both $\nu = 0.2$ and 0.5 is influenced by $\frac{E_1}{E_2}$ and $\frac{h}{a}$ showing greatest curvature for minimum values of both these variables (Fig.1(a)).

RADIAL STRAIN ON SURFACE (Fig.2)

In the figure, the strains are expressed in terms of the modulus of the top layer, resulting in identical scales for the figures. The surface radial strains are most strongly influenced by the shear loadings and the magnitude is of great significance in initiating fatigue failure of asphalt pavement layers (Pell 1965). The pattern of behaviour varies very little as the modular ratio and layer depth alters. Poisson's ratio has a large influence on the centreline and close to the edge of the load ($\frac{r}{a} = 1.25$), the smaller value of 0.2 leading to a larger numerical value of strain.

Strain reversal occurs for the combination of vertical load and inward shear, values being successively negative, positive, negative as the load traverses a particular point in the pavement. To a lesser extent this occurs with vertical and traction load, but the strain pattern is asymmetrical, values varying from positive to negative only as the load traverses the system. It is also noted that the maximum value of positive (compressive) strain for this loading case does not occur on the load axis, as appears from Fig.2 but at about a value of $\frac{r}{a} = 1.25$ on the other side of the centreline. For vertical load alone, the strain values are small at all locations when $\nu = 0.5$.

VERTICAL STRESS AT INTERFACE (Fig.3)

In these results the pattern of behaviour is strongly influenced by the $\frac{h}{a}$ value. Except on the centreline Poisson's ratio has a significant effect only when both the modular ratio and layer depth are large (Fig.3d) and then only for $\frac{r}{a} = 1.25$.

Again the inward shear load causes a significant increase only on the centreline, whereas for the horizontal load the change is a maximum away from the centreline, and for deep layers is very considerable even at large radii (Figs.3b, 3d). However, in these cases the absolute magnitude of the stress is small so that the consequences of the asymmetrical loading on the lower layer are reduced.

MULTIPLE LOADED AREAS

To indicate the interaction of two footings or tyre loads the cases shown in Figs.4, 5, 6 have been prepared. These are for the

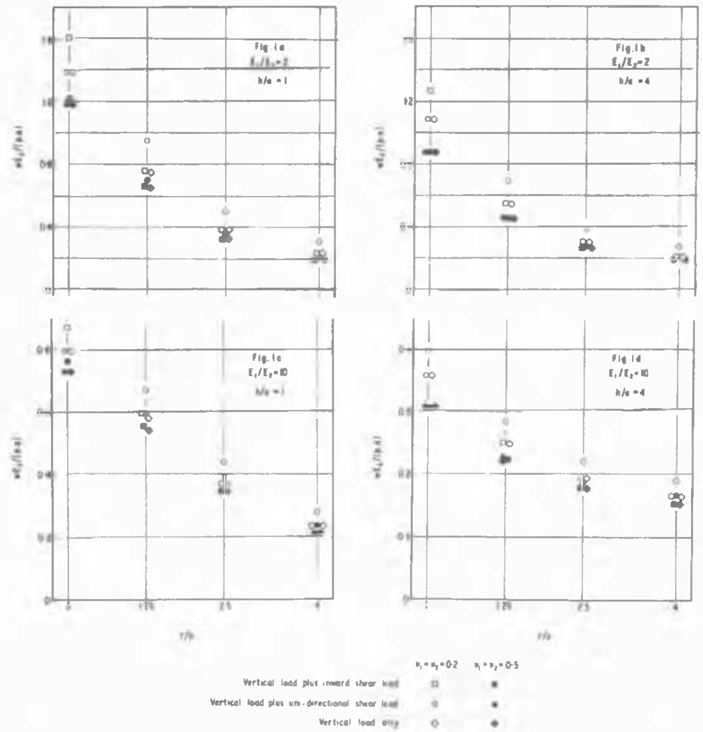

Fig.1. Vertical Displacement ($\frac{w.E_2}{p.a}$) (at depth z = 0) vs. Radius (r/a)

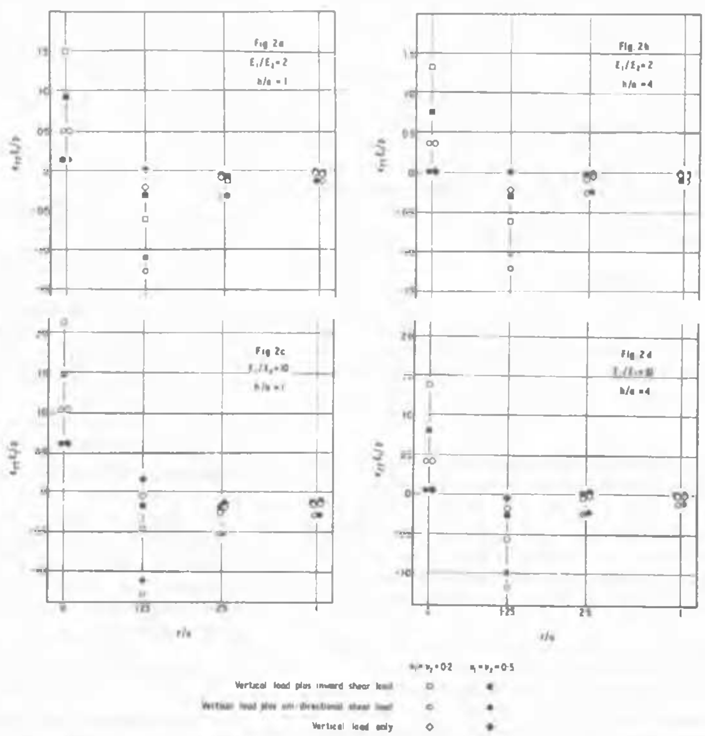

Fig.2. Radial Direct Strain ($\frac{\epsilon_{rr}.E_1}{p}$) (at depth z = 0) vs. Radius (r/a)

case of two identical vertical loads of uniform pressure.

Spacings of 2.5 and 5 radii between the loads have been selected, modular ratios of 2 and 10, layer depths of 4 and 1 radii and Poisson's ratios of 0.2 and 0.5 in both layers being used as the other variables. The vertical displacements shown in Fig.4 and the vertical strain shown in Fig.5 are expressed in terms of the modulus of the lower layer, whereas the radial strain shown in Fig.6 involves the modulus of the surface layer.

VERTICAL SURFACE DEFLECTION (Fig.4)

As for the single loadings previously considered, the lower value of Poisson's ratio gives an increase in deflection close to the loads, in this case on the centrelines of each load, and midway between them. This is most marked for the modular ratio of two. In the cases considered the maximum deflection occurs beneath one of the loads, except for the thin layer, close spacing, high modular ratio combination shown (in Fig.4a), where the midpoint deflection is slightly greater than that under the loads. The efficiency of a stiffer upper layer in reducing deflections is limited to conditions close to the load unless the stiffer layer is deep. Even then, the improvement is only slight at $\frac{r}{a} = 2.5$ and vanishes entirely at $\frac{r}{a} = 5$.

VERTICAL STRAIN AT THE INTERFACE IN THE LOWER LAYER (Fig.5)

The vertical stress or strain in the lower layer (subgrade) of a flexible pavement has been used as a design criterion in the rational design procedure. (Dormon and Edwards, 1967.) The allowable stress is apparently related to measures such as C.B.R., whereas in terms of strains, a constant value of about 10^{-3} appears to hold for many soils.

Because of the rapid decay with distance from load, the values for the thin layer and wide spacing are negligible, except immediately beneath one load. At the closer spacing the values between the loads are appreciable but less than the beneath-load values. However, when the layer is thick the values between the loads may be greater than those at the loads when the spacing is small and slightly less when the spacing is greater. In all cases where the values are appreciable the influence of Poisson's ratio is also large, maximum values of strain being associated with $\nu = 0.2$. The increase in strain is proportionally greater for the modular ratio of ten.

RADIAL STRAIN AT INTERFACE (Fig.6)

For a combination of two tyre loadings the critical radial strain is likely to occur at the interface rather than at the surface. The shear loadings which lead to the most severe radial strains on the surface are less significant at the interface. Again, the values are expressed in terms of the modulus of the top layer, but the scales are different, depending on the layer thickness.

The influence of Poisson's ratio is again most noticeable at or between the loads, especially at low values of modular ratio. Higher values of Poisson's ratio lead to greater negative values of the radial strain, and are therefore of greatest significance in initiating fatigue failure of asphaltic pavements. The largest values occur for greatest modular ratios reflecting the 'beam action' of the upper layer.

The pattern of strains at and between the loads varies considerably. The deeper layers have negative strains (tensile) at and between the loads, whereas for the shallower layers, the strain may become positive between the loads, especially for large spacings of the load or for low modular ratio and close spacing.

CONCLUSIONS

Solutions for stresses, strains and displacements in a two layer elastic system subjected to various surface loadings have been computed. In addition to the influence of modular ratio of the layers and layer depth, the results show the influence of the Poisson's ratios of the layers. Although it is often assumed to have a value of 0.5, a reduction to 0.2 results in a significant increase in surface deflection, in the maximum tensile radial strain on the surface, and in the compressive strain at the top of the lower layer.

The influence of unidirectional horizontal load and inwardly acting shear has been shown by combining them with the vertical load. The influence of the unidirectional load is asymmetrical, being zero on the load axis, positive on one side and negative on the other. Its effects away from the centreline persist for greater distances than the effects of the inward shear which are a maximum on the load axis. The greatest effect of the shear loadings is on the tensile radial strain on the surface. The inward shear case gives a severe strain reversal of two negative and one positive maxima at a particular point as the load passes; whereas, the unidirectional load gives only a positive and negative maximum as the load passes.

Dual wheel loadings or interaction of two foundations subjected to vertical loads have been examined for two different spacings of the loads. The most severe conditions for subgrade vertical strain or surface deflection occur beneath a load unless the layer depth is large and the spacing is small.

For combinations of loads such as these, the critical tensile strain occurs at the base of the top layer and its magnitude and location is influenced by all the variables

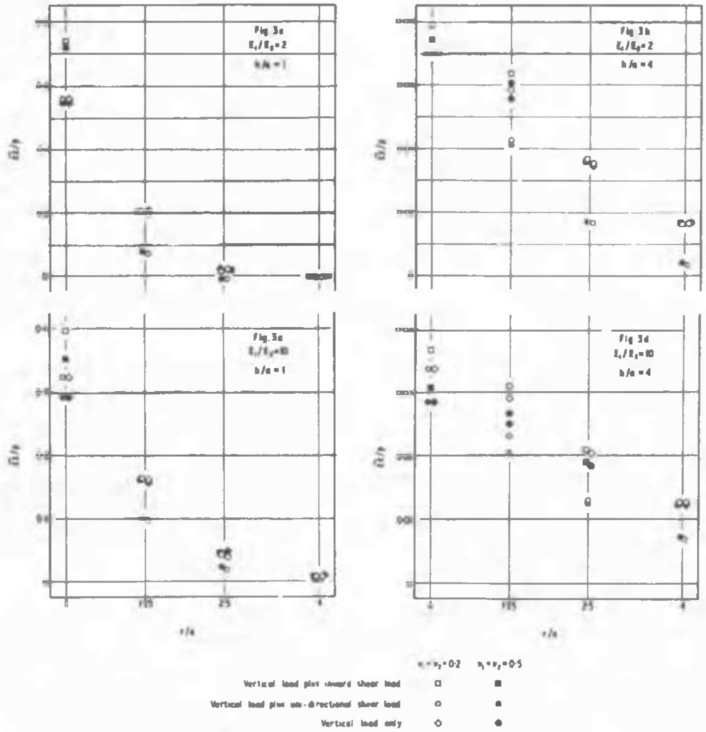

Fig.3. Vertical Direct Stress ($\frac{zz}{p}$) (at depth z = h) vs. Radius (r/a)

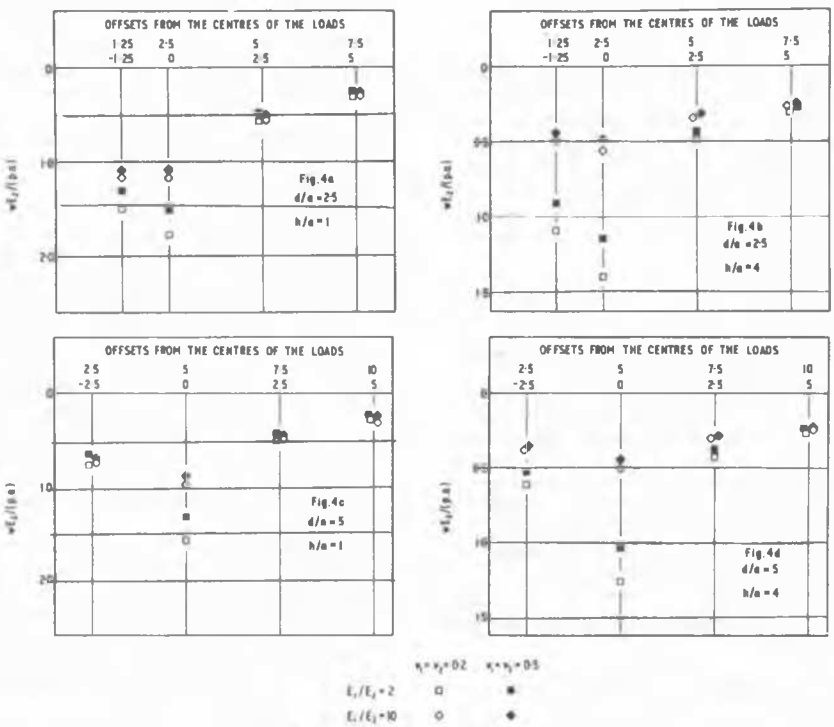

Fig.4. Vertical Displacement ($\frac{w.E_2}{p.a}$) (at depth z = 0) vs. Radius (r/a) Dual Loads

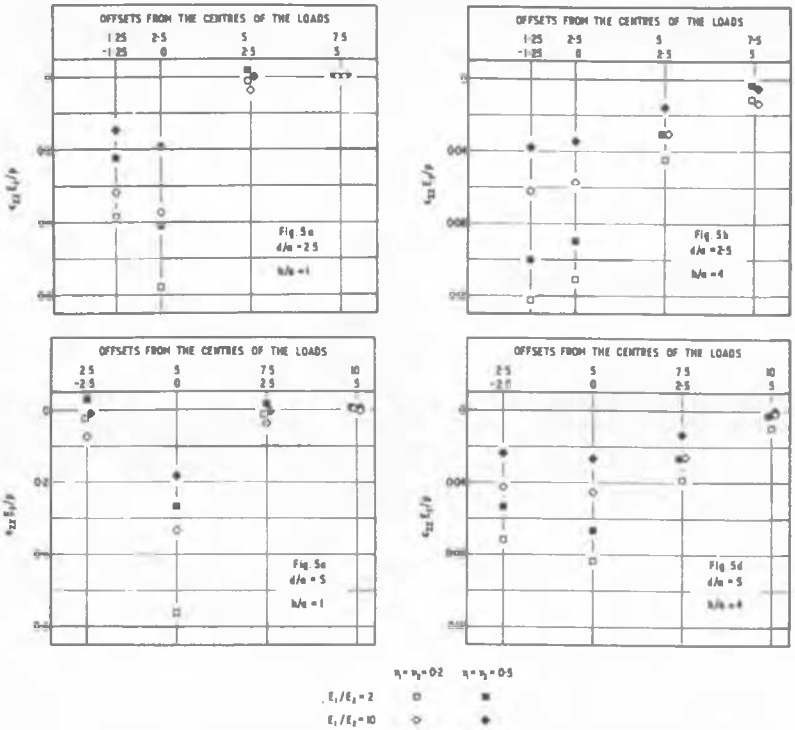

Fig.5. Vertical Direct Strain $(\frac{\varepsilon_{zz}.E_2}{p})$ (in bottom layer at depth z = h) vs. Radius (r/a) Dual Loads

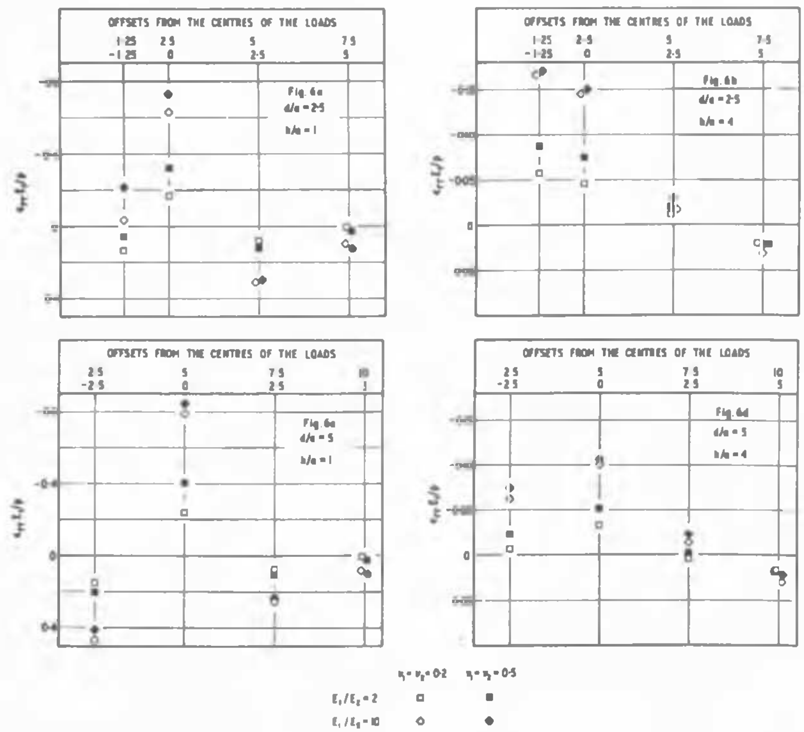

Fig.6. Radial Direct Strain $(\frac{\varepsilon_{rr}.E_1}{p})$ (at depth z = h) vs. Radius (r/a) Dual Loads

considered. Poisson's ratio has a very considerable effect on the radial and vertical strains investigated.

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