MECHANICS OF GRANULAR MATERIAL AS A PARTICULATED MASS

MECANIQUE DES MASSIFS PULVERULENTS EN TANT QUE MILIEUX GRANULAIRES

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SYNOPSIS The author's theory of mechanics of granular material, already published is briefly explained. His theory is based on the consideration that the mechanical properties of the material depend not only on the void ratio but also on the distribution of void ratio in the material. Calculating the number of possible configurations of voids, the entropy of the material was derived. With the help of this calculated entropy, relationships between the angle of internal friction and the void ratio were obtained, their validity was ascertained by experimental data. The critical state line proposed by Roscoe et al was explained by his theory.

Deformation characteristics of the granular material is discussed and some remarks about the deformation of the material are given.

INTRODUCTION

The classical mechanics of granular materials such as sands and gravels deals with the relationship between stresses at failure in which the material is assumed as a continuous medium having frictional resistance. Experiments tell us that the failure does not occur suddenly but the material deforms considerably before failure and slip surface which appears in the material at failure develops progressively. Such phenomena cannot be discussed by the classical theory and the theory thereof is left undiscovered. The basic understanding of such phenomena would not be certain when the study of the mechanical properties of granular material remains in the region of continium mechanics. It is quite natural as a first step to study the mechanical behaviour of the material on some idealized simplest model, for example, a regular assemblage of equal spheres, but it can easily be pointed out that such models are too much simplified ones to understand the complicated behaviour of the material. The present author adopted a more general model than the above mentioned simplest ones and on such a model he has tried to explain the mechanical properties of the material especially the failure problems. The details of his theory should be reffered to his original papers already published. In this paper, a brief explanation of the theory will be given and some considerations which have not yet been published will be added.

ABRIDGED EXPLANATION OF THE AUTHOR'S THEORY

It is a broadly recognized opinion that the mechanical behaviour of the granular material is dependent of the amount of void in the material. The present author thinks that the behaviour depends not only on the void ratio as a whole but also on the distribution of void in the material. The distribution of some quantity is described by its frequency curve and the character of this curve is expressed by the average and the deviation and other higher order momenets. Reversely speaking, many distributions can be found which give a fixed average and a fixed deviation of the quantity. As a first step or as a first approximation a proposal that the mechanical properties of the granular material depend on the average void ratio and the deviation thereof is assumed as valid. Hence the granular materials which have the same average void ratio and the same deviation of void ratio have similar mechanical properties even if the distribution of void is not exactly the same.

I) Basic considerations

If we classify elementary parts in the material according to their void ratios and the volumes of elements which have the same void ratio are summed up, we can get the following table.

Table 1

$$\begin{matrix} e_1 & e_2 & \cdots\cdots & e_n \\ V_1 & V_2 & \cdots\cdots & V_n \\ N_1 & N_2 & \cdots\cdots & N_n \end{matrix} \quad \cdots\cdots (1)$$

In the above table, $e_1, e_2, \ldots\ldots, e_n$ are void ratios arranged in ascending order, V_i ($i = 1, 2, \ldots\ldots, n$) are the total volume of elements which have the void ratio of e_i and N_i ($i = 1, 2, \ldots\ldots, n$) are the total volume of grains which are contained in elements which have the void ratio e_i calculated in terms of the number of grains.
The table shows a state of the granular material, in another word, the distribution of voids in the material.
The following conditions are identically satisfied.

$$\left.\begin{aligned} V_1 + V_2 + \cdots\cdots + V_n &= V \\ N_1 + N_2 + \cdots\cdots + N_n &= N \\ N_i V_o (1 + e_i) &= V_i \end{aligned}\right\} \cdots\cdots (2)$$

, where V is the total volume, N the total number of grains in the volume V, v_o is the volume of each grain assumed equal.
As shown in a paper already published, even when the size of each grain is not equal, problems can be treated in similar way if we change the coefficients adequately.
Putting

$$\bar{e} = \frac{1}{n}\sum e_i, \quad e_i = \bar{e}+\epsilon_i, \quad s = \frac{1}{n}\sum \epsilon_i^2 \cdots (3)$$

We can calculate the number Z of possible configurations of voids in the volume for fixed values of $\bar{e}$ and s by combinatorial consideration, the number Z depends on N_i. We can obtain the ratio of N_i to N so as to make Z be stationary, the void ratio as a whole e and the deviation of voids s being fixed. Such Z is written as W. This W gives the number of the configurations of voids in the material, that is, the number of states of the material when the void ratio e and the deviation of voids s are fixed.
In such calculation, quantities such as

$$\sum \epsilon_i^3, \quad \sum \epsilon_i^4, \cdots, \sum N_i \epsilon_i^3, \quad \sum N_i \epsilon_i^4, \quad (4)$$

are assumed small enough to be neglected compared with e and s.
Following the way of consideration in statistical mechanics, we can take log W as a quantity proportional to the entropy S of the material. Hence we have

$$S=K''\log W=K'\left[(1+e)\log(1+e)-e\log e-\frac{s}{2e(1+e)}\right] \quad (5)$$

, where K' and K" are constants.

II) Relationships between the angle of internal friction and void ratio.

When the work done by forces applied from without of the material is ΔA, and the energy stored as an internal energy is ΔU and the loss of energy from the material as heat is ΔQ, the law of conservation of energy gives

$$\Delta A = \Delta U + \Delta Q \cdots\cdots (6)$$

and ΔQ is written as

$$\Delta Q=T\Delta S=K\Delta\left[(1+e)\log(1+e)-e\log e-\frac{s}{2e(1+e)}\right] \quad (7)$$

, where K is a constant.
In case of failure, the increment of the internal energy can be neglected, hence the expression (6) can be written as

$$\Delta A=K\Delta\left[(1+e)\log(1+e)-e\log e-\frac{s}{2e(1+e)}\right] \quad (8)$$

If an assumption concerning the transformation of variables e and s to e and γ, γ being the increment of shearing strain after failure is permitted, and if the experimental evidence found in model tests with steel balls, that is, the value of s when the material starts to fail has a constant value s_o which seems dependent on the initial configuration of particles in the material, is also valid in actual granular material, we can find stress components as follows,

(i) plane stress

$$\left.\begin{aligned} P=\frac{\sigma_1+\sigma_3}{2}&=-K(1+e)\left[\log\frac{1+e}{e}+\frac{s_o}{2}\frac{1+2e}{e^2(1+e)^2}\right] \\ \tau=\frac{\sigma_1-\sigma_3}{2}&=-Kk_1\left[\log\frac{1+e}{e}+\frac{s_o}{2}\frac{1+2e}{e^2(1+e)^2}\right] \end{aligned}\right\} (9)$$

(ii) axial symmetry

$$\left.\begin{aligned} P=\frac{\sigma_1+2\sigma_3}{3}&=-K(1+e)\left[\log\frac{1+e}{e}+\frac{s_o}{2}\frac{1+2e}{e^2(1+e)^2}\right] \\ \frac{\sigma_1-\sigma_3}{3}&=-Kk_2\left[\log\frac{1+e}{e}+\frac{s_o}{2}\frac{1+2e}{e^2(1+e)^2}\right] \end{aligned}\right\} (10)$$

(iii) plane strain

$$\left.\begin{aligned}\frac{\sigma_1+\sigma_3}{2} &= -K(1+e)\left[\log\frac{1+e}{e}+\frac{s_0}{2}\,\frac{1+2e}{e^2(1+e)^2}\right]\\ \frac{\sigma_1-\sigma_3}{3} &= -Kk_3\left[\log\frac{1+e}{e}+\frac{s_0}{2}\,\frac{1+2e}{e^2(1+e)^2}\right]\end{aligned}\right\}\quad(11)$$

, where k_1, k_2 and k_3 are constants.
Above said assumptions cannot be verified directly by experiments. However, if the validity of expressions deduced from the theory is confirmed by experiments, we can affirm that the assumptions are permissible.
From equations (9), (10) and (11), the relationships between the angle of internal friction ϕ and void ratio e are derived. They are

$$\sin\phi = \frac{k_1}{1+e} \quad (\text{ plane stress }) \quad (12)$$

$$\sin\phi = \frac{3k_2}{2(1+e)+k_2} \quad (\text{axial symmetry}) \quad (13)$$

$$\sin\phi = \frac{3k_3}{2(1+e)} \quad (\text{ plane strain }) \quad (14)$$

These relationships are found to be valid when they are compared with experimental data of shearing tests on sands and gravels supplied by many authors.

III) Critical state line proposed by Roscoe et al.

Even when our theory is confirmed by comparison of the expressions (12), (13) and (14) with experimental data, the validity of terms in bracket in the expressions (9), (10) and (11) is not certain. The space curve in space (p, τ, e), proposed by Roscoe et al and named the critical state line, is the locus of point in the space which represents the state of granular material at failure. Each of expressions (9), (10) and (11) can be considered as an equation of a space curve, which represents the state of the material at failure. Hence if this curve has the same character as the critical state line which was found by experimental data, this gives a powerful verification of our theory. By plotting the curves expressed by (9), (10) or (11), the above anticipation was confirmed at least approximately. This fact gives the way of finding the parameters which appear in our theory, but unfortunately it could not yet be tried because of the lack in experimental data. In addition to this, it will give a physical interpretation of parameters contained in Roscoe's theory. Since Roscoe's theory is mainly based on experimental findings, some of parameters used in his theory have a little unnatural character from the physical standpoint, hence such interpretation would be important.

DEFORMATION CHARACTERISTICS OF GRANULAR MATERIAL

When the granular material deforms, the void ratio and its deviation change. If the quantities in (4) are negligibly small also in this case, the expression (6) is valid, hence we have

$$\begin{aligned}\Delta A &= \Delta U + K\Delta\left[(1+e)\log(1+e) - e\log e - \frac{s}{2e(1+e)}\right]\\ &= \Delta U + K\left[\left\{\log\frac{1+e}{e}+\frac{s(1+2e)}{2e^2(1+e)^2}\right\}\Delta e - \frac{1}{2e(1+e)}\Delta s\right]\end{aligned}\quad(15)$$

The quantity ΔU is the increment of the internal energy. When the granular material is deformed, a part of the deformation is elastic and the other part is non-elastic, the former corresponds to ΔU, the latter to the second of the right hand term of (15). Every term of (15) is positive, hence we get

$$K\left[\left\{\log\frac{1+e}{e}+\frac{s(1+2e)}{2e^2(1+e)^2}\right\}\Delta e - \frac{1}{2e(1+e)}\Delta s\right] \geq 0 \quad (16)$$

This is not other than the law of increase of entropy. As K is positive, we have

$$\left[\left\{\log\frac{1+e}{e}+\frac{s(1+2e)}{2e^2(1+e)^2}\right\} - \frac{1}{2e(1+e)}\frac{\Delta s}{\Delta e}\right]\Delta e \geq 0 \quad (17)$$

When the volume dilates by deformation, we get

$$2e(1+e)\left\{\log\frac{1+e}{e}+\frac{s(1+2e)}{2e^2(1+e)^2}\right\} \geq \frac{\Delta s}{\Delta e} \quad (18)$$

When the volume contracts by deformation, we get

$$2e(1+e)\left\{\log\frac{1+e}{e}+\frac{s(1+2e)}{2e^2(1+e)^2}\right\} \leq \frac{\Delta s}{\Delta e} \quad (19)$$

If the deformation ie elastic, $\Delta A = \Delta U$, hence we have

$$2e(1+e)\left\{\log\frac{1+e}{e}+\frac{s(1+2e)}{2e^2(1+e)^2}\right\} = \frac{\Delta s}{\Delta e} \quad (20)$$

Integrating this equation, we have

$$s = 2e(1+e)\left\{(1+e)\log(1+e) - e\log e - C'\right\} \quad (21)$$

, where C' is an integration constant. The equation (21) can be rewritten as

$$(1+e)\log(1+e) - e\log e - \frac{s}{2e(1+e)} = C' \quad (22)$$

From equation (5) and (22), we obtain

$$\log W = K'C' = C \quad \cdots\cdots\cdots\cdots\cdots\cdots (23)$$

, where C is a constant.
We write

$$Z=(1+e)\log(1+e)-e\log e-\frac{s}{2e(1+e)} \quad \cdots \quad (24)$$

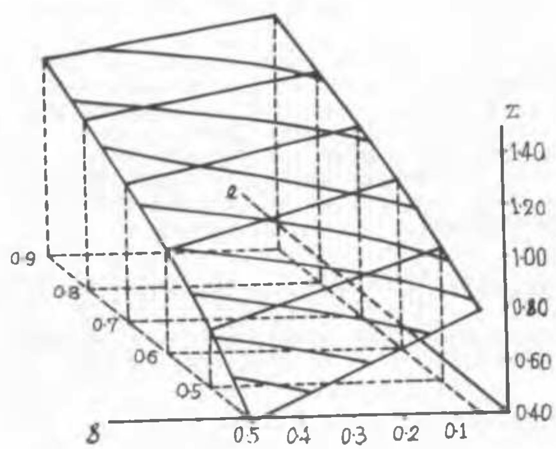

Fig. 1 Entropy surface

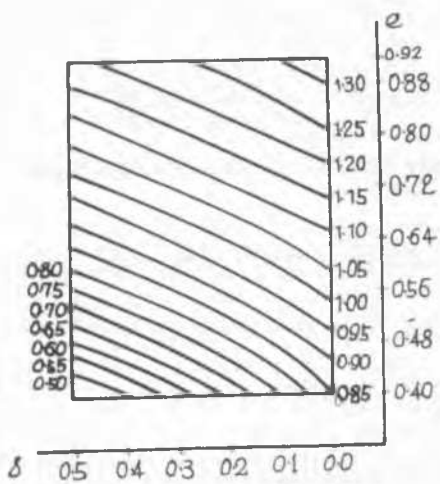

Fig. 2 Projected entropy surface

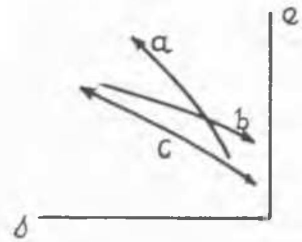

Fig. 3 The travelling lines of representative point

Fig. 4 Rotation of a particle

The equation (24) is that of a curved surface in space (e, s, z), which is shown in the figure 1. In the figure 1, contour lines on the surface are also shown.
The projection of the surface on the plane (e, s) with its contour lines is shown in the figure 2.

As a state of the granular material is specified by two variables e and s, the point which represents the state of the material travels along a space curve on the surface during the process of deformation. In the figure 3, a part of the projection is written enlarged, the curve c is one contour line.
If the deformation of the granular material is elastic, that is, the work done by forces applied from without of the material is completely stored in form of internal energy, the representative point travels along a contour line and can move in both directions shown by arrows, line c in the figure 3 is such a line.
If the deformation is non-elastic, the representative point goes along a curve like a in case of dilating, the curve a has smaller inclination to the e-axis than contour line c, whereas the representative point goes along a curve like b which has steeper inclination to the e-axis when the volume is decreasing during deformation.

Both these two latter cases the representative point should climb up the curved surface shown in the figure 1, because the law of increase of entropy should be valid. From the figure 3, it can be said that the deviation of void ratio increases with dilatation and it decreases with contraction, this increase of deviation of void ratio is smaller than its decrease for equal amount of increment or decrement of void ratio. When the material contracts, the distribution of void ratio in the material increase its uniformity quite rapidly.

REMARKS ABOUT THE PARAMETERS WHICH SPECIFY THE DEFORMATION

The displacement of each particle in the assemblage of particles is divided into three types, these are,

(i) relative change of distance between particles
(ii) relative rotation of particle around the other (like the rotation of earth around the sun)
(iii) rotation of each particle (like the rotation of earth around its axis)

On the other hand, from the point of view of continium mechanics, the deformation of the granular material is consisted of the rigid body displacement as a whole and of strain of each part of the material. The strain is expressed by its six components and has tensor characteristics. To unite the theories of deformation of granular material; one is the theory in which the material is assumed as a continuous medium, another is the theory in which the material is considered as a particulate mass; the relationship between the above mentioned three types of motion of particles and the strain components in the material should be obtained. This problem is a very difficult one and the author cannot yet answer. However some remarks will be given about the motion of particles. The first type of displacement mainly contribute to the dilatation or contraction of the assemblage as a whole which can easily be understood. This kind of displacement is mainly connected with the strain invariant of the first order. When a particle shown by hatching in the figure 4 (a) turned to the position shown in the figure 4 (b) or vise versa, the displacement of the category (ii) took place, however the configurations shown in these two figures are both one of the possible configurations in the volume. Hence the change of configuration of such character is included in those considered in our theory. And this kind of displacement would have connection with the strain invariants of first, second and possibly the third order.

The third type of motion was not yet considered in our theory. Since our theory which did not take the third type of motion into account could effectively explain the results of experiment to some extent, the effect of the rotation of particles seems to be small. When the material is composed of frictionless spheres, the rotation of each sphere would have no concerns with the deformation of material, this is an extreme case, however when the constituting spheres are frictional the rotation of spheres would contribute to the deformation of the material. The author believes that it is worth while to study more about the effect of the motion of this type.

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