A PROBABILISTIC FORMULATION OF SETTLEMENT-CONTROLLED DESIGN

PROBABILITES DES TASSEMENTS ET CRITERES DE PROJET

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SYNOPSIS. The probability distributions of settlement and rotation of rectangular foundations on randomly compressible, layered soils, are analyzed, and results are given for the extreme cases of flexible and rigid foundations. The determination of the statistical parameters of compressibility is also discussed.

Two possible ways of applying the results to the design of foundations are shown. One of them is based on the concept of allowable settlements and rotations while the other is an optimization criterion permitting a design that minimizes the expectation of total cost. The two criteria are applied to the analysis of a particular case and it is found that, under a reasonable set of assumptions, they lead to similar results.

Charts are given that permit application of the method to practical problems with a computational effort little greater than that involved in conventional settlement analyses.

INTRODUCTION

Natural soil deposits exhibit variations of mechanical properties which, for the sake of convenience, might be divided into two types. One of them includes systematic or clear-cut changes which are readily identified by conventional exploration techniques: an example of this type of variation is the change of compressibility with depth in either "homogeneous" or stratified soil deposits. The other type of variation does not show any systematic trend nor a deterministic character, and is best visualized as a random variation of properties: this is clearly apparent when one compares test results from a series of borings within an arbitrary area.

Random variations of compressibility are often responsible for rotations of structures founded on soils which, from a deterministic viewpoint, might be considered as homogeneous in the horizontal direction. In many cases those rotations affect the stability or the serviceability of the structure, or else those of adjacent and appurtenant constructions.

There seems to be no rational procedure to estimate probable settlements or tilting due to erratic deviations from homogeneity within the foundation subsoil. Such a method would be useful in practice, since even the most uniform natural soil layers show irregularly distributed heterogeneities. On the other hand, it is clear that neither such complex variations nor the geologic factors from which they arise can be defined in a complete, deterministic way. Hence, the problem should be approached on a probabilistic basis.

The aim of this work is two-fold: (1) to derive the probability distributions of settlement and rotation of a rectangular foundation, accounting for random variations of compressibility, and (2) to propose criteria for the use of these distributions in the rational design of foundations.

Notation. Letter symbols are defined where they first appear, and are arranged alphabetically in the last section of the Appendix.

PROBABILISTIC FORMULATION OF THE SETTLEMENT PROBLEM

In the present paper only the most usual case will be considered, i.e. that of rectangular foundations centrally loaded. The physical assumptions on which the problem is to be formulated are stated and discussed below.

Hypotheses

1. All variations in compressibility occurring in the horizontal direction are random.

This might simply be considered as an expression of the fact that present knowledge is not precise enough to allow of an exact description of such variations on a deterministic basis, no matter how thoroughly the site investigation is carried out.

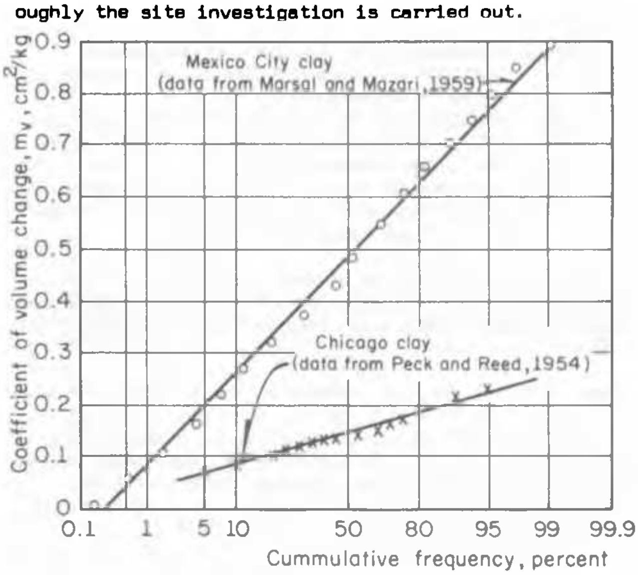

Fig 1. Cummulative frequency of the coefficient of volume change, m_v (stress level p = 1 kg/cm2)

Furthermore, there is some experimental evidence showing that, within a given natural soil stratum, the coefficient of volume change, m_v, behaves as a normally distributed random variable. This is shown in Fig 1, where data for Mexico City clays and Chicago clays have been plotted on probability paper. The fact that points corresponding to each of these clay deposits lie approximately along a straight line means that their frequency distribution is normal. Compressibility data for plotting Fig 1, was taken from Marsal and Mazari (1959) and Peck and Reed (1954).

2. The component of the foundation settlement arising from the random component of compressibility, can be approximated by a rigid movement.

This hypothesis is reasonable for, at least, those structures where the detrimental consequences of tilting are most severe, such as towers, elevated tanks, tall or slender buildings, and machine foundations, since all of them have high vertical rigidities.

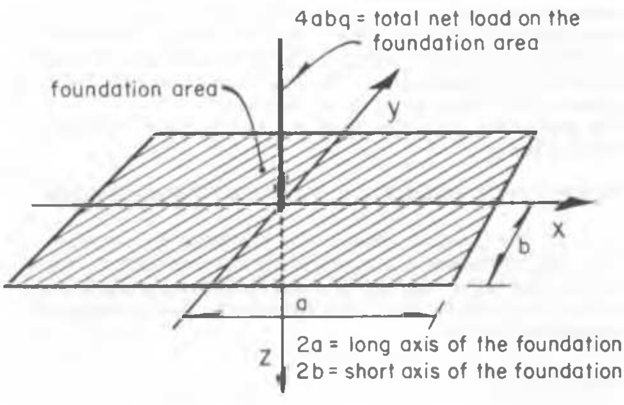

Fig 2. Geometry of the foundation area

Then, in the frame of reference shown in Fig 2, the settlement of the foundation is described by

$$\rho(x,y) = \rho_0(x,y) + \rho_1 + \theta_a x + \theta_b y \quad \ldots . 1$$

where

- $\rho_0(x,y)$ = deterministic component of settlement
- ρ_1 = vertical displacement due to the random component of compressibility
- θ_a = rotation in the direction of the long axis of the foundation due to the random component of compressibility
- θ_b = rotation in the direction of the short axis of the foundation due to the random component of compressibility

3. The relationship between settlement, ρ, coefficient of volume change, and effective vertical stress increment Δp, under every point of the foundation is

$$\rho = \int_0^H m_v \, \Delta p \, dz \quad \ldots\ldots 2$$

where H is the total thickness of the compressible strata.

If the appropriate value of m_v is used in Eq 2, the validity of this relationship has been demonstrated empirically (Skempton and Bjerrum, 1957; Rutledge, 1964; Seed, 1964).

Solution

The design of structural members requires a knowledge of $\rho_0(x, y)$, whose computation on deterministic grounds is a problem dealt with elsewhere (Sommer, 1965; Flores, 1968). On the other hand, an analysis of the overall behavior of a soil-structure system for the purpose of its rational design, is possible only if the probability distributions of θ_a and θ_b, and that of the average settlement, $\bar{\rho}$, can be estimated.

The mathematical derivation of those distributions is presented in the Appendix. If, as usual, the settlement computation is carried numerically after subdivision of the compressible strata into N horizontal sublayers such that each of the variables m_v and Δp is approximately the same throughout the thickness of the corresponding sublayer, then the results are as follow.

a) The average settlement, $\bar{\rho}$, is a normally distributed random function. Its expected value and its variance are given by Eqs 3 a and b, respectively:

$$E[\bar{\rho}] = q \sum_{i=1}^{N} f_i \quad \ldots\ldots 3a$$

$$\mathrm{var}[\bar{\rho}] = (q^2/16ab) \sum_{i=1}^{N} f_i K_i v_i^2 / \alpha_i^2 \quad \ldots\ldots 3b$$

in which, the subscript i identifies quantities corresponding to the i-th sublayer, and

$$\left.\begin{aligned} f_i &= \bar{C}_{ri} H_i \alpha_i / 2.3\,(p_i + Q) \\ v_i^2 &= A_0 \,\mathrm{var}[C_{ri}] \,/\, \bar{C}_{ri}^2 \end{aligned}\right\} \quad \ldots\ldots 3c$$

- A_0 = cross-section area of specimens in which determinations of the compression ratio* C_{ri}, were made
- $\bar{C}_{ri}$ = mean value of the compression ratio (see Eqs A-13b)
- $\mathrm{var}[C_{ri}]$ = variance of the compression ratio (see Eqs A-13b)
- H_i = thickness of the sublayer
- K_i = coefficient plotted in Fig 3
- p_i = vertical stress in the subsoil due to overlying soil, taking the depth of foundation as the datum
- Q = gross pressure on the soil-foundation contact area
- q = net pressure increment of the soil-foundation contact area
- α_i = coefficient plotted in Fig 6

b) The rotations in the directions of the long and short axes, θ_a and θ_b, respectively, are normally distributed random functions. Their expectations and variances are

$$E[\theta_a] = E[\theta_b] = 0 \quad \ldots\ldots 4a$$

$$\left.\begin{aligned} \mathrm{var}[\theta_a] &= (9q^2/16a^3 b) \sum_{i=1}^{N} f_i^2 K_i r_{ai}^2 v_i^2 / \alpha_i^2 \\ \mathrm{var}[\theta_b] &= (9q^2/16ab^3) \sum f_i^2 K_i r_{bi}^2 v_i^2 / \alpha_i^2 \end{aligned}\right\} \quad \ldots\ldots 4b$$

in which r_{ai}^2 and r_{bi}^2 are the coefficients plotted in Figs 4 and 5 respectively.

The statistical soil parameters $\bar{C}_{ri}$ and $\mathrm{var}[C_{ri}]$

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* In general, the compression ratio is defined as $C_r = C_c/(1 + e_0)$, where C_c is the compression index (or the recompression index, when applicable) and e_0 is the initial void ratio.

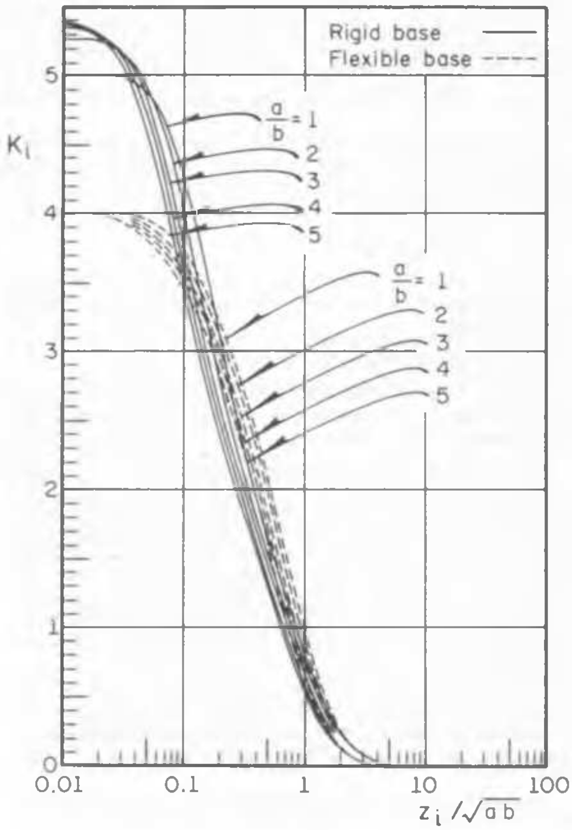

Fig 3. The dimensionless parameter K_i

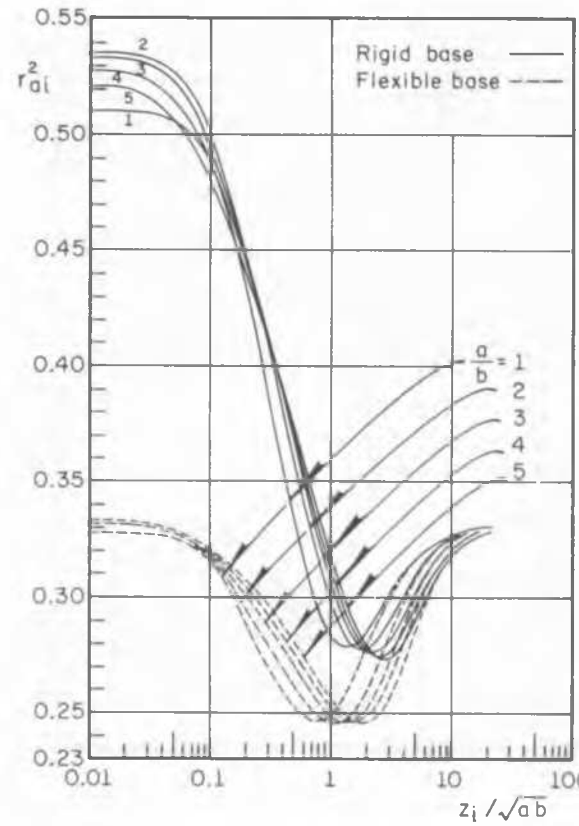

Fig 4. The dimensionless parameter r_{ai}^2

appearing in the results, can be computed with a good approximation only on the basis of a number of individual C_{ri} values greater than that usually determined for conventional settlement analyses. Yet, this should not be considered as a limitation of the present statistical approach, since the compression ratio is, more than any other engineering property, closely related to the natural water content of a clay. Consequently, a large number of values of C_{ri} can be determined at reasonable expense on the basis of that relationship (Terzaghi and Peck, 1948; Peck and Reed, 1954).

From Eqs 3 and the properties of the normal distribution, it follows that, with an arbitrary probability P, the average settlement of a foundation lies within the interval

$$q\left[f - u(P)\sqrt{F_0}\right] \le \bar{\rho} \le q\left[f + u(P)\sqrt{F_0}\right] \qquad \ldots 5$$

where

$$f = \sum_{i=1}^{N} f_i$$

$$F_0 = (1/16ab) \sum_{i=1}^{N} f_i^2 K_i v_i^2 / a_i^2$$

and u(P) is the value in the standard normal distribution such that the probability of a deviation numerically greater than u(P) is P.

Similarly, from Eqs 4, with an arbitrary probability P the rotations of the foundation are

$$\left.\begin{array}{l} |\theta_a| \le q\, u(P)\sqrt{F_1} \\ |\theta_b| \le q\, u(P)\sqrt{F_2} \end{array}\right\} \qquad \ldots\ldots 6$$

where

$$F_1 = (9/16a^3 b) \sum_{i=1}^{N} f_i^2 K_i r_{ai}^2 / v_i^2$$

$$F_2 = (9/16ab^3) \sum_{i=1}^{N} f_i^2 K_i r_{bi}^2 / v_i^2$$

Figs 4 and 5 show that r_{ai}^2 and r_{bi}^2 for a rigid foundation are always greater than those for a flexible one. Then, from Eq 6, the probability of rotations exceeding a certain value increases with the vertical rigidity of the structure, other factors being equal.

DESIGN CRITERIA

Once the probability distributions of settlement and rotation are known, several approaches to design are possible. Two of them will be briefly outlined below.

<u>A criterion based on allowable values of settlement and rotation</u>

The design of every foundation involves some consideration regarding the settlement that can be allowed without endangering the stability or the serviceability of the structure under design, or those of neighboring constructions.

The average settlement allowable in buildings is

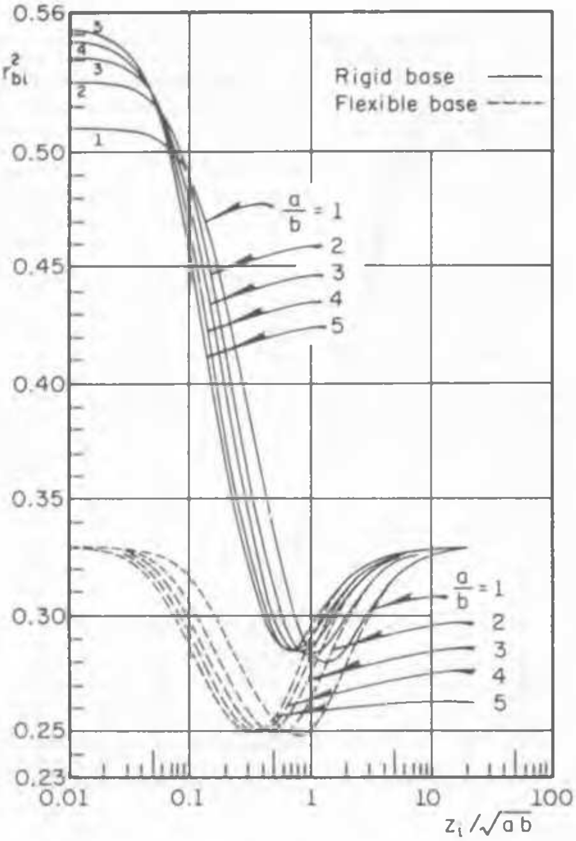

Fig 5. The dimensionless parameter r_{bi}^2

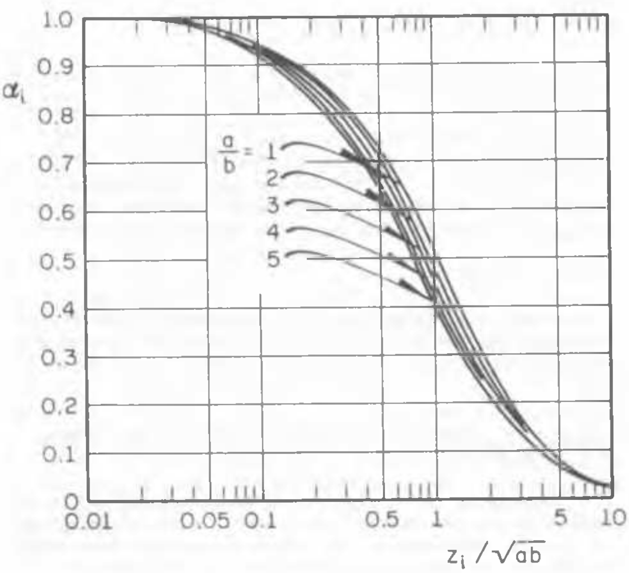

Fig 6. The coefficient α_i

usually limited by the permissible differences in elevation between some portions of the structure and their surroundings, or by the flexibility of connections for utilities such as water and sewerage pipelines, or else, by the amount of settlement which will not cause intolerable damage to structures nearby.

Calling $|\rho_p|$ the permissible vertical movement of the structure, Eq 5 implies that the average vertical displacement will be smaller than $|\rho_p|$, with a probability P, if the following inequality is satisfied

$$|q| \leq |\rho_p| / \left[f + u(P)\sqrt{F_0} \right] \qquad \ldots . 7$$

Similarly, if θ_p denotes the permissible rotation for a given structure, the design should satisfy the condition

$$\theta = \sqrt{\theta_a^2 + \theta_b^2} \leq \theta_p$$

Taking Eqs 6 into account, it is seen that, with a probability P, the rotation of the structure is within the permissible range if

$$|q| \leq \theta_p / \left[u(P)\sqrt{F_1 + F_2} \right] \qquad \ldots .. 8$$

In the cases of tall, relatively rigid structures, the dominant consideration in limiting the allowable tilting is generally human sensibility. In fact, according to Skempton and MacDonald (1956), the value of θ where tilting of high, rigid buildings might become visible is close to 1/250, whereas structural damage probably starts to be of concern for values of θ approaching 1/150.

More generally, if the shift of the line of action of the loads due to tilting is negligible, the permissible rotation of rigid structures certainly depends on the height of the structure, and only on that.

On this basis, the following value is proposed for the permissible rotation of structures where human perception of tilting is the dominant factor*

$$\theta_p = 1/(100 + 3h) \qquad \ldots . 9$$

Here, h is the height of the structure in meters. It is seen that Eq 9 gives $\theta = 1/100$ for $h = 0$, which is about the limit of perceptible deviations from horizontality in a floor; and $\theta h \leq 0.33$ m for every h. For intermediate values of h, Eq 9 gives values of θ in agreement with the observations of Skempton and MacDonald (1956). It also excludes the possibility of rotations endangering stability, since the maximum horizontal displacement of the tallest structure is limited to 0.33 m.

Regarding the allowable rotations of machine foundations, Bjerrum (1963) mentions that 1/750 is the limit where difficulties are to be feared. In the lack of more specific information, this limit might be used for θ_p in the case of machine foundations.

When using the design approach based on allowable values of settlement and rotation, the probabilities of not exceeding those values should be selected at a level consistent with the implication of each event, i.e. excessive settlement or tilting.

A criterion based on cost minimization

When the necessary statistical information is available, a better approach to design is based on the condition of minimum of expected cost (see for

* This equation was suggested to the authors by Dr. E. Rosenblueth, of the Universidad Nacional Autónoma de México.

example Turkstra, 1962; Rosenblueth, 1969), accounting for all possible sources of costs and their corresponding probabilities. Yet, the method has limitations because of the lack of quantitative information regarding the relationship between cost and damage, and the difficulties of evaluating some possible outcomes of the design in monetary terms.

However, if these limitations are kept in mind, the results of a cost-minimization approach based on reasonable assumptions are of interest.

For the case under discussion, it will be considered that the cost of the structure is given by

$$C_T = C_o + C_\rho + C_\theta$$

where

C_o = initial cost

C_ρ = present value* of the cost due to settlement

C_θ = present value* of the cost due to tilting

It will be further assumed that

$$C_o = C_1 + C_2 D_f$$
$$C_\rho = C_3 \bar{\rho}^2 \qquad \ldots\ldots\ldots 10$$
$$C_\theta = C_4 \theta^2$$

where C_1 to C_4 are constants and D_f is the depth of foundation.

Thus, since $D_f = (Q - q)/\gamma$, the expected total cost is

$$E[C_T] = C_1 + C_2(Q-q)/\gamma + C_3 E[\bar{\rho}^2] + C_4 E[\theta^2]$$

Here, from Eqs 3

$$E[\bar{\rho}^2] = \mathrm{var}[\bar{\rho}] + (E[\bar{\rho}])^2 = q^2(f^2 + F_o)$$

and, similarly, from Eqs 4

$$E[\theta^2] = E[\theta_a^2 + \theta_b^2] = \mathrm{var}[\theta_a] + \mathrm{var}[\theta_b] = q^2(F_1 + F_2)$$

* Assuming the design decision is made at time $t = 0$, the present value of the cost due to damage occurring at $t = t_1$ is its equivalent cost at $t = 0$. Then, if λ is the rate of interest, the present value is the cost at $t = t_1$ times $e^{-\lambda t_1}$

Use of these in $E[C_T]$ gives:

$$E[C_T] = C_1 + C_2(Q-q)/\gamma + q^2[C_3(f^2+F_0) + C_4(F_1+F_2)] \quad \ldots.. 11$$

If $H \gg D_f$, the functions f, F_o, F_1 and F_2 are but very slightly sensitive to changes in D_f, which means that they can be considered independent of q. Thus, the condition for a minimum of $E[C_T]$ being $\partial E[C_T] / \partial q = 0$, the following is obtained for q_{op}, the net pressure increment that gives a minimum expected total cost:

$$q_{op} = C_2/2\gamma[C_3(f^2+F_0) + C_4(F_1+F_2)] \quad \ldots\ldots 12$$

A NUMERICAL EXAMPLE

Consider a structure 50 m in height with 2a=20m, 2b = 10 m. Suppose three floors are required for parking facilities so that the space from excavation down to any depth $D_f \le 9$ m is, in principle, usable.

The gross weigth of the foundation-structure system is estimated to be $W = 3400 + 70\ D_f$ (tons), with D_f in meters.

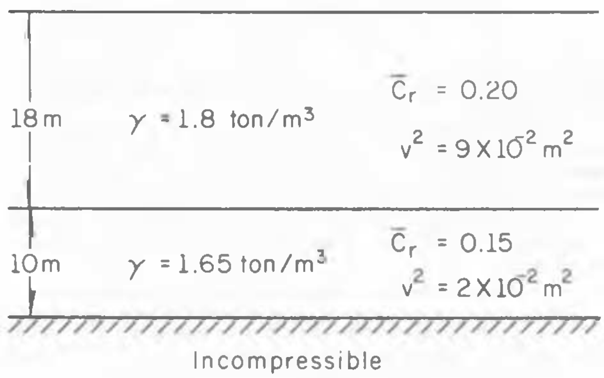

Fig 7. Soil profile and properties for the example

Table I. Computation of f, F_o, F_1 and F_2

H_i (m)	$z_i/\sqrt{ab}$	α_i	p_i (ton/m²)	$p_i + Q$ (ton/m²)	$\bar{C}_{ri}$	f_i (10^{-4} m³/ton)	K_i	r_{ai}^2	r_{bi}^2	v_i^2 (10^{-2} m²)	$f_i^2 \frac{K_i}{\alpha_i^2} v_i^2$ (10^{-6} m⁸/ton²)	$f_i^2 \frac{K_i}{\alpha_i^2} v_i^2 r_{ai}^2$ (10^{-6} m⁸/ton²)	$f_i^2 \frac{K_i}{\alpha_i^2} v_i^2 r_{bi}^2$ (10^{-6} m⁸/ton²)
2	0.14	0.902	1.80	19.80	0.20	79.2	3.55	0.482	0.456	9	24.5	11.8	11.2
"	0.42	0.740	5.40	23.40	"	54.9	1.82	0.378	0.325	"	9.0	3.4	2.9
"	0.71	0.600	9.00	27.00	"	38.6	1.20	0.336	0.290	"	4.5	1.5	1.3
"	0.99	0.500	12.60	30.60	"	28.4	0.80	0.302	0.281	"	2.3	0.7	0.2
"	1.27	0.410	16.20	34.20	"	20.8	0.63	0.286	0.283	"	1.4	0.4	0.1
"	1.56	0.335	19.80	37.80	"	15.4	0.50	0.280	0.288	"	1.0	0.3	0.1
"	1.84	0.290	23.40	41.40	"	12.1	0.38	0.277	0.295	"	0.6	0.2	0.1
"	2.12	0.240	27.00	45.00	"	9.2	0.25	0.278	0.303	"	0.3	0.1	-
"	2.40	0.215	30.60	48.60	"	7.6	0.20	0.280	0.305	"	0.2	0.1	-
"	2.68	0.180	33.90	51.90	0.15	4.5	0.15	0.283	0.306	2	-	-	-
"	2.97	0.160	37.20	55.20	"	3.7	0.11	0.286	0.307	"	-	-	-
"	3.25	0.130	40.50	58.50	"	2.9	0.09	0.290	0.311	"	-	-	-
"	3.53	0.115	43.80	61.80	"	2.4	0.07	0.294	0.315	"	-	-	-
"	3.82	0.100	47.10	65.10	"	2.0	0.05	0.297	0.318	"	-	-	-
						Σ = 281.8					43.8	18.5	15.9

$$f = \sum_{i=1}^{n} f_i = 2.81 \times 10^{-2}\ \mathrm{m^3/ton}$$

$$F_o = \frac{1}{16ab} \sum_{i=1}^{n} f_i^2 \frac{K_i}{\alpha_i^2} v_i^2 = \frac{1}{16 \times 10 \times 5}\ 43.8 \times 10^{-6} = 5.5 \times 10^{-8}\ \mathrm{m^6/ton^2}$$

$$F_1 = \frac{9}{16a^3b} \sum_{i=1}^{n} f_i^2 \frac{K_i}{\alpha_i^2} v_i^2 r_{ai}^2 = \frac{9}{16 \times 1000 \times 5}\ 18.5 \times 10^{-5} = 2.1 \times 10^{-9}\ \mathrm{m^4/ton^2}$$

$$F_2 = \frac{9}{16ab^3} \sum_{i=1}^{n} f_i^2 \frac{K_i}{\alpha_i^2} v_i\ r_{bi}^2 = \frac{9}{16 \times 10 \times 125}\ 15.9 \times 10^{-6} = 7.2 \times 10^{-9}\ \mathrm{m^4/ton^2}$$

The soil profile and properties are shown in Fig 7.

The following is to be determined:

a) The minimum D_f for which the probability of a rotation within the permissible range given by Eq 9 is 0.99

b) The minimum D_f for which the probability of an average settlement less than 15 cm is 0.95

c) How the results for (a) and (b) compare with the optimum depth of foundation for the hypotheses of costs given by Eqs 10 under the following conditions:

c1) Since the excavated space is usable, the initial cost of the project is estimated to increase only about 3 percent per meter of excavation. Then in Eqs 10, $C_2/C_1 \simeq 3 \times 10^{-2}\ m^{-1}$

c2) An average settlement of 0.3 m is estimated to imply a cost whose present values is about 10 percent of the initial cost of the project. Then in Eqs 10, $C_3/C_1 \simeq 1.14\ m^{-2}$

c3) A rotation of 1/100 is estimated to imply a cost with a present value of about 15 percent of the initial cost of the project. Then, in Eqs 10, $C_4/C_1 \simeq 1.5 \times 10^3$

Solution for the criterion of allowable settlement and rotation

The values of the parameters involved in the solution are computed in table I, assuming infinite vertical rigidity of the structure. From these values the following is obtained:

1. For requirement (a) of the example, $P = 0.99$, then $u(P) = 2.58$ and, from Eq 9 $\theta_p = 1/250$. Thus, from Eq 8:

$$|q| \leq 1/(250 \times 2.58 \times 9.65 \times 10^{-5}) = 16.1\ ton/m^2$$

2. For requirement (b) of the example, $P = 0.95$, then $u(P) = 1.96$. Thus, from Eq 7:

$$|q| \leq 0.2/(2.81 \times 10^{-2} + 1.96 \times 2.35 \times 10^{-4}) = 7.1\ ton/m^2$$

Therefore, requirement (b) prevails and the minimum depth of foundation for the criterion of allowable settlement and rotation is computed from $D_f = (Q - q)/\gamma$, with $[q] \leq 7.1\ ton/m^2$ which gives $D_f \geq 6.8$ m.

Solution for the criterion of cost minimization

Substitution of the pertinent data in Eq 12 gives

$$q_{op} = 3 \times 10^{-2}/2 \times 1.8\left[1.14(7.84 \times 10^{-4} + 5.5 \times 10^{-8}) + 1.5 \times 10 \times 9.3 \times 10^{-9}\right]$$

$$\therefore\ q_{op} = 9.1\ ton/m^2$$

from which, $D_f = 5.5$ m

Comparison of results

Notice that, under the assumptions adopted for the analysis of this particular example, the criterion of cost minimization and that of allowable settlement and rotation, give solutions which are similar to each other: a difference in D_f little greater than 20 percent results between the two criteria. Furthermore, the computations involved are so simple that several analyses can be made with alternative hypotheses (and in a practical case this should be done) in order to judge the sensitivity of the results to these hypotheses.

It is also apparent that for typical urban structures on compressible soils with a coefficient v_1^2 of the orden of 0.1 m^2 or smaller*, the controlling parameter is f, while F_0, F_1 and F_2 have practically no effect on design decisions.

CONCLUSIONS

1. From statistical considerations, the average settlement of a foundation can be regarded as a normally distributed random function with mean and variance given by Eqs 3

2. Similarly, the rotations around the principal axes of the foundation are normally distributed about zero and their variances are given by Eqs 4

3. Other things being equal, the probability of a rotation exceeding a certain value increases with the rigidity of the foundation

4. The results permit a rational approach to the design of foundations whose settlement and rotation are to be kept within tolerable values. Furthermore, when information is obtainable, or assumptions can be made regarding the potential costs of tilt and settlement, the results given can be used to arrive at a design that minimizes the expectation of total cost

5. The methods of analysis suggested involve computational work that is but little greater than that required in a conventional settlement analysis.

ACKNOWLEDGEMENTS

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REFERENCES

BARKAN, D. D. 1962. "The Effect of Area and Shape of a Foundation Base on the Coefficient of Elastic Uniform Compression", Dynamics of Bases and Foundations, Mc Graw-Hill, New York, pp. 21-25

BJERRUM, L. 1963. Discussion of Session VI, Proc. European Conf. on Soil Mech. and Found. Engrg., Wiesbaden, Vol. 2, pp. 135-137

ELORDUY, J. et al. 1967. "Dynamic Response of Bases of Arbitrary Shape Subjected to Periodic Vertical Loading", Proc. Intl. Symposium on Wave Propagation and Dynamic Properties of Earth Materials, Albuquerque, pp. 105-121

FADUM, R. E. 1948. "Influence Values for Estimating Stresses in Elastic Foundations", Proc. 2nd. Intl. Conf. on Soil Mech. and Found. Engrg., Vol. 3, pp. 77-84

FLORES, A. 1968. "Análisis de cimentaciones sobre suelo compresible", Ingeniería, México, Vol. 38, N° 3, pp. 339-361

MARSAL, R. J. and MAZARI, M. 1959. "Análisis estadístico de propiedades", El subsuelo de la ciudad de México, Instituto de Ingeniería, México, pp. 241-293

PARZEN, E. 1964. "Integration and Differentiation of Stochastic Processes", Stochastic Processes, Holden-Day, San Francisco, pp. 78-87

PECK, R. B. and REED, W. C. 1954. "Engineering Properties of Chicago Subsoils", Univ. of Illinois Bulletin, Vol. 51, N° 44, pp. 62

ROSENBLUETH, E. 1969. "Safety and Structural Design", Chapter 19 of Reinforced Concrete Engineering (edited by B. Bresler), to be published by Mc Graw-Hill

* For the sake of reference, values of v_1^2 for Mexico City clay and Chicago clay, derived from the data in Fig 1, are $1.7 \times 10^{-3}\ m^2$ and $0.8 \times 10^{-3}\ m^2$, respectively.

RUTLEDGE, P. C. 1964. "Summary and Closing Address", Design of Foundations for Control of Settlement, ASCE, Evanston, pp. 579-587

SEED, H. B. 1964. "Settlement Analyses, A Review of Theory and Testing Procedures", Design of Foundations for Control of Settlement, ASCE, Evanston, pp. 531-540

SKEMPTON, A. W. and BJERRUM, L. 1957. "A Contribution to the Settlement Analysis of Foundations on Clay", Géotechnique, Vol. 7, pp. 168-178

SKEMPTON, A. W. and MACDONALD, D. H. 1956. "The Allowable Settlements of Buildings", Proc. Institution of Civil Engrs., Vol. 5, N° 3, pp 727-768

SOMMER, H. 1965. "A Method for the Calculation of Settlements, Contact Pressures and Bending Moments in a Foundation, Including the Influence of the Flexural Rigidity of the Superstructure", Proc. VI Intl. Conf. on Soil Mech. and Found. Engrg., Vol. 2 pp. 197-201

TERZAGHI, K. 1955. "Evaluation of Coefficients of Subgrade Reaction", Géotechnique, Vol. 5, pp. 297-326

TERZAGHI, K. and PECK, R. B. 1948. "Program for Subsoil Exploration", Soil Mechanics in Engineering Practice, J. Wiley, New York, pp. 285-311

TURKSTRA, C. J. 1962. "A Formulation of Structural Design Decisions", Ph. D. Dissertation, Univ. of Waterloo, 155 pp.

APPENDIX. MATHEMATICAL DEVELOPMENTS

Determining the probability of settlement and rotation

The following are equations arising from the hypotheses of the paper:

$$\rho(x,y) = \rho_o(x,y) + \rho_1 + \theta_a x + \theta_b y \quad \ldots . 1$$

$$\rho(x,y) = \int_0^H m_v(x,y)\,\Delta p(x,y)\,dz \quad \ldots . 2$$

Settlement analyses are usually made by numerical integration of Eq 2, after subdivision of the compressible strata into a number of horizontal sublayers such that m_v and Δp are both approximately constant throughout the thickness of each sublayer, under a certain point of the foundation area.

From hypothesis 1, the coefficient of volume change for the i-th sublayer may be written:

$$m_{vi} = m^o_{vi} + m'_{vi} \quad \ldots\ldots\ldots A\text{-}1$$

where

m^o_{vi} = mean value of m_{vi}, a constant for each sublayer

m'_{vi} = deviation of m_{vi} from the mean, a random variable for each sublayer

Similarly, the net vertical stress increment may be written

$$\Delta p_i = \Delta p^o_i + \Delta p'_i \quad \ldots\ldots A\text{-}2$$

where Δp^o_i is the deterministic components of Δp_i, and $\Delta p'_i$ is its deviation from Δp^o_i.

Since m_{vi} is a normally distributed random variable, it follows that m'_{vi} is normally distributed as well. Its mean and covariance are, respectively

$$\left.\begin{aligned} &E\left[m'_{vi}\right] = 0 \\ &\mathrm{cov}\left[m'_{vi}(\underset{\sim}{V_1}), m'_{vi}(\underset{\sim}{V_2})\right] = \mathrm{cov}\left[m_{vi}(\underset{\sim}{V_1}), m_{vi}(\underset{\sim}{V_2})\right] = s_i^2\,\delta(\underset{\sim}{V_1} - \underset{\sim}{V_2}) \end{aligned}\right\}\ldots A\text{-}3$$

where $\underset{\sim}{V_1}$, $\underset{\sim}{V_2}$ are the position vectors of arbitrary volume elements within the corresponding sublayer, $\delta(\underset{\sim}{V_1} - \underset{\sim}{V_2})$ is the Dirac delta function and s_i^2 is an empirical parameter whose determination is discussed in a later section of this Appendix.

Combining Eqs 1, 2, A-1 and A-2, and eliminating the deterministic components:

$$\rho_1 + \theta_a x + \theta_b y = \sum_{i=1}^{N}\left[m'_{vi}\Delta p^o_i + (m^o_{vi} + m'_{vi})\,\Delta p'_i\right]H_i \quad \ldots . A\text{-}4$$

where N is the number of sublayers used in the numerical integration.

From the concept of the coefficient of subgrade reaction, the second term in the right-hand side of Eq A-4 may be written:

$$\sum_{i=1}^{N}(m^o_{vi} + m'_{vi})\,\Delta p'_i H_i = \Delta p'_o / k_s$$

where $\Delta p'_o$ is the value of $\Delta p'_i$ at z = 0, and k_s is an appropriate coefficient (Terzaghi, 1955). Using this in Eq A-4 yields:

$$\Delta p'_o = k_s\left[\rho + \theta_a x + \theta_b y - \sum_{i=1}^{N} m'_{vi}\Delta p^o_i H_i\right] \quad \ldots . A\text{-}5$$

From equilibrium conditions:

$$\left.\begin{aligned} &\int_{-a}^{a}\int_{-b}^{b}\Delta p_o(x,y)\,dx\,dy = 4\,abq \\ &\int_{-a}^{a}\int_{-b}^{b}\Delta p_o(x,y)\,x\,dx\,dy = 0 \\ &\int_{-a}^{a}\int_{-b}^{b}\Delta p_o(x,y)\,y\,dx\,dy = 0 \end{aligned}\right\} \quad \ldots\ldots A\text{-}6a$$

and

$$\left.\begin{aligned} &\int_{-a}^{a}\int_{-b}^{b}\Delta p^o_o(x,y)\,dx\,dy = 4\,abq \\ &\int_{-a}^{a}\int_{-b}^{b}\Delta p^o_o(x,y)\,x\,dx\,dy = 0 \\ &\int_{-a}^{a}\int_{-b}^{b}\Delta p^o_o(x,y)\,y\,dx\,dy = 0 \end{aligned}\right\} \quad \ldots\ldots A\text{-}6b$$

where Δp_o is the pressure increment on the foundation area, Δp^o_o is the value of Δp^o_i at z = 0 and q is the average pressure increment at the depth of foundation.

From Eqs A-2 and A-6 it follows that

$$\left.\begin{aligned} &\int_{-a}^{a}\int_{-b}^{b}\Delta p'_o(x,y)\,dx\,dy = 0 \\ &\int_{-a}^{a}\int_{-b}^{b}\Delta p'_o(x,y)\,x\,dx\,dy = 0 \\ &\int_{-a}^{a}\int_{-b}^{b}\Delta p'_o(x,y)\,y\,dx\,dy = 0 \end{aligned}\right\} \quad \ldots\ldots A\text{-}7$$

Substitution of Eq A-5 in Eq A-7 results in a system of three equations with three unknowns, from which

$$\left.\begin{aligned} &\rho_1 = (1/4\,ab)\int_{-a}^{a}\int_{-b}^{b}\left[\sum_{i=1}^{N} m'_{vi}\Delta p^o_i(x,y)\,H_i\right]dx\,dy \\ &\theta_a = (1/I_y)\int_{-a}^{a}\int_{-b}^{b}\left[\sum_{i=1}^{N} m'_{vi}\Delta p^o_i(x,y)\,H_i\right]x\,dx\,dy \\ &\theta_b = (1/I_x)\int_{-a}^{a}\int_{-b}^{b}\left[\sum_{i=1}^{N} m'_{vi}\Delta p^o_i(x,y)\,H_i\right]y\,dx\,dy \end{aligned}\right\}\ldots A\text{-}8$$

where I_x and I_y are the moments of inertia of the foundation area with respect to x and y.

Notice that Eqs A-8 are independent of k_s; then the actual value of this coefficient in Eq A-5 is irrelevant.

From Eq A-8 and the Central Limit Theorem, ρ_1, θ_a and θ_b have normal distributions. From Eq A-3 and the rules of integration of stochastic processes (Parzen, 1964):

$$\mathrm{var}\left[\rho_1\right]=(1/16a^2b^2)\int_{-a}^{a}\int_{-b}^{b}\left\{\sum_{i=1}^{N}\left[s_i\,\Delta p_i^o(x,y)H_i\right]^2\right\}dx\,dy$$

$$\left.\begin{aligned}
\therefore\ \mathrm{var}\left[\rho_1\right]&=(1/16a^2b^2)\sum_{i=1}^{N}s_i^2H_i^2\left\{\int_{-a}^{a}\int_{-b}^{b}\left[\Delta p_i^o(x,y)\right]^2dx\,dy\right\}\\
\mathrm{var}\left[\theta_a\right]&=(1/I_y^2)\sum_{i=1}^{N}s_i^2H_i^2\left\{\int_{-a}^{a}\int_{-b}^{b}\left[\Delta p_i^o(x,y)\right]^2x^2dx\,dy\right\}\\
\mathrm{var}\left[\theta_b\right]&=(1/I_x^2)\sum_{i=1}^{N}s_i^2H_i^2\left\{\int_{-a}^{a}\int_{-b}^{b}\left[\Delta p_i^o(x,y)\right]^2y^2dx\,dy\right\}\\
E\left[\rho_1\right]&=E\left[\theta_a\right]=E\left[\theta_b\right]=0
\end{aligned}\right\}\cdots \text{A-9}$$

The integrals in the right-hand side of Eqs A-9 depend on the geometry of the foundation area and on the pressure distribution at the mean depth z_i of the corresponding sublayer. For a specific problem, i.e. for a foundation of given geometry and rigidity, those integrals are functions of z_i only, and they can be written as follows:

$$\left.\begin{aligned}
\int_{-a}^{a}\int_{-b}^{b}\left[\Delta p_i^o(x,y)\right]^2dx\,dy&=a\,b\,q^2K_i\\
\int_{-a}^{a}\int_{-b}^{b}\left[\Delta p_i^o(x,y)\right]^2x^2dx\,dy&=a^3bq^2K_i\,r_{ai}^2\\
\int_{-a}^{a}\int_{-b}^{b}\left[\Delta p_i^o(x,y)\right]^2y^2dx\,dy&=ab^3q^2K_i\,r_{bi}^2
\end{aligned}\right\}\cdots\cdots \text{A-10}$$

Here, K_i, r_{ai}^2 and r_{bi}^2 are dimensionless parameters depending on a/b, $z_i/\sqrt{ab}$ and on the contact-pressure distribution. They have been computed by numerical integration of Eqs A-10 and are plotted in Figs 3 to 5 for both, infinitely rigid and infinitely flexible foundations. The numerical integration of Eqs A-10 in the case of flexible bases was performed using Fadum's (1948) solution for Δp_i^o. The method of integration for the case of rigid foundations has been developed and described by Elorduy et al (1965).

Substitution of Eqs A-10 into Eqs A-9 results in the following:

$$\left.\begin{aligned}
\mathrm{var}\left[\rho_1\right]&=(q^2/16\,a\,b)\sum_{i=1}^{N}s_i^2H_i^2K_i\\
\mathrm{var}\left[\theta_a\right]&=(9q^2/16a^3b)\sum_{i=1}^{N}s_i^2H_i^2K_i\,r_{ai}^2\\
\mathrm{var}\left[\theta_b\right]&=(9q^2/16ab^3)\sum_{i=1}^{N}s_i^2H_i^2K_i\,r_{bi}^2\\
E\left[\rho_1\right]&=E\left[\theta_a\right]=E\left[\theta_b\right]=0
\end{aligned}\right\}\cdots\cdots \text{A-11}$$

where, ρ_1, θ_a and θ_b are normally distributed random functions.

Now, let $\bar{\rho}$ be the average of $\rho(x, y)$ and $\bar{\rho}_0$ that of $\rho_0(x, y)$. Then from Eq 1, $\bar{\rho}=\bar{\rho}_0+\rho_1$ and, since ρ_1 has been found to be normally distributed, the same holds true for $\bar{\rho}$, its mean and variance being

$$\left.\begin{aligned}
E\left[\bar{\rho}\right]&=\bar{\rho}_0\\
\mathrm{var}\left[\bar{\rho}\right]&=\mathrm{var}\left[\rho_1\right]
\end{aligned}\right\}\cdots\cdots \text{A-12a}$$

It is known that, for rectangular plates, the difference in $\bar{\rho}_0$ between the extreme case of zero and infinite foundation flexibility is not larger than three percent (Barkan, 1963). Therefore, for practical purposes and for every degree of foundation rigidity, $\bar{\rho}_0$ can be estimated from Eq 2 using $m_v = m_{vi}^o$ and $\Delta p = \overline{\Delta p_i^o}$, where

$$\overline{\Delta p_i^o}=(1/4ab)\int_{-a}^{a}\int_{-b}^{b}\Delta p_i^o(x,y)\,dx\,dy=\alpha_i q$$

is the average of the stress increment for the i-th sublayer, corresponding to a uniform load distribution over the foundation area. Then in Eq A-12a

$$\bar{\rho}_0=\sum_{i=1}^{N}m_{vi}^o\,\overline{\Delta p_i^o}\,H_i=\sum_{i=1}^{N}m_{vi}^o q\,\alpha_i H_i \quad\cdots\cdots \text{A-12b}$$

The coefficient α_i has been computed as a function of a/b and $z_i/\sqrt{ab}$ and is given in Fig 6.

Eqs A-11 together with Eqs A-12a and b constitute the mathematical solution to the proposed problem.

Determining the statistical soil parameters

Let $\bar{m}_{vi}$ represent experimental values of m_{vi} from laboratory tests on samples of the i-th sublayer. Then

$$\bar{m}_{vi}=(1/A_0)\int_{A_0}m_{vi}(x,y)\,dA$$

where A_0 is the cross-section area of the test specimen for which $\bar{m}_{vi}$ was determined.

Therefore

$$\mathrm{var}\left[\bar{m}_{vi}\right]=(1/A_0)^2\mathrm{var}\int_{A_0}m_{vi}(x,y)\,dA$$

From the rules of integration of stochastic processes (Parzen, 1964) the variance of the integral in the right-hand side is

$$\mathrm{var}\int_{A_0}m_{vi}dA=\int_{A_0}\left\{\int_{A_0}\mathrm{cov}\left[m_{vi}(\underset{\sim}{V_1}),m_{vi}(\underset{\sim}{V_2})\right]dA_1\right\}dA_2$$

Introducing Eq A-3 into the last integration:

$$\mathrm{var}\int_{A_0}m_{vi}dA=\int_{A_0}\left\{\int_{A_0}s_i^2\,\delta(\underset{\sim}{V_1}-\underset{\sim}{V_2})\,dA_1\right\}dA_2=\int_{A_0}s_i^2\,dA=A_0s_i^2$$

Substitution of this in the equation for $\mathrm{var}\left[\bar{m}_{vi}\right]$ yields

$$s_i^2=A_0\,\mathrm{var}\left[\bar{m}_{vi}\right]$$

Now, $\bar{m}_{vi}$ may be written in terms of C_{ri}, the compression ratio, as follows

$$\bar{m}_{vi}=(C_{ri}/\overline{\Delta p_i^o})\log_{10}(1+\overline{\Delta p_i^o}/p_{oi})$$

where p_{oi} is the effective vertical stress in situ for sublayer i, and $\overline{\Delta p_i^o}=\alpha_i q$ has been previously defined.

Then, m_{vi}^o and s_i^2 become:

$$\left.\begin{aligned}
m_{vi}^o&=(\bar{C}_{ri}/\alpha_i q)\log_{10}(1+\alpha_i q/p_{oi})\\
s_i^2&=A_0\mathrm{var}\left[C_{ri}\right]\left[(1/\alpha_i q)\log_{10}(1+\alpha_i q/p_{oi})\right]^2
\end{aligned}\right\}\cdots \text{A-13a}$$

Here,

$$\left.\begin{aligned}
\bar{C}_{ri}&=\frac{1}{n}\sum_{j=1}^{n}C_{ri}^j\\
\mathrm{var}\left[C_{ri}\right]&=\left[1/(n-1)\right]\sum_{j=1}^{n}\left[C_{ri}^j-\bar{C}_{ri}\right]^2
\end{aligned}\right\}\cdots\cdots \text{A-13b}$$

C_{ri}^j (j = 1, 2, ... n) being a set of n experimental values of the compression ratio for the i-th sublayer.

Simplifying the results

In the general case, the in situ vertical stress p_{oi} should be written:

$$p_{oi}=p_i+\gamma D_f$$

where p_i is the vertical stress in the subsoil taking the depth of foundation D_f as the datum, and γ is the average unit weight of the excavated soil. Furthermore,

$$D_f=(Q-q)/\gamma$$

where Q is the gross pressure on the soil-foundation contact area. Therefore

$$p_{oi}=p_i+Q-q$$

and, in Eqs A-13a

$$\log_{10}(1+\alpha_i q/p_{oi})=\log_{10}\left\{\left[p_i+Q-q(1-\alpha_i)\right]/(p_i+Q-q)\right\}$$

whose expansion in a Taylor's series gives

$$\log_{10}\left\{\left[p_i+Q-q(1-\alpha_i)\right]/(p_i+Q-q)\right\}=\frac{\alpha_i}{2.3}\left\{\left[q/(p_i+Q)\right]-\left[(2-\alpha_i)/2\right]\left[q/(p_i+Q)\right]^2+\cdots\right\}$$

In many cases, the ratio $q/(p_i + Q)$ will be much smaller than unity (in fact, the heavier the structure and the more compressible the foundation soil, the smaller that ratio will be). Thus the first term of the series will generally suffice as an approximation, i.e.,

$$\log_{10}(1+\alpha_i q/p_{oi}) = \alpha_i q/2.3(p_i+Q)$$

Then, from Eqs A-11 to A-14 the following results are finally obtained:

a) The average settlement, $\bar{\rho}$, is a normally distributed random function. Its expectation and variance are

$$E\left[\bar{\rho}\right] = q\sum_{i=1}^{N} f_i \qquad \ldots\ldots 3a$$

$$\mathrm{var}\left[\bar{\rho}\right] = (q/16ab)\sum_{i=1}^{N} f_i^2 K_i v_i^2/\alpha_i^2 \qquad \ldots\ldots 3b$$

where

$$\left.\begin{aligned} f_i &= \bar{C}_{ri} H_i \alpha_i/2.3(p_i+Q) \\ v_i^2 &= A_o \,\mathrm{var}\left[C_{ri}\right] \Big/ \bar{C}_{ri}^2 \end{aligned}\right\} \qquad \ldots\ldots 3c$$

b) The rotations in the directions of the long and short axes, θ_a and θ_b respectively, are normally distributed random functions. Their expectations and variances are

$$E\left[\theta_a\right] = E\left[\theta_b\right] = 0 \qquad \ldots\ldots 4a$$

$$\left.\begin{aligned} \mathrm{var}\left[\theta_a\right] &= (9q^2/16a^3b)\sum_{i=1}^{N} f_i^2 K_i r_{ai}^2 v_i^2/\alpha_i^2 \\ \mathrm{var}\left[\theta_b\right] &= (9q^2/16ab^3)\sum_{i=1}^{N} f_i^2 K_i r_{bi}^2 v_i^2/\alpha_i^2 \end{aligned}\right\} \ldots\ldots 4b$$

<u>Notation</u>

a	= half the length of the foundation area
A	= area
A_o	= cross-section area of consolidation specimens
b	= half the width of the foundation area
C_o	= initial cost of the project
C_{ri}	= compression ratio for the i-th sublayer
$\bar{C}_{ri}$	= mean value of C_{ri}
C_T	= total cost of the project
C_θ	= present value of the cost due to tilting
C_ρ	= present value of the cost due to settlement
C_1 to C_4	= constants (see Eqs 23)
D_f	= depth of foundation
E[ ]	= mathematical expectation of
f, F_o, F_1, F_2,	functions (see Eqs 5 and 6)
h	= height of the structure, <u>in meters</u>
H	= total thikness of compressible subsoil
H_i	= thickness of the i-th sublayer
i, j,	= integers
I_x, I_y,	= moments of inertia of the foundation area
k_s	= coefficient of subgrade reaction
K_i	= dimensionless parameter (see Fig 3)
m_{vi}	= coefficient of volume change for the i-th sublayer
m°_{vi}	= mean value of m_{vi}
m'_{vi}	= deviation of m_{vi} from the mean
$\bar{m}_{vi}$	= experimental value of m_{vi}
n	= number of experimental values of C_{ri}
N	= number of sublayers used in the numerical integration
p_i	= effective vertical stress in the subsoil taking the depth of foundation as the datum
p_{oi}	= effective vertical stress in situ
P	= a probability
q	= average net pressure increment on the foundation area
Q	= gross pressure on the foundation area
q_{op}	= optimum value of q, for cost minimization
r_{ai}^2	= a dimensionless parameter (see Fig 4)
r_{bi}^2	= a dimensionless parameter (see Fig 5)
s_i^2	= a measure of the variance of m_{vi} (see Eq A-3)
u (P)	= value in the standard normal distribution such that the probability of a deviation numerically greater than u(P) is P
v_i	= a measure of the coefficient of variation of m_{vi} (see Eqs 3c)
var[ ]	= variance of
$\underset{\sim}{V}$	= position vector of an elemental volume of soil
x, y, z,	coordinates (see Fig 2)
α_i	= a dimensionless parameter (see Fig 6)
γ	= unit weight of the soil
$\delta(\underset{\sim}{V}_1-\underset{\sim}{V}_2)$	= Dirac delta function
$\Delta p_i(x,y)$	= net vertical stress increment, in sublayer i
$\Delta p_i^\circ(x,y)$	= deterministic component of Δp_i
$\Delta p_i'(x,y)$	= deviation of Δp_i from the mean
$\Delta p_o(x,y)$	= value of Δp_i at z = 0
$\Delta p_o^\circ(x,y)$	= deterministic component of Δp_o
$\Delta p_o'(x,y)$	= deviation of Δp_o from the mean
θ	= rotation of the foundation
θ_a	= rotation of the foundation in the direction of a
θ_b	= rotation of the foundation in the direction of b
θ_p	= permissible value of θ
$\rho(x,y)$	= settlement at point (x, y)
$\rho_o(x,y)$	= deterministic component of $\rho(x, y)$
ρ_1	= uniform settlement due to the random component of compressibility
$\bar{\rho}$	= the average of $\rho(x, y)$ over the foundation area
$\bar{\rho}_o$	= the average of $\rho_o(x, y)$ over the foundation area