A ONE-DIMENSIONAL MODEL FOR PROGRESSIVE FAILURE.
UNE MODELE UNIDIMENSIONNELLE POUR LA RUPTURE PROGRESSIVE

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SYNOPSIS A mathematical model for the progressive failure of a single layer bonded to a rigid base is developed for elastic, plastic, and strain softening behavior of the bonding material. Inclinations of the slope and frictional strength are included. The solutions are presented in dimensionless form and in terms of conventional soil mechanics parameters. The initiation of a failure surface is shown to depend largely upon the swelling potential of the soil and the initial lateral stresses, and can occur even if the conventional factor of safety against failure at peak strength is high. The extent of propagation of the failure surface depends further on the residual factor of safety. A simple chart allows these interactions to be studied easily.

INTRODUCTION

Skempton (1964) and Bjerrum (1967) have described the failure of slopes cut into overconsolidated clays or clay shales and have shown that such failures can occur as the result of progressive softening and straining of the clay over the years after the cut was made.

Before a cut is made into such soil, there exist large, horizontal, compressive stresses, which are released at the surface of the slope by the digging of the cut. Consequently, the clay tends to swell; it may then lose strength and yield plastically. If the plastic yielding causes further loss of strength, further plastic flow may occur until the collapse of the slope. There is a complicated interaction between the stresses caused by the weight of material in the slope, the stresses released by excavation, the swelling of the soil, the initially available strength of the soil (peak shear strength), the strain-softening of the soil, and the final strength of the soil (residual shear strength.)

In addition to laboratory studies of the peak and residual strengths, swelling potential, and other material properties, theoretical analyses are needed to understand how these various factors interact. The geometry of the problem is clearly two or three dimensional, but much insight can be gained by studying a simpler model to see the effect of the various factors.

BASIC MODEL

The one dimensional model consists of a single layer of thickness, h, of elastic material whose extensional strain is related to the tensile stress (or the decrease in compressive stress) by a modulus, E. This modulus can also be considered as a swelling potential. The layer has an initial compressive stress, σ_o, and is bonded to a rigid base, as is shown in Fig. 1. Initially α is taken as zero.

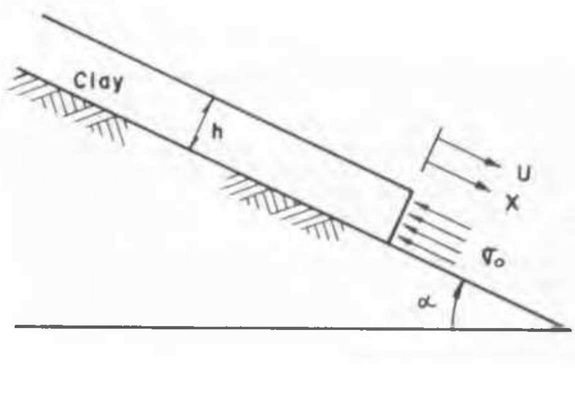

Fig. 1 Basic Model

If σ_o is removed, the layer will tend to swell to the right. How far it will move depends on the nature of the bond between the layer and the base. Using the convention illustrated in Fig. 1 and calling displacements u, and locations x, one can develop the equilibrium of an infinitesimal element, which leads to the basic equation:

$$Eh \frac{d^2u}{dx^2} = \tau \qquad (1)$$

where τ is the shear stress in the bond

SOLUTIONS FOR VARIOUS STRESS-STRAIN RELATIONS

There are many possible relations between the shear stress carried by the bond, τ, and the relative displacement between the layer and the rigid base, u. One of the simplest occurs if there is no relative displacement until a critical shear stress, c, is reached, after which there is no further resistance to motion along the bond. Such a rigid-plastic bonding material gives equations that are easily solved to give displacements along the failure surface:

$$u = \frac{c}{2Eh} x^2 + \frac{\sigma_o}{E} x + \frac{\sigma_o^2 h}{2Ec} \qquad (2)$$

and the extent of the plastic failure surface:

$$x_{CR} = -\frac{\sigma_o h}{c} \qquad (3)$$

The negative sign arises because the surface must propagate to the left in the -x direction.

For linearly elastic relation between τ and u in which there is never any yielding, the basic equation is again easily solved to give:

$$u = \frac{\sigma_o}{E} \left(\frac{Eh}{k}\right)^{1/2} \exp\left[\left(\frac{k}{Eh}\right)^{1/2} x\right] \qquad (4)$$

A combination of the two previous stress-strain relations gives a linearly elastic - perfectly plastic material, illustrated in Fig. 2. The two previous solutions are modified to insure compatibility and equilibrium at the section where the bond changes from plastic to elastic.

Figure 3 shows a linearly elastic - strain softening material which has a peak shear strength, C_p, and a residual strength, C_r. The solution for this case is virtually identical to the solution for the perfectly plastic material except that the ratio between peak and residual strength is unity for the perfectly plastic material.

INCLINED SLOPES AND FRICTIONAL MATERIALS

The basic model can be modified to represent an inclined layer by taking α not equal to zero. The weight of material in the layer now enters the calculations, for, if it has a unit weight of γ there will be a vertical force on each Δx of the bonding material equal to $\gamma h \Delta x$. This will give a component of stress $\gamma h \cos\alpha$, normal to the slope and a component, $\gamma h \sin\alpha$, parallel to the slope. If the available shear strength, C_p, or C_r, is then reduced by the amount necessary to hold the slope in place before σ_o was removed, namely, $\gamma h \sin\alpha$, a new set of strengths, C_p' and C_r' are found. These strengths can be used in all the equations found above because the rest of the problem is identical to that for the flat layer.

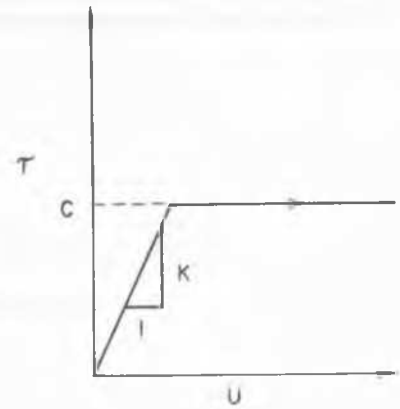

Fig. 2 Elastic Perfectly Plastic Soil

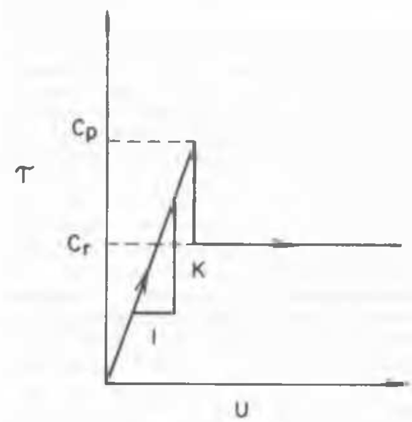

Fig. 3 Strain Softening Soil

It is now possible that C_r' may go to zero if C_r is equal to $\gamma h \sin \alpha$. This corresponds to a critical h or critical vertical depth $Z = h/\cos \alpha$, for which a failure surface, once started, would propagate to infinity. Such a case cannot occur for a horizontal layer.

A frictional material can also be considered simply by replacing C_p and C_r by their equivalent frictional shear strengths in terms of cohesion, C_p or C_r, and friction, ϕ_p or ϕ_r. The new strengths are here called S_p and S_r to distinguish them from the cohesive components. For inclined slopes the strengths are again reduced by $\gamma h \sin \alpha$ to become S'_p and S'_r. The solution includes all previous solutions.

It is possible that a failure surface may not start at all if the strength is large enough or other conditions are met. By examining the solution for the elastic material and considering when the stresses will exceed S_p, a criterion of first yield can be found.

In order to study the effects of varying the parameters it is convenient to restate the equations in dimensionless form. The following four dimensionless quantities were chosen as the basic ones.

$\frac{\sigma_o}{S'_p}$ - the ratio of in situ horizontal stress to peak shear strength available

$\frac{S'_p}{S'_r}$ - the ratio of peak to residual available shear strength

$\frac{E}{kh}$ - the ratio of moduli in the soil layer and in the bond with base

$\frac{\sigma_o}{E}$ - the ratio of in situ horizontal stress to swelling modulus

The use of these parameters allows the solutions to be summarized as follows:

First yield occurs if:

$$\frac{\sigma_o}{S'_p} \geq \left[\frac{E}{kh}\right]^{1/2} \tag{5}$$

The extent of the failure surface is:

$$\left(\frac{x}{h}\right)_{CR} = \left[-\frac{\sigma_o}{S'_p} + \left(\frac{E}{kh}\right)^{1/2}\right]\frac{S'_p}{S'_r} \tag{6}$$

The elastic displacements are:

$$\frac{u}{h} = \frac{E}{kh}\cdot\frac{\sigma_o}{E}\cdot\frac{S'_p}{\sigma_o}\exp\left[\left(\frac{kh}{E}\right)^{1/2}\frac{\sigma_o}{S'_p}\right] \tag{7}$$

$$\left[\frac{S'_p}{S'_r} - \frac{S'_p}{S'_r}\right]\exp\left[\left(\frac{kh}{E}\right)^{1/2}\frac{x}{h}\right]$$

and the plastic displacements are:

$$\frac{u}{h} = \frac{1}{2}\frac{S'_r}{S'_p}\frac{S'_p}{\sigma_o}\frac{\sigma_o}{E}\left[\left(\frac{x}{h}\right)^2 - \left(\frac{x}{h}\right)^2_{CR}\right] + \frac{\sigma_o}{E}\left[\left(\frac{x}{h}\right) - \left(\frac{x}{h}\right)_{CR}\right] + \frac{E}{kh}\frac{\sigma_o}{E}\frac{S'_p}{\sigma_o} \tag{8}$$

If no yield ever occurs, the elastic displacements are:

$$\frac{u}{h} = \frac{\sigma_o}{E}\left(\frac{E}{kh}\right)^{1/2}\exp\left[\left(\frac{kh}{E}\right)^{1/2}\frac{x}{h}\right] \tag{9}$$

EXPRESSION IN TERMS OF CONVENTIONAL SOIL MECH ANICS PARAMETERS

The horizontal in situ stress can also be expressed as $k_o \gamma Z$ where k_o is the coefficient of lateral earth pressure at rest, γ is the unit weight of the soil, and Z is the distance from the surface. The total force over a unit thickness of a face would be $1/2\, k_o \gamma h^2$ so σ_o can be replaced by $1/2\, k_o \gamma h$.

One could analyze the stability of a slope, such as those considered here, by the conventional means of taking the ratio of the resisting shear forces to the driving forces caused by the weight of the soil.

The factor of safety for peak strength would be:

$$FS_p = \frac{C_p + \gamma h \cos\alpha \tan\phi p}{\gamma h \sin\alpha} \tag{10}$$

and that for residual strength would be:

$$FS_r = \frac{C_r + \gamma h \cos\alpha \tan\phi r}{\gamma h \sin\alpha} \tag{11}$$

The factor of safety against the first yield can be found from equation (5), which defines the conditions necessary to initiate the failure surface. Substitution of the conventional soil mechanics parameters gives the following equation for the factor of safety against first yield, FS_y:

$$FS_y = \left(\frac{2 \sin\alpha}{K_o}\right)\left(\frac{E}{kh}\right)^{1/2}(FS_p - 1) \tag{12}$$

Substitution of these parameters into equation (6) for the extent of the failure surface, and some algebraic manipulations leads to

$$\left(\frac{x}{h}\right)_{CR} \frac{2\sin\alpha}{K_o} = \left[\frac{1}{FS_r - 1}\right]\left[FS_y - 1\right] \qquad (13)$$

which expresses the extent of the failure surface in terms of the angle of the slope, K_o, and the factor of safety. This enables the analysis to be summarized in a dimensionless plot, Fig. 4.

In Fig. 4 the ordinate is the factor of safety against first yield. If the latter is greater than one, no failure begins. If the residual factor of safety and the factor of safety against yield are both less than one, total failure occurs. In the region where yielding can occur but where ultimate failure cannot, lines are plotted for constant values of

$$\left(\frac{x}{h}\right)_{CR} \frac{2\sin\alpha}{K_o}$$

Figure 4 can be used by calculating FS_r and FS_p, finding the dimensionless parameter

$$\left(\frac{x}{h}\right)_{CR} \frac{2\sin\alpha}{K_o}$$

and then computing $\left(\frac{x}{h}\right)_{cr}$ from known values of α and k_o. Examples are shown in the next section.

Of the parameters used so far the shear constant k is the hardest to define rationally and to measure. If one has experimental evidence of the relative displacement at peak strength, u_y, the parameter k can be redefined as the ratio of peak available shear to relative displacement at peak shear. This leads to a new equation for FS_y:

$$FS_y = \left[\frac{E u_y \sin\alpha \, (FS_p - 1)}{\gamma}\right] \frac{2}{K_o h} \qquad (14)$$

EXAMPLE PROBLEM

As an example of the use of the chart of Fig. 4, a problem was chosen with dimensions and material properties comparable to those for a real case. The following properties were not changed throughout the calculations:

$k_o = 2$
$\alpha = 5°$
$\gamma = 100$ pcf
$\phi_p = 21.6°$
$C_p = 1300$ psf
$\phi_r = 9.5°$
$C_r = 900$ psf

The values of ϕ and C were chosen from the data on Cucaracha shale reported by Hirsch-

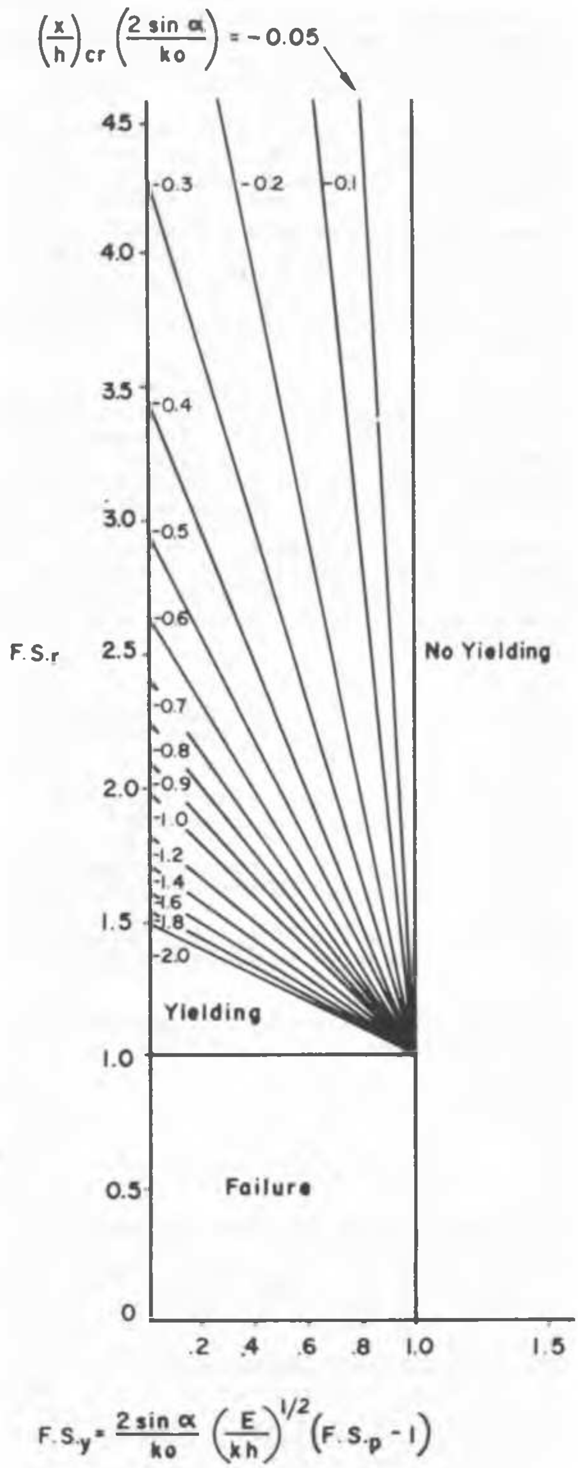

Fig. 4 Dimensionless Results

feld, et. al. (1965). Table I summarizes the values of E, h, and u_y and the resulting factors of safety FS_p, FS_r and FS_y. The lengths of the failure surface are in the last column. At heights of 100 feet such slopes have failed in the field.

It can be seen that changes of material constants within a reasonable range do not affect the length of propagation severely. However, even for slopes with high FS_p and FS_r values, there is likely to be extensive propagation of the failure surface. If the layer is thinner, the extent of the failure surface decreases, and eventually there is no first yield.

CONCLUSIONS

The one dimensional model indicates that factors other than those considered in the conventional stability analysis of slopes may have a dominant effect on whether failure will start. Once failure starts, the length of propagation becomes much more dependent on the conventional parameters of ϕ, c, and slope depth.

The analysis results in a simple chart that can be used at least qualitatively to evaluate the possible behavior of a long slope.

Clearly, two dimensional analyses are essential to an understanding of the problem in the field. These will require numerical procedures because of their greater complexity, and work on this is now in progress.

ACKNOWLEDGEMENT

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REFERENCES

Bjerrum, Laurits (1967), "The Third Terzaghi Lecture: Progressive Failure in Slopes of Overconsolidated Plastic Clay and Clay Shales", Proceedings, ASCE Journal of the Soil Mechanics and Foundations Division, Vol. 93, No. SM5, pp.3-49

Hirschfeld, R.C., Whitman, R.V. and Wolfskill, L.A. (1965), "Engineering Properties of Nuclear Craters," Research Report R65-53, M.I.T. Department of Civil Engineering

Skempton, A.W. (1964), "Long Term Stability of Clay Slopes," Geotechnique, Vol. 14, No. 2, pp. 77-102

TABLE I

Problem No.	E	H	U_y	FS_p	FS_r	FS_y	Failure Surface Dimensionless	Failure Surface ft.
1	10^5 psf	100 ft.	0.01 ft.	6.00	2.94	0.0208	0.505	568
2	10^7	100	0.01	6.00	2.94	0.208	0.4	458
3	10^7	100	0.04	6.00	2.94	0.416	0.3	344
4	10^7	75	0.04	6.50	3.28	0.585	0.18	152
5	10^7	50	0.04	7.50	3.96	0.953	0.016	9.1
6	10^7	40	0.04	8.25	4.48	1.26	----	--