Some Stress-Strain Relationships for Soils

Quelques Relations entre contraintes et déformations des sols

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SUMMARY

By means of a simple assumption concerning the distribution of the deformation states of particle contacts in a potential rupture surface before loading, an equation is developed for the relationship between shear stresses and strains at first loading. Some empirical relationships of a still simpler form are also indicated. One of them is combined with an empirical formula for isotropic compression, giving rise to a more general formula, which is also applicable to oedometer and triaxial tests.

SOMMAIRE

Au moyen d'une supposition simple concernant la répartition des états de déformation aux contacts des grains sur un plan de rupture potentiel avant chargement, on a élaboré une équation établissant la relation contrainte-déformation au premier chargement. Quelques relations empiriques plus simples sont aussi indiquées. Une d'entre elles est combinée à une formule empirique pour la compression isotrope, résultant en une formule plus générale qui est aussi applicable aux essais à l'oedomètre et aux essais triaxiaux.

ASSUMPTIONS

Consider a sample of a soil, consisting of separate particles, which has been deposited in any natural or artificial way. In a potential rupture surface any two particles in contact with each other will probably already have slid back and forth several times during deposition.

In an arbitrary kind of shear test we now apply an increasing sliding or shear deformation δ and measure the corresponding average shear stress τ. This gives the dotted curve in Fig. 1, corresponding to "first loading." After a deformation δ_f the average shear stress has reached it ultimate value τ_f, corresponding to failure. If we increase the deformation beyond δ_f, τ would remain constant ($=\tau_f$). If we now reverse the deformations, we get the full curve through point 3 which, theoretically, reaches the opposite failure stress $-\tau_f$ after a deformation δ_f. This curve corresponds to "full unloading." If, finally, we reverse the deformations a second time, we get the full curve through point 4, corresponding to "full reloading." By partial unloading or reloading we may arrive at any point inside the hysteresis loop formed by the two full curves mentioned above.

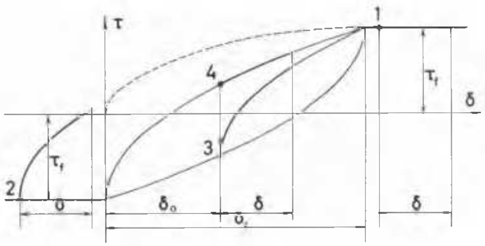

FIG. 1. Stress-strain diagram with hysteresis loop.

From the above it follows that, after deposition but before first loading in the shear test, any existing particle contact in the potential rupture surface corresponds to a point inside or on the hysteresis loop, or on one of its two straight-lined extensions (coresponding to $\pm\tau_f$).

An unknown fraction α of the particle contacts corresponds to points in or on the hysteresis loop, and the remaining fraction $1 - \alpha$ to points on its extensions (half on each). The fraction α is distributed statistically in some unknown way, but for simplicity we shall here assume an even distribution, not over the area of the loop, but over the two boundaries (half on each).

By first loading in the shear test ultimate failure will not be reached until all particle contacts in the failure surface have been brought to failure. This means, as already stated above, that the deformation δ_f necessary for failure in the first loading must be identical with the deformation necessary for complete reversal in unloading or reloading.

GENERAL FORMULA FOR SHEAR DEFORMATIONS

Experience shows that the unloading and reloading curves can usually be expressed in the simple form:

$$\tau/\tau_f = 2(\delta/\delta_f)^n \tag{1}$$

where n must be determined experimentally. By means of this, and the above assumptions, we shall now deduce an equation for the curve corresponding to *first loading* (dotted in Fig. 1). For each particle contact we can find an expression for the additional shear force τ developed by an additional shear deformation δ. For a particle contact 1, which already was at failure (in the same direction) before the test, we find of course:

$$\tau_1 = 0. \tag{2}$$

For a particle contact 2, which was at failure, but in the opposite direction equation (1) gives the following force:

$$\tau_2 = 2\tau_f(\delta/\delta_f)^n \qquad (\delta \leq \delta_f). \tag{3}$$

For a particle contact 3 on an unloading curve we get:

$$\tau_3 = 2\tau_f(\delta/\delta_f)^n \qquad (\delta \leq \delta_f - \delta_0) \tag{4a}$$

and

$$\tau_3 = 2\tau_f[(\delta_f - \delta_0)/\delta_f]^n \qquad (\delta \geq \delta_f - \delta_0). \tag{4b}$$

By integration we can find the average value for the unloading curve:

$$\tau_3 = \frac{1}{\delta_t}\int_0^{\delta_t-\delta} 2\tau_t\left(\frac{\delta}{\delta_t}\right)^n d\delta_0 + \frac{1}{\delta_t}\int_{\delta_t-\delta}^{\delta_t} 2\tau_t\left(\frac{\delta_t-\delta_0}{\delta_t}\right)^n d\delta_0$$

$$= \frac{2\tau_t}{\delta_t^{n+1}}\left[\delta^n(\delta_t-\delta) + \frac{\delta^{n+1}}{n+1}\right]. \qquad (5)$$

Finally, for a particle contact 4 on a reloading curve we get:

$$\tau_4 = 2\tau_t\left[\left(\frac{\delta+\delta_0}{\delta_t}\right)^n - \left(\frac{\delta_0}{\delta_t}\right)^n\right] \qquad (\delta \le \delta_t - \delta_0) \quad (6a)$$

and

$$\tau_4 = 2\tau_t\left[1 - \left(\frac{\delta_0}{\delta_t}\right)^n\right] \qquad (\delta \ge \delta_t - \delta_0). \quad (6b)$$

The average value for the reloading curve is:

$$\tau_4 = \frac{1}{\delta_t}\int_0^{\delta_t-\delta} 2\tau_t\left[\left(\frac{\delta+\delta_0}{\delta_t}\right)^n - \left(\frac{\delta_0}{\delta_t}\right)^n\right] d\delta_0$$

$$+ \frac{1}{\delta_t}\int_{\delta_t-\delta}^{\delta_t} 2\tau_t\left[1 - \left(\frac{\delta_0}{\delta_t}\right)^n\right] d\delta_0$$

$$= \frac{2\tau_t}{\delta_t^{n+1}}\left[\delta\delta_t^n - \frac{\delta^{n+1}}{n+1}\right]. \qquad (7)$$

According to our assumptions, the total shear force will be:

$$\tau = \tfrac{1}{2}(1-\alpha)(\tau_1+\tau_2) + \tfrac{1}{2}\alpha(\tau_3+\tau_4)$$

$$= \frac{\tau_t}{\delta_t^{n+1}}\{(1-\alpha)\delta^n\delta_t + \alpha[\delta^n(\delta_t-\delta) + \delta\delta_t^n]\}. \qquad (8)$$

This gives the final formula:

$$\frac{\tau}{\tau_t} = \left(\frac{\delta}{\delta_t}\right)^n + \alpha\left(\frac{\delta}{\delta_t}\right)\cdot\left[1 - \left(\frac{\delta}{\delta_t}\right)^n\right]. \qquad (9)$$

The constants n and α, which both must lie between 0 and 1, should be determined experimentally in any given case. It should be noted that equation (9) is valid only for *first loading*. For unloading and reloading, equation (1) applies.

In most practical cases the best results seem to be obtained with $\alpha = 1$, i.e.:

$$\tau/\tau_t = (\delta/\delta_t) + (\delta/\delta_t)^n - (\delta/\delta_t)^{n+1} \qquad (10)$$

This equation will be used in the following.

SPECIAL CASES

With $n = 1$, ½, and ⅓ respectively we find:

$$\tau/\tau_t = (\delta/\delta_t)[2 - (\delta/\delta_t)], \qquad (10a)$$

$$\tau/\tau_t = \sqrt{(\delta/\delta_t)}[1 - (\delta/\delta_t)] + (\delta/\delta_t), \qquad (10b)$$

and

$$\tau/\tau_t = \sqrt[3]{(\delta/\delta_t)}[1 - (\delta/\delta_t)] + (\delta/\delta_t). \qquad (10c)$$

Corresponding curves are drawn in Figs. 2, 3, and 4 respectively (in the latter the full curve). Unloading and reloading curves are also shown. The different curves seem to apply to different kinds of soils, for example Fig. 2 often applies to soft clay, Fig. 3 to loose sand, and Fig. 4 to dense sand. Moreover, the curves seem to apply, not only to direct shear tests, but to almost any test, in which shear stresses play a dominant role, for example triaxial tests, plate loading tests, pile loading tests.

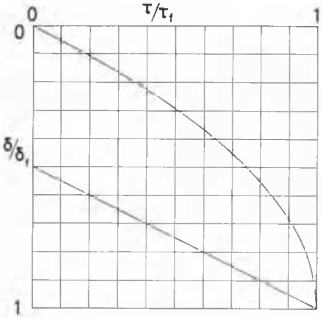

FIG. 2. Curve corresponding to equation (10a).

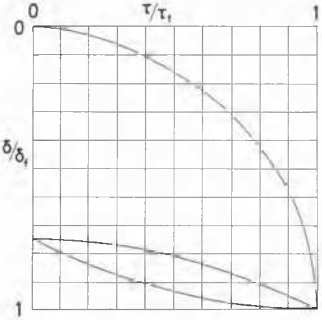

FIG. 3. Curve corresponding to equation (10b).

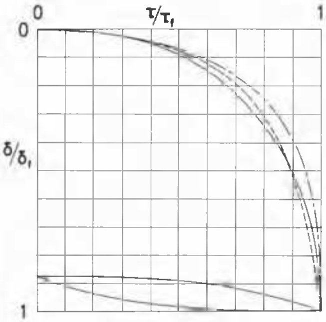

FIG. 4. Curve corresponding to equations (10c), (12c), and (13a).

A DEFINITION OF FAILURE

If the $\tau - \delta$ curve has a maximum, it is natural to define failure as corresponding to this maximum. However, in many cases no such maximum occurs, and failure is then defined more or less arbitrarily. A suitable definition can be found, when it is observed (from the full curve in Fig. 4) that $\delta = 0.5\delta_f$ corresponds to $\tau = 0.9\tau_f$. This gives the definition already proposed by the writer (Brinch Hansen, 1963b): Failure corresponds to the load, at which the deformation is twice the deformation at 90 per cent of the load. It is also worth noticing that $\delta = 0.1\delta_f$ corresponds to $\tau = 0.5\tau_f$. This leads to the simple rule: When the factor of safety is about 2, the deformation at working load will be approximately 10 per cent of the deformation at failure.

SIMPLER FORMULAS

Although the developed formulas are not very complicated, they have, with the exception of (10a), the drawback that δ/δ_f cannot be expressed explicitly in τ/τ_f. It may, therefore, be preferable to employ still simpler formulas, usually of the "hyperbolic" type. The simplest formula of this type was proposed by Kondner (1963). His formula may be written as:

$$\tau/\tau_f = (a+1)\delta/(a\delta + \delta_f) \tag{11a}$$

$$\delta/\delta_f = \tau/[(a+1)\tau_f - a\tau]. \tag{11b}$$

For small stresses this implies a linear relationship. However, in reality this relationship is more often parabolic, which fact led the writer (Brinch Hansen, 1963a) to propose a formula of the following type:

$$\tau/\tau_f = \sqrt{[(b+1)\delta/(b\delta + \delta_f)]} \tag{12a}$$

$$\delta/\delta_f = \tau^2/[(b+1)\tau_f^2 - b\tau^2]. \tag{12b}$$

In order to fulfil the above-mentioned 90 per cent criterion, we have to put $b = 3$, with which we obtain:

$$\tau/\tau_f = \sqrt{[4\delta/(3\delta + \delta_f)]} \tag{12c}$$

$$\delta/\delta_f = \tau^2/(4\tau_f^2 - 3\tau^2). \tag{12d}$$

The corresponding curve is shown with a dotted line on Fig. 4. In a majority of cases the writer has found this relation to be the best approximation to experimental results.

Another useful formula, proposed by the writer (Brinch Hansen, 1963b) is:

$$\frac{\tau}{\tau_f} = \frac{2\sqrt{\delta\delta_f}}{\delta + \delta_f} \qquad \frac{\delta}{\delta_f} = \left[\frac{\tau_f}{\tau} - \sqrt{\left(\frac{\tau_f}{\tau}\right)^2 - 1}\right]^2. \tag{13a-b}$$

The special feature of this formula, as opposed to (11) and (12), is that it gives a maximum at failure. The corresponding curve is shown with a dash-dotted line on Fig. 4. It should be noticed that (12c,d) and (13) give the same results for small stresses.

ISOTROPIC COMPRESSION

When we consider first loading only, the results of an isotropic compression test can usually be expressed approximately by the equation:

$$\epsilon_v = \epsilon_1 + \epsilon_2 + \epsilon_3 = 3A(\sigma_m/M)^m \tag{14}$$

where M is a constant "deformation modulus" for the soil in question, whereas A and m may depend upon the water content w or the void ratio e.

ANISOTROPIC COMPRESSION

The mean normal stress σ_m may still be assumed to produce a volume decrease as expressed by (14). The deviator stress $\sigma_1 - \sigma_3 = \sigma_1 - \sigma_2$ has two effects. First, it produces shear deformations $\epsilon_1 - \epsilon_3 = \epsilon_1 - \epsilon_2$, and second it produces a volume dilatation, which is usually proportional to the shear deformation. In an oedometer test ($\epsilon_2 = \epsilon_3 = 0$) the stress ratio $\sigma_3/\sigma_1 = K_0$ is found to be approximately independent of σ_1 (or σ_m). This means that the parenthesis in (14) must also be a factor for the shear deformation and dilatation.

If we define a "stress angle" v by the equation:

$$\sin v = (\sigma_1 - \sigma_3)/(\sigma_1 + \sigma_3 + 2c\cot\phi) \tag{15}$$

we have, according to Coulomb's failure criterion, $v_f = \phi$.

We can then combine (12b), (14), and (15) as follows:

$$\epsilon_1 - \epsilon_3 = 3B(\sigma_m/M)^m \frac{1}{(k\sin\phi/\sin v)^2 - 1}. \tag{16}$$

Further we get:

$$\epsilon_v = \epsilon_1 + 2\epsilon_3 = 3A(\sigma_m/M)^m - \beta(\epsilon_1 - \epsilon_3). \tag{17}$$

It is then easy to derive:

$$\epsilon_1 = (\sigma_m/M)^m \cdot \left[A + \frac{B(2-\beta)}{(k\sin\phi/\sin v)^2 - 1}\right] \tag{18}$$

and

$$\epsilon_2 = \epsilon_3 = (\sigma_m/M)^m \cdot \left[A - \frac{B(1+\beta)}{(k\sin\phi/\sin v)^2 - 1}\right]. \tag{19}$$

The dimensionless quantities m, β, k, A and B, as well as the deformation modulus M must, of course, be determined experimentally. They are, with the exception of M, functions of w or e.

Similar relationships may be developed using, instead of (12b), one of the other equations for δ.

THE OEDOMETER TEST

The condition $\epsilon_2 = \epsilon_3 = 0$ gives the following formula for the stress angle v_0 in the oedometer test:

$$\sin v_0 = \frac{k\sin\phi}{\sqrt{\frac{B}{A}(1+\beta) + 1}}. \tag{20}$$

The coefficient $K_0 = \sigma_3/\sigma_1$ can then be found from (15) which, in the case of $c = 0$, gives:

$$K_0 = (1 - \sin v_0)/(1 + \sin v_0) \tag{21a}$$

and

$$\sin v_0 = (1 - K_0)/(1 + K_0). \tag{21b}$$

By means of (20) it is possible to write equations (18)–(19) as follows:

$$\epsilon_1 = A(\sigma_m/M)^m \cdot \left[1 + \eta\frac{(k\sin\phi/\sin v_0)^2 - 1}{(k\sin\phi/\sin v)^2 - 1}\right] \tag{18a}$$

and

$$\epsilon_2 = \epsilon_3 = A(\sigma_m/M)^m \cdot \left[1 - \frac{(k\sin\phi/\sin v_0)^2 - 1}{(k\sin\phi/\sin v)^2 - 1}\right]. \tag{19a}$$

If only the deformations during the compression test are wanted, we must deduct the (isotropic) deformations due to the consolidation pressure σ_c:

$$\epsilon_1 = \left(\frac{\sigma_m}{M}\right)^m \cdot \left[A + \frac{B(2-\beta)}{(k \sin \phi/\sin v)^2 - 1}\right] - \left(\frac{\sigma_c}{M}\right)^m \cdot A \quad (22)$$

$$\epsilon_2 = \epsilon_3 = \left(\frac{\sigma_m}{M}\right)^m \cdot \left[A - \frac{B(1+\beta)}{(k \sin \phi/\sin v)^2 - 1}\right] - \left(\frac{\sigma_c}{M}\right)^m \cdot A. \quad (23)$$

Equations (18a)–(19a) can, of course, be changed similarly.

CONCLUSION

Comparatively simple equations have been developed, which should be suitable for describing, in a purely empirical way, the actual relationships between stresses and strains in soils, corresponding to the usual laboratory tests.

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