Principal Stress Ratios and Their Infuence on the Compressibility of Soils

Les Rapports des contraintes principales et leur influence sur la compressibilité des sols

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SUMMARY

First, some simple analyses are carried out to obtain numerical values for the principal stress ratios as a function of mobilized shear stress, both on effective and total stress bases. Secondly, based on the theory of elasticity, an investigation is made of the influence of the principal stress ratio on shear strain and deformation modulus of elastic materials. This analysis appears to throw some new light on the determination of initial deformation of clays (under undrained conditions). Finally, the influence of the principal stress ratio on the compressibility characteristics of sand is described from the experience gained from some 50 laboratory tests. For principal stress ratios higher than $K' = 0.55$ the influence resembles elastic behaviour, but for decreasing K' below 0.5 the influence of shear stress appears to play a much more predominant role in sand than in elastic materials.

SOMMAIRE

Premièrement, quelques simples analyses ont été effectuées afin de trouver des valeurs numériques pour les rapports des contraintes principales en fonction de la contrainte tangentielle mobilisée, sur la base des contraintes effectives et des contraintes totales. Deuxièmement, fondée sur la théorie de l'élasticité, une étude de l'influence des rapports des contraintes principales sur la déformation de cisaillement et le module de déformation pour les matériaux élastiques a été effectuée. Cette analyse semble apporter une lumière nouvelle sur la détermination de la déformation initiale de l'argile (dans l'état non drainé). Finalement, on décrit l'influence des rapports des contraintes principales sur les caractéristiques de compressibilité pour le sable, en partant des résultats d'environ 50 essais expérimentaux. Quand il s'agit des rapports des contraintes principales plus grands que $K' = 0{,}55$, il semble que l'influence corresponde au comportement élastique, tandis que pour K' inférieur à 0,5, il semble que l'influence de la contrainte tangentielle joue un rôle plus dominant pour le sable que pour les matériaux élastiques.

PRINCIPAL STRESS RATIO

AS AN INTRODUCTION a study will be made of the actual range of numerical values of the principal stress ratios needed in the laboratory for triaxial compressibility tests on soils. Fig. 1 shows a cylindrical soil specimen subjected to effective principal stresses σ'_1 and $\sigma'_2 = \sigma'_3$. Herein the ratio between the minor and major effective principal stress is denoted by K', hence

$$K' = \sigma'_3/\sigma'_1. \tag{1}$$

When the ratio is required on total stress basis, it is denoted by K.

The ratio K' (or K) can, in general, be expressed in terms of a nominal state of equilibrium, defined by the shear strength and a factor of safety. Thus the shear stress τ on a plane rising at an angle α will be expressed as

$$\tau = \tau_f/F \tag{2}$$

where $1/F$ represents the degree of mobilization of the shear strength τ_f. For given shear-strength properties the principal stress ratio is a function of the shear-strength parameters and the degree of mobilization. Based on this principle a comprehensive study of K' and K has been undertaken at our institute, from which two cases will be considered below. In both the shear strength is expressed by one parameter.

In the first case the shear strength is given by the expression ($c = 0$)

$$\tau_f = \sigma' \tan \phi \tag{3}$$

where σ' is the effective normal stress on the shear plane considered. Combining Eqs (2) and (3) it is readily seen that the minimum value of F corresponds to the requirement

$$\tau/\sigma' = \tan \phi/F = \text{maximum} \tag{4}$$

when ϕ is assumed constant.

It is also well known that Eq (4) leads to critical shear planes given by (for $K' < 1$)

$$\alpha_n = \pm (\pi/4 + \phi_n/2) \tag{5}$$

where ϕ_n is defined by

$$\tan \phi_n = \tan \phi/F. \tag{6}$$

Moreover, the value of K' is equal to

$$K' = \tan^2(\pi/4 - \phi_n/2). \tag{7}$$

This value of K' is plotted *versus* ϕ in Fig. 1, for three different values of F, 1.0, 1.3, and 1.6; for comparison Jaky's value of

$$K'_0 = 1 - \sin \phi \tag{7a}$$

is included in the figure.

Two examples illustrate the application of the graph. A large horizontal sand layer with $\phi = 35°$ is at rest with a nominal safety factor of $F_0 = 1.6$ corresponding to $K'_0 = 0.42$, according to point S in Fig. 1. This sand is now loaded over a very large area (to avoid rotation of the principal stresses), and the shear stresses are consequently increased.

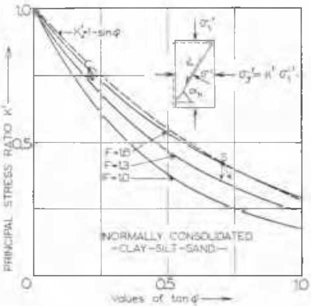

FIG. 1. Effective principal stress ratio K' as function of mobilized shear stress $\tan \phi_n = \tan \phi / F$ for $c = 0$.

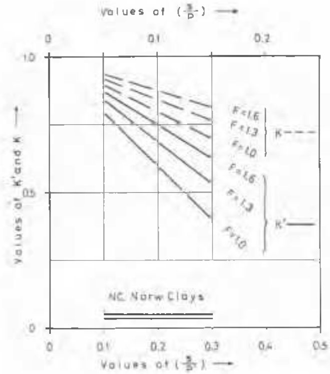

FIG. 2. *In-situ* values of K and K' as functions of shear strength ratios (s/p and s/p') and safety factor F.

Let the new condition correspond to $F_1 = 1.3$. Assuming no increase in ϕ' due to consolidation, the new state of equilibrium yields $K'_1 = 0.36$ as shown by the point of the vertical arrow. The average K' to be used in the laboratory for compressibility determination with constant effective stress ratio could therefore be chosen as the mean $K' = 0.39$. If the consolidation of the sand leads to an increased ϕ', then the situation is as illustrated by the inclined arrow from point S.

If we were dealing with a very soft clay layer where $\tan \phi = 0.2$, the same at rest ($F_0 = 1.6$) and loaded conditions ($F_1 = 1.3$) would give $K'_0 = 0.77$ and $K'_1 = 0.73$, giving an average $K' = 0.75$ if there was no increase in $\tan \phi$ due to consolidation. If an increase in ϕ took place, the inclined arrow from point C illustrates the situation.

For the second case, the shear strength is given in terms of total stresses; it is assumed constant and isotropic at each depth but may vary from one depth to another. Hence, for a given depth

$$\tau_f = s = \text{constant}. \tag{8}$$

In such cases, it is well known that for a principal stress element the maximum shear stress at $\alpha_n = \pm 45°$ equals

$$\tau = \tfrac{1}{2}(\sigma_1 - \sigma_3) = \tfrac{1}{2}(\sigma'_1 - \sigma'_3) = s/F. \tag{9}$$

Hence

$$\sigma_3 = \sigma_1 - 2(s/F). \tag{10}$$

For a semi-infinite body let $\sigma_1 = p$ designate the total overburden pressure. Then the total principal stress ratio becomes

$$K = 1 - (2s/Fp). \tag{11}$$

If $\sigma'_1 = p'$ denotes the effective overburden, then the effective principal stress ratio becomes, according to (9),

$$K' = 1 - (2s/Fp') \tag{11a}$$

The values of K and K' are plotted in Fig. 2 *versus* (s/p) and (s/p') for different values of F.

To illustrate the application of Fig. 2, let us assume that the shear strength on total stress basis (undrained) is isotropic at each depth, but varies linearly with depth. For the given soil profile then, s/p and s/p' are constants.

For normally consolidated Norwegian marine clays the ratio s/p' (determined by vane tests) often ranges between 0.1 and 0.3, according to Bjerrum (1954). Hence s/p is of the order of magnitude of 0.05 to 0.15 for $\gamma \sim 2\gamma'$. For a given at rest condition corresponding to $F_0 = 1.6$ one would find for $s/p' = 0.1$ that $K' = 0.87$ and $K = 0.94$, while for $s/p' = 0.3$ the corresponding values are $K' = 0.62$ and $K = 0.81$. These high values are seen to correspond to low values of $\tan \phi$ in Fig. 1 as would be expected. A diagram correlating K', $\tan \phi$, F, and s/p' considered both as isotropic and anisotropic quantities has been established, but is considered to be beyond the scope of this article.

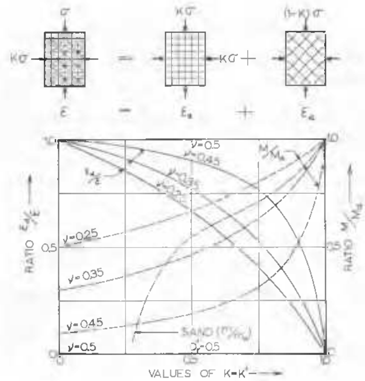

FIG. 3. Influence of principal stress ratio on shear strain and deformation modulus for elastic materials.

COMPRESSIBILITY OF ELASTIC MATERIALS UNDER PRINCIPAL STRESS CONDITIONS

Fig. 3 shows a cylindrical specimen of an elastic body subjected to a major principal stress σ and a minor and intermediate principal stress $K\sigma$ (total stresses equals effective stress, hence $K = K'$).

TABLE I. UNDRAINED, INITIAL DEFORMATION MODULI OF CLAY (LADEMOIN, TRONDHEIM)

Depth (meters)	Unconfined E_i(tons/sq.m.)	$K = 0.8$ M_i(tons/sq.m.)	Ratio M_i/E_i	Routine data
4.2–5.0	145	430	3.0	w_P = 18–19 per cent
10.2–11.0	255	920	3.6	w_L = 31–36 per cent
14.2–15.0	295	1250	4.2	w = 32–33 per cent
18.2–19.0	475	>1250	<2.6	($S_t \sim 5$–15)
	AVERAGE		3.3	

From the theory of elasticity the vertical strain ϵ can be expressed as follows:

$$\epsilon = (\sigma/E - \nu/E)(K\sigma + K\sigma)$$
$$= (1 - 2\nu K)\sigma/E. \qquad (12)$$

This strain can be divided into two components (Fig. 3), one due to all-round pressure ϵ_a, and one due to deviator stress ϵ_d, where the latter equals

$$\epsilon_d = (1 - K)(\sigma/E). \qquad (13)$$

Hence, the ratio between the strain component caused by deviator stress (shear stress) and the total strain becomes

$$\epsilon_d/\epsilon = (1 - K)/(1 - 2\nu K). \qquad (14)$$

This ratio is plotted in Fig. 3 for various values of ν and K, from which it is seen that the strain due to deviator stress is the most significant component for small K' values (sand $K' = 0.3$ to 0.5) and also for larger K' values (clay $K' = 0.5$ to 0.8) when $\nu \geqq 0.4$.

Since the shear stresses appear to be responsible for a major part of the vertical compression further investigation of the influence of K on the compressibility is indicated.

If K is kept constant during a compressibility test on an elastic specimen the vertical stress-vertical strain curve is linear, and the tangent modulus $M = d\sigma/d\epsilon$ for this curve is seen to be constant, Eq (12), and equals

$$M = E/(1 - 2\nu K). \qquad (15)$$

If K varies from one test to another (but is kept constant for each test) a reference value is needed for comparison, and let the value M_a for all-round pressure ($K = 1.0$) be chosen, i.e.,

$$M_a = E/(1 - 2\nu). \qquad (16)$$

Hence,

$$M/M_a = (1 - 2\nu)/(1 - 2\nu K). \qquad (17)$$

This ratio is plotted in Fig. 3 *versus* $K = K'$ for different ν, from which it is seen that M/M_a increases for increasing K.

This consideration can be helpful in clarifying an important issue in connection with the modulus of initial deformation of clays (initial settlement under undrained conditions). From experience, the Norwegian Geotechnical Institute has concluded that for estimation of initial settlements of clay the modulus derived from the stress-strain curve of the unconfined compression has to be multiplied by 3 or 4 to correspond with observations.

This experience is now readily explained both by theory and by laboratory experiments. The slope of the unconfined stress-strain curve, where $K = 0$, yields $M = E_i$. In nature, however, the actual K (total stresses, undrained conditions) is of the average order of magnitude of 0.75 to 0.85 for NC clays (Fig. 2), hence from Eq (15)

$$M_i = E_i/(1 - 2\nu K). \qquad (15a)$$

For example, if $K = 0.75$–0.85 and $\nu = 0.40$ to 0.45, $M_i = (2.5$–$4.2)E_i$ which appears to cover the experiences of NGI. It should therefore appear logical to carry out K tests (0.75 to 0.85) for the determination of M_i to be used for estimating initial settlements, since ν is in reality unknown.

For further illustration the results of some laboratory undrained K tests are given in Table I for one clay profile. The results are in good agreement with the above findings, inasmuch as the average test value of M_i/E_i is equal to 3.3.

TESTS ON SAND

In a previous paper (Janbu, 1963), the author suggested using the tangent modulus M of the $\sigma' - \epsilon$ curve as a measure of the compressibility of soils, hence $M = d\sigma'/d\epsilon$. For the types of sand tested for constant K' the tangent modulus varied with stress according to the formula

$$M = m\sigma_a(\sigma'/\sigma_a)^{1-a} \qquad (18)$$

where σ_a = reference stress $\sim$ 1 atmosphere = 1 kg/sq.cm., σ' = effective vertical stress, m = modulus number (pure number), and a = stress exponent (pure number). Equation (18) is closely related to the formula proposed by Ohde (1939).

For a vertical stress increase from σ'_0 to $\sigma' = \sigma'_0 + \Delta\sigma'$ one obtains the vertical strain ϵ by integration of $d\sigma'/M$ between these stress limits. Hence

$$\epsilon = (1/ma)[(\sigma'/\sigma_a)^a - (\sigma'_0/\sigma_a)^a] \qquad (19)$$

from which it is seen that the reference stress σ_a is introduced solely for the purpose of obtaining a dimensionally correct equation.

The main object of the experimental analysis (Janbu, 1963) was to obtain typical numerical values for m and a for various types of soils. The main question dealt with below is how a and m vary when K' varies from test to test, while K' is kept constant for each test. Only the experimental results of two types of sand will be given here. Some characteristic properties of the sands are given in Table II. For both sands grains larger than 1 mm in diameter were removed by sieving. The tests were drained and were carried out in triaxial cells of the NGI type. Initially, some tests were run for the purpose of obtaining

TABLE II. CHARACTERISTIC PROPERTIES OF SAND A AND SAND B

Index properties	A sand	B sand
Specific gravity γ_s	2.73 g/cu.cm.	2.75 g/cu.cm.
Maximum porosity n_{max}	46.2 per cent	43.5 per cent
Minimum porosity n_{min}	34.0 per cent	30.0 per cent
Grain size D_{10}	0.13 mm	0.11 mm
Grain size D_{60}	0.42 mm	0.42 mm
Grain size D_{100}	1.00 mm	1.00 mm
Angle of friction ϕ for n = 35.5 per cent	41.8°	
Angle of friction ϕ for n = 31.9 per cent		43.50

a rate of loading sufficiently slow to secure complete drainage.

Typical $\sigma' - \epsilon$ curves and $M - \sigma'$ curves are shown in Fig. 4 for two K' values for B sand. Before a compressibility test with constant K' was started, the sample had to be subjected to a small initial all-round pressure ($K' = 1.0$), so that the actual point of zero strain is unknown. For the study of the tangent modulus however, this has no influence except perhaps very near zero stress.

When a test curve ($\sigma' - \epsilon$), or preferably ($M - \sigma'$), is obtained, both parameters, a and m, can be computed by reading off M and σ' at any two points. The obtained values of m and a are generally almost independent of which points are selected, but in practice the use of average values obtained from a few sets of points selected within the actual stress range is suggested. The values of m and a for the curves in Fig. 4 are given in the figure.

Fig. 5 shows by a limited number of average curves how the stress exponent (a) and the modulus number (m) vary with porosity for different principal stress ratios K' for both types of sand. The figure illustrates clearly how the modulus number increases with increasing density and with increasing K'. A similar, but not so marked, variation is observed for the stress exponent a; and the per cent variations are moderate.

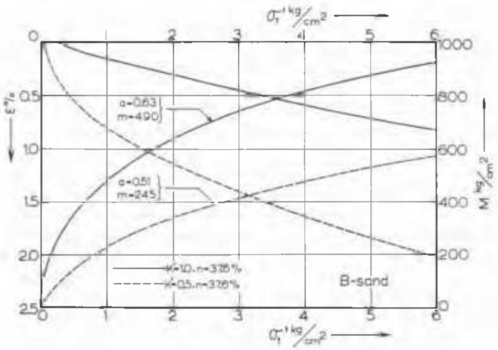

FIG. 4. Typical variation of strain (ϵ) and deformation modulus (M) *vs.* stress ($\sigma'_1 = \sigma'$).

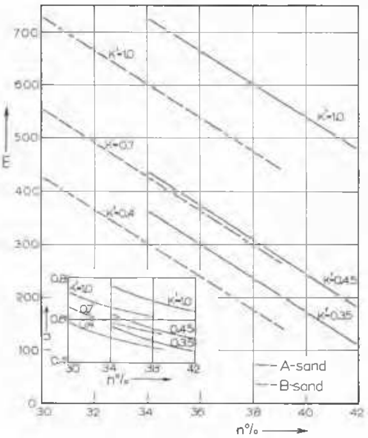

FIG. 5. Diagram showing stress exponent (a) and modulus number (m) as a function of porosity (n) for principal stress ratios $K' = 0.35 - 1.0$.

In order to demonstrate how compressibility depends on the principal stress ratio for both sands, it is advantageous to use the relative density D'_r on an n basis,

$$D'_r = (n_{max} - n)/(n_{max} - n_{min}) \qquad (20)$$

instead of the porosity itself.

For the two types of sand the ratio between the modulus number m and the reference value m_a for $K' = 1.0$ is plotted in Fig. 6 *versus* effective principal stress ratio K' for three different relative densities, $D'_r = 0.25$, 0.50, and 0.75. This figure demonstrates that the variation of m/m_a versus K' is similar to the elastic behaviour for $K' >$ about 0.55 (see also Fig. 3). However, for $K' < 0.5$ the ratio

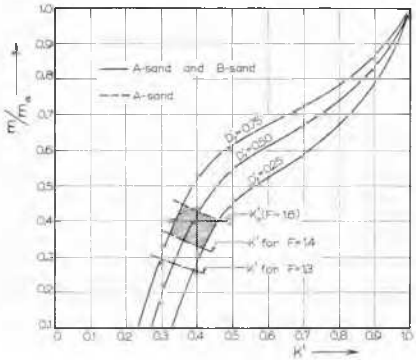

FIG. 6. Ratio m/m_a as a function of K' for relative densities $D'_r = 0.25$ to 0.75.

drops much more rapidly than for elastic materials (theoretically $m/m_a = 0$ for $K' = \tan^2(45° - \phi/2)$).

In order to point out the applicable range of the curves for normally consolidated sands the points corresponding to $F \sim 1.6$ (K'_0 − conditions), $F = 1.4$ and $F = 1.3$ have been obtained for A sand for each of the three relative densities. This has been achieved from an estimated $\phi - n$ relationship for the sand, based on one determined ϕ value (Table II). K'_0 is calculated from formula (7a). The corresponding curves for B sand are not shown in the figure, but seem to be located somewhat lower than those for A sand.

In most cases of normally consolidated sands the K' condition for a specific case of loading would be located between the unloaded condition (K'_0) and the loaded condition corresponding to an average safety factor with respect to shear failure, say $F = 1.3$ to 1.4. For instance for A sand with an average $F = 1.4$ the applicable range of m/m_a would be the shaded area in Fig. 6.

Finally, a consideration will be given to the accuracy of using K' = constant for compressibility determinations of sand for practical application to settlement analysis. For this purpose consider a normally consolidated, medium dense sand with $D'_r = 0.5$, and let m/m_a be a relative measure of the compressibility since the a variation is fairly small. Before loading $K'_0 = 0.41$, (corresponding to $F \sim$

1.6) giving $m/m_a = 0.43$ (Fig. 6). After loading $F = 1.4$ corresponding to $K'_1 = 0.36$, and $m/m_a = 0.35$. Using an average $K' = 0.385$ = constant in a triaxial compression test one would have found $m/m_a = 0.39$ which is only $\pm$ 10 per cent different from the limiting ratios for the K'_0 and K'_1 ($F = 1.4$) conditions.

The actual compressibility is believed to be located between the two limits for K'_0 and K'_1, so that the error of using a constant, average K' should therefore theoretically be less than the limiting errors ± 10 per cent. The theoretical errors of constant K' tests are therefore in most cases negligible in comparison to the practical difficulties of obtaining reliable information about the *in-situ* relative densities and shear-strength properties of sand deposits.

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