Stochastic Processes in the Grain Skeleton of Soils

Phénomènes stocastiques dans la structure granulaire des sols

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SUMMARY

It is recognized in soil mechanics that the grain structure of a soil does not react instantaneously to changes in stresses. However, little attention has been given to this fact in theoretical developments. Time lags are attributed generally to interference between the fluid and solid phases. This paper is concerned with transient phenomena in the grain skeleton during one-dimensional compression. Movements of particles are considered the consequence of erratic impulses transferred to each grain by the neighbouring ones, coupled with constant action. Furthermore, particles are assumed to move in a viscous medium. The process is analysed by stochastic methods, and it is concluded that the process is governed by the diffusion equation (Fokker-Planck). A solution is offered for specified boundary conditions, and results discussed. The proposed stochastic theory may find application in the study of transient phenomena, like consolidation of soils.

SOMMAIRE

Il est reconnu dans la mécanique des sols que la structure granulaire d'un sol ne réagit pas instantanément aux variations de contraintes. Ces retards sont généralement attribués à l'interférence entre les phases fluide et solide. Toutefois, peu d'attention a été, jusqu'à maintenant, accordée à ce fait dans les travaux théoriques. Cette communication traite des phénomènes transitoires dans une structure granulaire, sous l'action d'une compression unidirectionnelle. Les mouvements de particules sont considérés comme une conséquence des impulsions erratiques transmises à chaque grain par les voisins, plus une action constante. En outre, on suppose que les particules se meuvent en milieu visqueux. Le phénomène est analysé par un calcul de probabilités et il en est conclu que ce phénomène est régi par l'équation de diffusion (Fokker-Planck). Une solution est offerte pour les conditions limites spécifiées et les résultats en sont discutés. Cette théorie stocastique proposée peut trouver son application dans l'étude des phénomènes transitoires tels que la consolidation des sols.

DEFINITIONS AND GENERAL RELATIONSHIPS

LET US CALL V_T the total volume of a soil element which, due to variations in the state of stress, undergoes a volumetric change ΔV. In the direction of reference axes x, y, and z, deformations of the volume element are λ_x, λ_y and λ_z, respectively. Corresponding strains will be defined as follows:

$$\epsilon_v = \Delta V / V_T \tag{1}$$

$$\epsilon_x = \lambda_x / l_x,\ \epsilon_y = \lambda_y / l_y,\ \epsilon_z = \lambda_z / l_z \tag{2}$$

l_x, l_y, and l_z being the initial dimensions of the soil element. Except for terms of higher order, such strains are related by the expression,

$$\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z \tag{3}$$

Assuming that volumetric changes ΔV are equal to variations in the volume of voids, ΔV_v, the correlation between strains and void ratios, e, is:

$$\epsilon_v = -\Delta e/(1+e). \tag{4}$$

As known, Δe is obtained by dividing ΔV_v by the total volume of grains V_s. The ratio V_s/V_T will be called concentration of the solid matter, q. This soil characteristic in terms of void ratio is equal to

$$q = 1/(1+e). \tag{5}$$

From (5), changes in concentration are:

$$\Delta q = -\Delta e/(1+e)^2 = \epsilon_v/(1+e) \tag{6}$$

On the other hand, the grain concentration, n_v, is defined as the number of particles per unit of volume. Assuming that the average volume of individual grains v_s is known; then,

$$q = n_v v_s \tag{7}$$

(For determination of v_s see Marsal, 1963).

IDEALIZED SOIL STRUCTURE

Soils are two-phase or three-phase systems. A grain skeleton encloses voids which are filled with a fluid. This fluid may be composed of gases and a liquid. In further considerations of this paper, the actions of neither gas nor liquid phases will be taken into account. The above-mentioned skeleton constitutes a discrete body composed of grains having a great variety of shapes and sizes. Also the mineral composition of these particles may change within wide ranges. Due to this fundamental character of soils, one is forced to work with statistical values of their physical properties. It is also required that the number of particles per unit volume be large enough that asymptotic formulae derived from a stochastic analysis become acceptable.

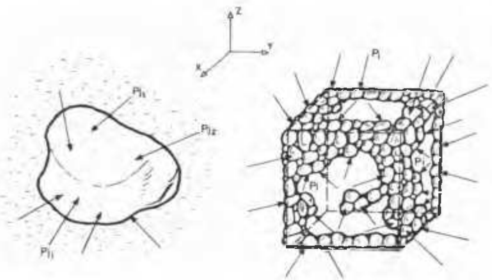
Among the mechanical soil properties of statistical nature, the void ratio, concentration of solids, and coefficient of compressibility are the essential ones in the calculations that follow. All of them depend on the shape and size of grains as well as on the particle arrangement in the skeleton. Substantial breakage of grains may occur in some cases and affect these soil properties.

From the stochastic analysis discussed herein, two other properties of the solid phase will evolve: the coefficient of diffusion and the drift. Both are statistical parameters which take into account grain displacements induced in the soil skeleton by changes in stresses.

In this paper, the solid phase is considered isotropic. The analysis will be confined to the one-dimensional case.

STATIC EQUILIBRIUM OF A SOIL

Let P_{ji} be the intergranular forces acting on a particle j and assume that this grain is in equilibrium (Fig. 1A). Also consider that x, y, z is the Cartesian frame of reference. According to the principles of statics, the sum of all components and moments of forces P_{ji} must be zero. Now detach a cubic element from a granular mass which is in equilibrium. Intergranular forces cancel out inside the element, but along its outer faces there are sets of vectors P_i as shown in Fig. 1B. Magnitudes and directions of P_i vary from contact to contact in a manner that would be extremely difficult to predict.



A B

FIG. 1. A, intergranular forces acting on a particle; B, cubic element and contact forces.

Let us call S_x, T_{xy}, T_{xz} the components of forces P_i acting on the element face normal to axis x (see Fig. 2A). Accordingly, S_y, T_{yx}, T_{yz} and S_z, T_{zx}, T_{zy}, are the components associated to faces (x, z) and (x, y), respectively. The intergranular stresses $\bar{\sigma}_x$, τ_{xy}, τ_{xz}, etc. will be defined as the arithmetic sums of force components per unit of total area. Therefore,

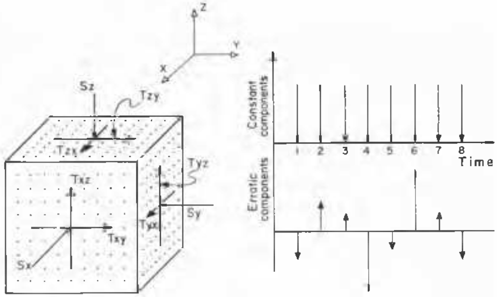
$$\bar{\sigma}_x = \sum_{i=1}^{N_s} S_{x\,i}, \quad \tau_{xy} = \sum_{i=1}^{N_s} T_{xy\,i}, \quad \tau_{xz} = \sum_{i=1}^{N_s} T_{xz\,i}, \text{ etc.} \tag{8}$$

where N_s is the average number of contacts per unit of total area. For determination of N_s and related matters see Marsal (1963).

Although direct information on contact forces P_i is not available, one can imagine that the magnitude of their components varies so irregularly, that they may be considered as random events. In addition, if such events are independent from each other, the effective stresses $\bar{\sigma}$ and τ are governed by distribution functions that converge to the normal type (central limit theorem, Cramér, 1946). Consequently, it is necessary to deal with average values of $\bar{\sigma}$ and τ as well as their respective variance in order to make statistical computations. It would be difficult to predict accurately the error which is involved when stresses are measured only by means of average values instead of using the distribution functions furnished by the theory of probabilities. However, for the problem dealt with in this paper, it will be assumed that the said error is not significant.

TRANSIENT PHENOMENA

Consider that a soil layer, infinite in extent and $2H$ thick, which after being in equilibrium under a vertical stress $\bar{\sigma}_1$ is subjected to a load increment $\Delta\bar{\sigma}$. Let us concentrate our attention on a single particle of the soil mass away from the boundaries. Prior to the application of $\Delta\bar{\sigma}$, contact forces P_i transmitted by adjacent grains to this particle satisfy static conditions of equilibrium. In order to investigate what happens after applying $\Delta\bar{\sigma}$ to the soil layer, we will make the following hypotheses: (*a*) movements of the particle are produced by unbalanced contact forces acting on it; (*b*) surrounding grains transfer contact forces and also operate as a homogeneous medium restricting the particle displacement, its rate being constant. Furthermore, actions on the particle are split in two parts: a constant intergranular force, and erratic contact forces, the resultant vector changing in magnitude and direction with time (Fig. 2B).



A B

FIG. 2. A, idealized cubic element and average components of intergranular forces. B, components of intergranular forces in the z direction.

The above-described model is similar to that adopted for the Brownian movement. Instead of the molecular activity of the liquid we have erratic forces transferred by the neighbouring grains. In both cases, the particle moves inside a Newtonian body. The axis normal to the surface of the layer is z. Since we are dealing with a one-dimensional case, components of the particle displacements in the x and y directions cancel out and are of no consequence in further analysis of the process. Due to the action of intergranular forces combined with the restriction to displacement in the granular mass, the particle walks back and forth by steps and is drifted downward along the z-axis. At this stage we make use of the stochastic method of analysis originally devised by Markov (1912), and further extended by Kolmogorov (1936), Feller (1957), Chandrasekhar (1943), and others. This method requires that the magnitude and direction of each displacement of the particle be independent of the preceding ones and that the probability of each step along the z line adjusts itself to a distribution function selected *a priori*. In our case, due to the character of the contact forces being exerted by neighbouring grains on the particle, the independence of events seems most likely. Equal probabilities for the forward and backward direction will be adopted in respect to the distribution law.

FREE-MOVING PARTICLES

According to Chandrasekhar (1943), the probability that a particle finds itself between z and $z + \Delta z$ at time t, is given by,

$$W(z, t) = \frac{1}{2(\pi D t)^{\frac{1}{2}}} \exp[-z^2/4Dt]; \tag{9}$$

where,

$$D = \tfrac{1}{2} n l^2 \tag{10}$$

n being the number of displacements per unit time and l^2 the mean square length of the steps performed by the grain. D will be called the coefficient of diffusion of the solid phase. It is of interest to investigate the amount of particle motion restricted by the presence of a reflecting barrier or absorbing walls (Chandrasekhar, 1943). In the first case, the probability that the grain is in the interval $(z, z + \Delta z)$ at time t, results in

$$W(z, t; z_1) = \frac{1}{2(\pi D t)^{\frac{1}{2}}} \{\exp(-z^2/4Dt) + \exp[-(2z_1 - z)^2/4Dt]\} \tag{11}$$

z_1 denoting the location of the reflecting barrier. From this expression, it is found that

$$(\partial W/\partial z)_{z=z_1} = 0. \tag{12}$$

When a perfect absorbing wall exists at z_2, the probability is given by the following equation:

$$W(z, t; z_2) = \frac{1}{2(\pi D t)^{\frac{1}{2}}} \{\exp(-z^2/4Dt) - \exp[-(2z_2 - z)^2/4Dt]\} \tag{13}$$

Hence,

$$W(z_2, t; z_2) = 0. \tag{14}$$

It is easy to demonstrate that expression (9) satisfies the well-known equation of diffusion

$$D(\partial^2 W/\partial z^2) = \partial W/\partial t \tag{15}$$

and that (12) and (14) correspond to boundary conditions. Fig. 3A shows the probability curves (Eq 9) for several times.

PARTICLE DRIFT AT CONSTANT SPEED

If the particle is not only acted upon by erratic impulses, but also by forces which drift it at a constant speed c, the probability function (Feller, 1957) results

$$W(z, t) = \frac{1}{2(\pi D t)^{\frac{1}{2}}} \exp[-(z - 2ct)^2/4Dt]. \tag{16}$$

This is the fundamental solution of the Fokker-Plank equation

$$D(\partial^2 W/\partial z^2) - 2c(\partial W/\partial z) = \partial W/\partial t. \tag{17}$$

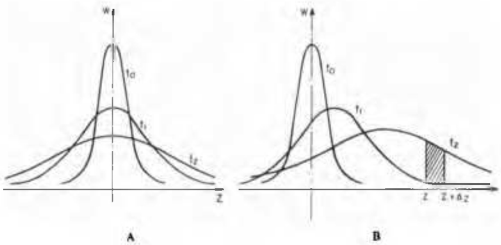


FIG. 3. A, probability curves as time elapses ($t_0 < t_1 < t_2$), free-moving particle. B, probability curves as time elapses, particle drift at constant speed.

Curves representing the probability function (16) are drawn in Fig. 3B. The solution of Eq (17), satisfying initial and boundary conditions of our problem, is the answer being sought.

PROBABILITIES VERSUS GRAIN CONCENTRATIONS

The probability of finding a single particle in the interval $(z, z + \Delta z)$ at time t has been introduced. However, one can imagine that function W also represents the fraction of a large number of particles located between z and $z + \Delta z$ at time t, the initial condition for all the grains being that $z = 0$ at $t = 0$. This interpretation of W permits the calculation of the physical meaning of this function. In fact, $W(z, t_1)\, \Delta Z$ gives the number of particles that are located between z and $z + \Delta z$ at time t_1 (see Fig. 3B). When Δt has elapsed, $t_2 = t_1 + \Delta t$. Then, the number of particles in the interval $(z, z + \Delta z)$ is equal to $W(z, t_2)\Delta Z$. The difference, $W(z, t_2)\, \Delta Z - W(z, t_1)\, \Delta Z$ shown in Fig. 3B by the shaded area between curves for t_1 and t_2, is the change in grain concentration Δn_v during the interval Δt. Therefore, function W identifies with n_v, and, according to Eq (7), it is proportional to q.

SOLUTION FOR THE ONE-DIMENSIONAL CASE

It follows that from the previous paragraph, the differential equations of the process involving a large number of particles is,

$$D\frac{\partial^2 q}{\partial z^2} - 2c\frac{\partial q}{\partial z} = \frac{\partial q}{\partial t}. \tag{18}$$

For an infinite layer $2H$ thick, assumed to be isotropic and homogeneous, boundary conditions are:

$$q(z, 0) = q_1;\ q(0, t) = q_2;\ q(2H, t) = q_2 \tag{19}$$

where q_1 and q_2 are the initial and final concentrations of the solid phase.

Eq (16) is a solution of the diffusion process with drift, but it does not satisfy conditions (19). In order to integrate Eq (18) let us homogenize the boundary conditions using

$$Q(z, t) = q(z, t) - q_2. \tag{20}$$

Then, transform variable Q by means of Eq (21), so that the differential equation becomes the known diffusion law (Eq 15).

$$Q(z, t) = U(z, t)\exp[cz/D - c^2t/D]. \tag{21}$$

Finally, the general solution is assumed to be

$$U(z, t) = \sum_{n=0}^{\infty} a_n \phi_n(z)\phi_n(t). \tag{22}$$

In which a_n are arbitrary constants. Each particular solution is of the form

$$U_n(z, t) = (a_n/c_n)\ (A_n \sin K_n z + B_n \cos K_n z) \exp(-K_n^2 Dt) \tag{23}$$

Applying boundary conditions to Eq (23), its coefficients can be evaluated, and we obtain finally

$$U(z, t) = (q_1 - q_2)\sum_{n=0}^{\infty} \frac{n\pi/2[1 - (-1)^n\exp(-2Hc/D)]}{(Hc/D)^2 + (n\pi/2)^2} \times \exp\left(-\frac{n^2\pi^2 Dt}{4H^2}\right) \sin\frac{n\pi z}{2H}. \tag{24}$$

Taking into account previous transformations, the general solution of (18) becomes:

$$q(z, t) = q_2 + (q_1 - q_2) \sum_{n=0}^{\infty} \frac{n\pi/2[1-(-1)^n \exp(-2Hc/D)]}{(Hc/D)^2 + (n\pi/2)^2} \times \exp\left[\left(-\frac{n^2\pi^2 Dt}{4H^2} - \frac{c^2 t}{D}\right)\right] \exp\left(\frac{cz}{D}\right) \sin\frac{n\pi z}{2H}. \quad (25)$$

We are also interested in average values of concentrations q for the layer $2H$ thick, as time elapses. Through integration of (25), the following equation is found:

$$q_a(t) = q_2 + (q_1 - q_2) \sum_{n=0}^{\infty} \frac{4n^2\pi^2\left[1 - (-1)^n Ch\frac{2Hc}{D}\right]}{\left[\left(\frac{2Hc}{D}\right)^2 + (n\pi)^2\right]^2} \times \exp\left[-\left(\frac{n^2\pi^2 D}{4H^2} + \frac{c^2}{D}\right)t\right]. \quad (26)$$

In the above expression Dt/H^2 and c^2t/D are dimensionless. Let us call them, diffusion and drift time factors, respectively.

$$T_1 = Dt/H^2,\ T_2 = c^2t/D \quad (27)$$

Hence

$$\sqrt{T_2/T_1} = Hc/D = J \quad (28)$$

J being independent of time.

TIME-COMPRESSION CURVES

Further application of the results so far obtained necessitates converting the concentration changes into strains. If we call $\Delta q = q_2 - q_1$ the total variation of q for a constant increment of intergranular pressure $\Delta\bar{\sigma}$, the change in concentration at time t is equal to:

$$\Delta q_a(t) = \Delta q - [q_2 - q_a(t)] = q_a(t) - q_1 \quad (29)$$

Substituting (26), affected by expressions (27) and (28) in the above equation, we have:

$$\Delta q_a = \Delta q\left\{1 - \sum_{n=0}^{\infty} \frac{4n^2\pi^2[1-(-1)^n Ch2J]}{[4J^2 + n^2\pi^2]^2} \times \exp\left[-\left(\frac{n^2\pi^2}{4} + J^2\right)T_1\right]\right\}. \quad (30)$$

Considering relationship (6) and that $-\Delta e = a_v\Delta\bar{\sigma}$ (a_v being the coefficient of compressibility), changes in concentration are expressed as follows:

$$\Delta q = m_v\Delta\bar{\sigma}/(1+e) = \epsilon_v/(1+e) \quad (31)$$

where m_v is the bulk modulus of compressibility and ϵ_v the volumetric strain. For the one-dimensional case, $\epsilon_v = \epsilon_z$; then,

$$\epsilon_z = m_v\Delta\bar{\sigma}\left\{1 - \sum_{n=0}^{\infty} \frac{4\pi^2 n^2[1-(-1)^n Ch2J]}{[4J^2 + n^2\pi^2]^2} \times \exp\left[-\left(\frac{n^2\pi^2}{4} + J^2\right)T_1\right]\right\}. \quad (32)$$

The product $m_v\Delta\bar{\sigma}$ is the final strain ϵ_f. Therefore, the strain ratio Λ is equal to

$$\Lambda = \epsilon_z/m_v\Delta\bar{\sigma}. \quad (33)$$

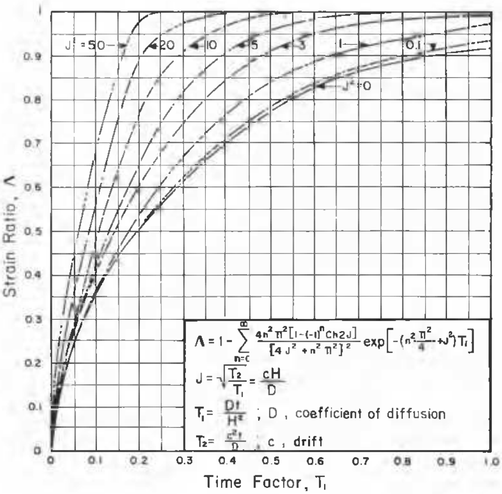


FIG. 4. Time-strain relationships for the grain skeleton.

For different values of J, function (33) was evaluated and results plotted in Fig. 4. When J decreases, Eq (33) tends to a limiting curve given by expression,

$$\Lambda = 1 - \sum_{n=0}^{\infty} \frac{8}{n^2\pi^2} \exp\left(-\frac{n^2\pi^2}{4}T_1\right). \quad (34)$$

As J increases, curves $\Lambda - T_1$ move toward the ordinate axis, the first portion being almost a straight line for $J^2 > 5$. Eq (33) is not a convenient function for further applications in soil mechanics. The series converges rather quickly when J is smaller than unity. However, convergence becomes slow for large values of parameter J, particularly when T_1 tends to zero. For $J^2 = 20$ and $T_1 = 0$ one hundred terms were required to find out that the series in Eq (32) has a limiting value of unity. This fact led to an investigation of approximate expressions for function (33). It seems that the following ones would be simple and adequate in most cases.

$$\Lambda = 1 - \frac{8}{[4J^2 + \pi^2]^2} \exp\left[-\left(\frac{\pi^2}{4} + J^2\right)T_1\right], \quad \text{for } J \leqslant 1 \quad (35)$$

$$\Lambda = 1 - (aT_1 + 1)\exp(-bT_1), \text{ for } J > 1. \quad (36)$$

Function (35) accounts for the first term of the series only. In Eq (36), a and b have to be selected so that the best fitting to (33) is obtained. For instance, $a = 6$ and $b = 11$ are suitable values for $J^2 = 10$.

CLOSING REMARKS

Based on the stochastic interpretation of phenomena occurring in the grain structure of a soil, strains of the skeleton alone are not instantaneous. As noted, interference between the solid and fluid phases were not considered in these theoretical developments. For the unidimensional case, the process in the granular skeleton is governed by the thickness of the soil layer and two parameters, the coefficient of diffusion and the drift. The later one seems to depend on physical characteristics of the grains, the void ratio, and the increment of effective pressure applied to the soil. The coefficient of diffusion is a property of the grain structure related to the number of random displacements per unit time and the mean square length of steps performed by the particles during the compression of the layer (refer to Eq 10). It is worth noting that the process is greatly influenced by the thickness of the layer (see Fig. 4). This result is of importance for applications, since compressibility properties of soils are obtained by testing samples whose thickness is very small as compared to that of the stratum in the prototype.

The theory herein described was developed in order to explain the so-called secondary compression of soils. This can be done combining the diffusion law for grains in the skeleton with Terzaghi's theory of consolidation, as proposed by the writer (Marsal, 1961).

ACKNOWLEDGMENTS

The author is grateful to Messrs. L. Ramïrez de Arellano and E. Moreno Gómez for their valuable help in preparing this paper.

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