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Calculation of Bed for Foundation with Ring Footing

Calcul des semelles de forme annulaire

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SUMMARY

Formulae have been obtained for calculating the settlement and reactive pressures of an absolutely rigid ring foundation subjected to an axial-symmetrical load, when the foundation bed of the model is a linear deforming half-space medium (theory of elasticity). Simultaneously, formulae have been obtained for the design moments required for determining the stresses in ring foundations having a high rigidity. For one case in which the ratio of the ring radii, n, is 0.6, tables of the design moments are set forth.

SOMMAIRE

Ce mémoire presente des formules pour le calcul du tassement et des pressions dans le sol sous une fondation annulaire absolument rigide soumise à des charges axiales et symétriques. Ces formules ont été développées, suivant la théorie de l'élasticité, en utilisant un modèle dans un milieu unidimensionnel. Le mémoire donne aussi des formules permettant le calcul des contraintes dans une semelle rigide de forme annulaire. Dans le cas où le rapport des rayons de l'anneau, n, est 0.6, l'auteur présente en tableaux les moments théoriques.

IT IS DIFFICULT to design ring foundations because of the absence of formulae for determining the deformation of the bed and the stresses in the foundations themselves. The required formulae for these calculations can be obtained from the following twin integral equations (Egorov, and Nichiporovich, 1961):

$$\left.\begin{aligned} w_0 &= \frac{2(1-\nu^2)}{E}\int_0^\infty D(\alpha)J_0(r\alpha)d\alpha \\ \sigma_0 &= \int_0^\infty \alpha D(\alpha)J_0(r\alpha)d\alpha, \end{aligned}\right\} \qquad (1)$$

where w_0 and σ_0 are the deflection and the reactive pressures directly under the foundation footing; $J_0(r\alpha)$ is Bessel's function of the first order and zero sequence; and E and ν are average values of the modulus of deformation and Poisson's ratio of the soil.

The twin integral equations have been obtained on the assumption that there is no friction under the foundation.

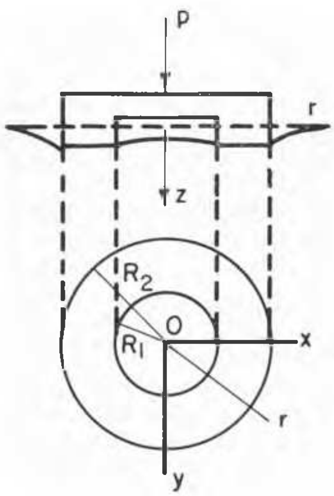

FIG. 1. Axial-symmetrical loading of ring foundation.

Over the whole length of the boundary plane $z = 0$, therefore, the tangential stresses are equal to zero (Fig. 1). Outside a settlement plate with $0 \leqslant r < R_1$ and $R_2 < r < \infty$ the stress σ_0 also equals zero. Inside a settlement plate with $R_1 < r < R_2$ the deflection w_0 is constant.

From these boundary conditions the unknown factor $D(\alpha)$ in Eq 1 is determined. In the case of a ring plate with an axial-symmetrical external load, it is sufficient to represent $D(\alpha)$ as follows:

$$D(\alpha) = \int_0^{\pi} F(\xi)J_0(\alpha\xi)d\phi, \qquad (2)$$

where

$$\xi = \sqrt{(a^2 + b^2 - 2ab\cos\phi)}$$

$$a - b = R_1,\ a + b = R_2.$$

Here R_1 and R_2 are respectively the internal and the external radii of the ring foundation.

Taking the value ξ as the variable of integration, Eq 2 can be represented as

$$D(\alpha) = \int_{R_1}^{R_2} \frac{2F(\xi)J_0(\alpha\xi)\xi d\xi}{\sqrt{(\xi^2 - R_1^2)(R_2^2 - \xi^2)}}. \qquad (3)$$

Upon substituting the second of the equations (1) with Eq 3 according to Hankel's theorem (Sneddon, 1951), the boundary conditions for the stress σ_0 under a ring plate are immediately met. According to this theorem outside a ring plate with $0 \leqslant r < R_1$ and $R_2 < r < \infty$, we have $\sigma_0 = 0$, while within this plate with $R_1 < r < R_2$ the reactive pressures $\sigma_0 = p(r)$ are determined from the simple formula

$$p(r) = \frac{2F(r)}{\sqrt{[(r^2 - R_1^2)(R_2^2 - r^2)]}}. \qquad (4)$$

For approximate solution of the problem under consideration it will be sufficient to represent the auxiliary function $F(r)$ as follows:

$$F(r) = C\sqrt{(r^2 - m^2R_2^2)}. \qquad (5)$$

The value of C is determined from the condition of equality of the external and internal forces, the formula being

$$P = \int_{R_1}^{R_2}\int_0^{2\pi} p(r) r dr d\theta. \tag{6}$$

Substituting in Eq 6 the value of $p(r)$ from Eq 4, and taking into account Eq 5, we obtain

$$C = \frac{P}{4\pi\sqrt{(1 - m^2)}R_2 E_0}, \tag{7}$$

where E_0 is a complete elliptical integral of the second order having the form

$$E_0 = \int_0^{\pi/2} \sqrt{(1 - k^2 \sin^2\theta)} d\theta$$

$$k^2 = \frac{1 - n^2}{1 - m^2}; \quad n = \frac{R_1}{R_2}.$$

The values of m depending upon the ratio of the ring radii, n, are determined from the condition that w_0 is constant, which the first of the equations (1) must satisfy within the interval $R_1 < r < R_2$. The complicated mathematical transformations used to calculate the values of m depending upon n are not set forth herein, however, they can be found in works by the author (Egorov, 1963).

Calculations show that within the interval $0 \leqslant n \leqslant 0.9$ it can be taken that $m = 0.8n$. To determine the settlement of a ring foundation, the first Eq 1 can be represented as follows:

$$w_0 = [P(1 - \nu^2)/ER_2]\omega(n). \tag{8}$$

The deflection factor $\omega(n)$ is given in Table I. With $n = 0$ we obtain the settlement of a circular (solid) foundation.

TABLE I. VALUES OF DEFLECTION FACTOR, $\omega(n)$ FOR ASSUMED VALUES OF n

$n = 0$	0.2	0.4	0.6	0.8	0.9	0.95
$\omega(n) = 0.5$	0.50	0.51	0.52	0.57	0.60	0.65

For this instance $\omega(0) = 0.5$. Table I shows that the settlements of circular and ring foundations are of the same order for an identical general force P, other conditions being equal, if the ratio of the ring radii is within the interval of 0 to 0.6. The same is true when a pair of forces is acting with a moment $M = Pe$, where e is the eccentricity. The solution of this problem is obtained from the twin integral equations (1), where $J_0(r\alpha)$ should be replaced with $(x/r)J_1(r\alpha)$ (Egorov and Nichiporovich, 1961).

Without considering this problem here, it can be recommended that the formula obtained previously for a circle be used when determining the inclination of a ring foundation with $0 \leqslant n \leqslant 0.6$.

$$f = 3(1 - \nu^2)M/4ER_2^3. \tag{9}$$

From Eqs 4–7 the formula for the reactive pressures under an absolutely rigid ring foundation in the case of an axial-symmetrical load can be derived. This formula will be

$$p(r) = \frac{P}{2\pi R_2\sqrt{(1 - m^2)}E_0}\sqrt{\frac{r^2 - m^2R_2^2}{(r^2 - R_1^2)(R_2^2 - r^2)}}. \tag{10}$$

From this formula two partial solutions can be obtained, one for a circle and one for a strip. In the case of a circle $R_1 = 0$ ($n = 0$) and $m = 0$, and Eq 10 will be reduced to

$$p(r) = P/[2\pi R_2\sqrt{R_2^2 - r^2}]. \tag{11}$$

In the case of a strip, the origin of co-ordinates should be transferred to the middle of the ring by substituting $r - R_1 = b + x$ and $R_2 - r = b - x$, where $2b = R_2 - R_1$ remains constant with the values of r, R_1, and R_2 having infinity as their limit. Hence,

$$r = (R_1/R_2) \to 1; \ (r/R_1) \to 1; \ (r/R_2) \to 1;$$
$$(P/2\pi R_2) \to P_1; \ E_0 = \pi/2.$$

By making the corresponding substitutions in Eq 10, with account taken of the previous ratios, we obtain the well-known formula for calculating the reactive pressures under a continuous foundation

$$p(x) = P_1/[\pi\sqrt{(b^2 - x^2)}], \tag{12}$$

where P_1 is the linear load and $2b$ is the width of the foundation.

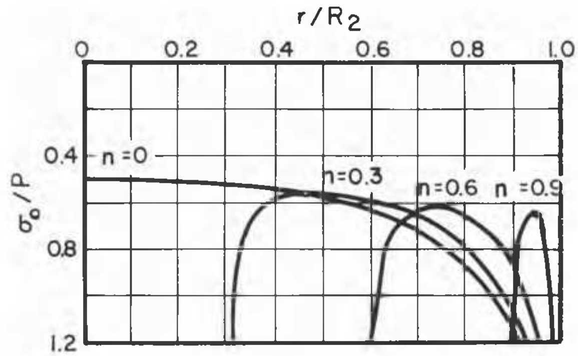

FIG. 2. Reactive pressures for one half of a ring foundation.

Fig. 2 shows the changes in the reactive pressures in the right-hand half of a ring foundation. The curves correspond to the values of $R_1 = 0$, 3, 6, 9 and $R_2 = 10$, which give $n = 0$ (circle), $n = 0.3$, $n = 0.6$ and $n = 0.9$. They have been plotted from Eq 10 reduced to the following:

$$p(r) = \frac{(1 - n^2)\sqrt{(\rho^2 - m^2)}p}{2\sqrt{(1 - m^2)}E_0\sqrt{[(\rho^2 - n^2)(1 - \rho^2)]}}$$
$$p = \frac{P}{\pi R_2^2(1 - n^2)}; \quad \rho = \frac{r}{R_2}. \tag{13}$$

The reactive pressures calculated from Eqs 12 and 13 for $n = 0.9$ coincide. Therefore for $n \geqslant 0.9$ ring foundations can be calculated by means of the formulae obtained for continuous strip footings.

As is well known, the stresses in a foundation slab having an axial-symmetrical form are determined from the following equation:

$$D\left(\frac{\partial^4 w}{\partial r^4} + \frac{2}{r}\frac{\partial^3 w}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^3}\frac{\partial w}{\partial r}\right) = q(r) - p(r). \tag{14}$$

The slab, having a high cylindrical rigidity (D), takes up the reactive pressures coinciding with the distribution of the stresses under an absolutely rigid foundation ($D = \infty$). For this reason the formulae obtained below for the design moments with $D = \infty$ can be used to determine the stresses appearing in slabs with a high rigidity. The distributions of the external axial-symmetrical vertical load $q(r)$ and the reactive pressures $p(r)$ along the footing of a ring foundation can be represented in the form of the integrals

$$q(r) = \int_0^\infty Q(\alpha) J_0(r\alpha)\alpha d\alpha;$$
$$p(r) = \int_0^\infty P(\alpha) J_0(r\alpha)\alpha d\alpha. \qquad (15)$$

In case of the action of an external linear load P_1 (in tons/m) distributed on the surface of a ring foundation slab along a circumference with a radius of R_0, we have

$$Q(\alpha) = \frac{P}{2\pi} J_0(\alpha R_0);$$
$$P = 2\pi R_0 P_1. \qquad (16)$$

For a uniformly distributed vertical external load, q (in tons/sq.m.) over the whole area of a ring foundation with the radii $R_1 < R_2$, we get

$$Q(\alpha) = \frac{P}{\pi(1-n^2)}\left[\frac{J_1(\alpha R_2)}{\alpha R_2} - n^2 \frac{J_1(\alpha)R_1}{\alpha R_1}\right],$$
$$P = \pi q R_2^2(1-n^2); \quad n = \frac{R_1}{R_2}. \qquad (17)$$

The factor being integrated in the second Eq 15 coincides with the factor $D(\alpha)$ in Eqs 1. Therefore from Eqs 2, 5, and 7 we have

$$P(\alpha) = \frac{P}{4\pi R_2 \sqrt{(1-m^2)}E_0} \times \int_0^\pi \sqrt{(\xi^2 - m^2R_2^2)} J_0(\alpha\xi) d\phi. \qquad (18)$$

The common integral of the differential equation (14) is expressed by the sum of two solutions:

$$w(r) = w_0(r) + w_1(r), \qquad (19)$$

where $w_0(r)$ is the common solution of homogeneous Eq 14 and $w_1(r)$ is the partial solution satisfying the righthand part of Eq 14. They have the form:

$$w_0 = C_0 + C_1 \ln r + C_2 r^2 + C_3 r^2 \ln r \qquad (20)$$

$$w_1 = \frac{1}{D}\int_0^\infty \frac{Q(\alpha) - P(\alpha)}{\alpha^3}\left[J_0(r\alpha) - 1\right] d\alpha. \qquad (21)$$

The integral constants in Eq 20 are determined from the given border conditions for the ring foundation slab. For example, in the case of a freely lying slab the lateral forces are equal to zero along the internal and external perimeters of the ring, which result in $C_3 = 0$. The constant C_0 does not have any influence on the value of the designed moments. To facilitate calculations let us assume that $C_0 = -C_1 \ln R_1$.

The radial and tangential moments are determined with the aid of Eqs 19 to 21 and are expressed as follows:

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r}\right] = \int_0^\infty \frac{Q(\alpha) - P(\alpha)}{2\alpha}\left[(1+\nu)J_0(\alpha r) - (1-\nu)J_2(\alpha r)\right] d\alpha + D\left[(1-\nu)\frac{C_1}{r^2} - 2(1+\nu)C_2\right]; \qquad (22)$$

$$M_t = -D\left[\nu\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right] = \int_0^\infty \frac{Q(\alpha) - P(\alpha)}{2\alpha}\left[(1+\nu)J_0(\alpha r) + (1-\nu)J_2(\alpha r)\right] d\alpha - D\left[(1-\nu)\frac{C_1}{r^2} + 2(1+\nu)C_2\right], \qquad (23)$$

where ν is Poisson's ratio for the material of the foundation. (For reinforced concrete $\nu = 1/6$.)

As the ring foundation slab freely rests on the bed which is being compressed, it should be assumed that $M_r = 0$ when $r = R_1$ and $r = R_2$. These conditions are sufficient for determining the constants C_1 and C_2 contained in Eqs 22 and 23. For them the following expressions have been obtained.

1. In case of a linear load P_1 in tons/m, distributed along a circumference with a radius of $R_0(R_1 \leqslant R_0 \leqslant R_2)$:

$$\left.\begin{aligned} C_1 &= \frac{PR_1^2}{8\pi D(1-n^2)(1-\nu)} \times [(1+\nu)(2\ln\alpha - A) - (1-\nu)(1-\alpha^2-B)]; \\ C_2 &= \frac{P}{16\pi D(1-n^2)(1+\nu)} \times [(1+\nu)n^2(2\ln\alpha - A) - (1-\nu)(1-\alpha^2-B)]; \end{aligned}\right\} \qquad (24)$$

$$A = \frac{1}{E_0}\int_0^{\pi/2} \sqrt{(1-k^2\sin^2\phi)} \times \ln[1-(1-n^2)\sin^2\phi] d\phi;$$

$$B = \frac{1-m^2}{3}\left[(1-k^2)\frac{F_0}{E_0} + (2k^2-1)\right];$$

$$P = 2\pi R_0 p_1; \quad k^2 = (1-n^2)/(1-m^2); \quad \alpha = R_0/R_2.$$

2. In case of a load q in tons/sq.m., uniformly distributed over the area of the entire ring foundation slab,

$$\left.\begin{aligned} C_1 &= -\frac{PR_1^2}{8\pi D(1-n^2)(1-\nu)} \times \left[(1+\nu)\left(1+\frac{2n^2}{1-n^2}\ln n + A\right) + (1-\nu)\left(\frac{1-n^2}{2} - B\right)\right]; \\ C_2 &= -\frac{P}{16\pi D(1-n^2)(1+\nu)} \times \left[(1+\nu)n^2\left(1+\frac{2n^2}{1-n^2}\ln n + A\right) + (1-\nu)\left(\frac{1-n^2}{2} - B\right)\right]; \end{aligned}\right\} \qquad (25)$$

$$P = \pi R_2^2(1 - n^2)q; \qquad n = \frac{R_1}{R_2}.$$

In Eq 24 F_0 and E_0 are complete elliptical integrals of the first and second order. After substituting the constants C_1 and C_2 with their values from Eqs 24 and 25 in Eqs 22 and 23 it remains necessary to calculate the integrals in the expression

$$K = \int_0^\infty \frac{Q(\alpha) - P(\alpha)}{2\alpha} \times [(1+\nu)J_0(\alpha r) - (1-\nu)J_2(\alpha r)]d\alpha. \quad (26)$$

Substituting $Q(\alpha)$ and $P(\alpha)$ in Eq 26 with their values from Eqs 16 and 18, we obtain, after integrating, two values for K depending upon the existing inequalities $r \leqslant R_0$ and $r \geqslant R_0$. Hence the radial moment M_r for the case of a linear load along the circumference with a radius of R_0 is determined by means of the following values of K:

$$\left.\begin{aligned} K &= \frac{P}{4\pi}\left[(1+\nu)\left(S - \ln\frac{R_0}{r}\right) + (1-\nu)T\right] \\ &\qquad (r \leqslant R_0) \\ K &= \frac{P}{4\pi}\left\{(1+\nu)S + (1-\nu)\left[T - \left(1 - \frac{R_0^2}{r^2}\right)\right]\right\} \\ &\qquad (r \geqslant R_0) \end{aligned}\right\} \quad (27)$$

where

$$S = \frac{1}{E_0}\left[\tfrac{1}{2}\int_0^{\frac{1}{2}(\pi-\beta)} \sqrt{(1 - k^2\sin^2\theta)} \times \ln[1 - (1-n^2)\sin^2\theta]d\theta - E\ln\rho\right];$$

$$T = \frac{1}{2E_0\rho^2}\left[\frac{1-m^2}{2}((1-k^2)(F_0 - F) + (2k^2 - 1)(E_0 - E)) + \frac{1}{3\sqrt{(1-m^2)}}\sqrt{[(1-\rho^2)(\rho^2 - n^2)(\rho^2 - m^2)]} - (E_0 - E)(1-\rho^2)\right];$$

$$F = \int_0^{\frac{1}{2}(\pi-\beta)} \frac{d\theta}{\sqrt{(1 - k^2\sin^2\theta)}};$$

$$E = \int_0^{\frac{1}{2}(\pi-\beta)} \sqrt{(1 - k^2\sin^2\theta)}d\theta;$$

$$k^2 = \frac{1-n^2}{1-m^2},$$

$$\beta/2 = \arccos\sqrt{\frac{1-\rho^2}{1-n^2}};$$

$$\rho = \frac{r}{R_2}.$$

The values of the complete elliptical integrals F_0 and E_0 are obtained from the values of F and E for $\beta = 0$.

For the case of a load q (in tons/sq.m.) equally distributed over the whole ring slab we get

$$K = \frac{P}{8\pi(1-n^2)}\left\{(1+\nu) \times [2\ln\rho + 1 - \rho^2 + 2(1-n^2)S] + (1-\nu)\left[n^2\left(1 - \frac{n^2}{2\rho^2}\right) - \frac{\rho^2}{2} + 2(1-n^2)T\right].\right\} \quad (28)$$

The definite integrals contained in the expressions for S and T are calculated according to tabulated values of elliptical integrals, except for the integrals containing expressions with natural logarithms. The latter have been calculated in accordance with Simpson's method.

For determining the tangential moment M_t in Eqs 27 and 28 in the second member of the expressions K before the factor $(1 - \nu)$ the sign minus should be taken instead of plus. This rule follows from the basic Eqs 22 and 23, where expressions are given for M_r and M_t.

As an example, the dimensionless values of the moments $\bar{M}_r$ and $\bar{M}_t$ have been calculated for $n = 0.6$ from the formulae:

$$M_r = P_1R_2\bar{M}_r, \qquad M_t = P_1R_2\bar{M}_t;$$
$$M_r = qR_2^2\bar{M}_r, \qquad M_t = qR_2^2\bar{M}_t,$$

where P_1 is the linear external load along circumference with radius R_0, q is the load uniformly distributed over the entire ring foundation slab, and R_2 is the external radius of slab.

TABLE II. VALUES OF $\bar{M}_r$ WITH $n = 0.6$

	For a linear load P_1					For a distributed load q
ρ	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	
0.6	0	0	0	0	0	0
0.7	−0.0077	+0.0547	+0.0303	−0.0048	−0.0474	+0.0056
0.8	−0.0103	+0.0246	+0.0610	+0.0144	−0.0530	+0.0078
0.9	−0.0118	+0.0014	+0.0043	+0.0413	−0.0419	+0.0035
1.0	0	0	0	0	0	0

TABLE III. VALUES OF $\bar{M}_t$ WITH $n = 0.6$

	For a linear load P_1					For a distributed load q
ρ	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	
0.6	+0.5159	+0.3460	+0.1226	−0.1551	−0.4880	+0.0452
0.7	+0.4292	+0.3030	+0.1096	−0.1307	−0.4191	+0.0377
0.8	+0.3675	+0.2622	+0.1096	−0.1054	−0.3962	+0.0329
0.9	+0.3166	+0.2241	+0.0905	−0.0873	−0.3472	+0.0295
1.0	+0.2810	+0.1978	+0.0767	−0.0817	−0.3204	+0.0271

Tables II and III contain respectively the values of $\bar{M}_r$ and $\bar{M}_t$ depending upon $\rho = r/\bar{R}_2$ and $\alpha = R_0/R_2$. In addition, at the end of these tables, values are given for $\bar{M}_r$ and $\bar{M}_t$ for the case of a distributed load q. These tables illustrate that when calculating ring foundation slabs the maximum moments arise in the direction of a tangent. From Table III, however, it follows that when an external linear load acts along circumference R_0 the value of M_t is near to zero with $n = 0.6$ and $\alpha \rightarrow 0.85$.

The maximum value of the radial moment M_r with $n =$

0.6 almost coincides with the calculated value in case of a plane problem with a width of the rigid strip $R_2 - R_1 = 2b$, when the linear force P_1 acts on the middle of the ring strip. In a book by Gorbunov-Posadov (1957) tables of moments are given for calculating continuous foundations, where $M = 0.318P_1b$. According to Table II the value of $\bar{M}_r$ given for $\rho = 0.8$ and $\alpha = 0.8$ corresponds to this instance. Hence we obtain $M_r = 2/(1-n)0.061P_1b = 0.305P_1b$.

Under the action of a load uniformly distributed over the whole ring foundation slab we obtain for $n = 0.6$ at the middle of the ring strip $M_r = 0.078qb^2$. At the same time in the formulation of the plane problem for the middle of the strip we have $M = 0.136qb^2$. It is therefore better to calculate ring foundation slabs for $n \leqslant 0.6$ by means of the formulae given above, than in the formulation of the plane problem.

In this paper the formulae of the design moments have been obtained by assuming that under the foundations the bed is deformed to an unlimited depth. Actually the design moments are influenced only by the deformation of the soil located within the compressed stratum of the bed. For this reason the values of the design moments obtained should be reduced by 20 per cent.

It should be noted that from the expressions of M_r and M_t given above when $n = 0$ a particular case of a round slab with a high rigidity is obtained, which was considered in detail in a booklet by Gorbunov-Posadov (1941).

In the Soviet Union the beds of structures are calculated according to the deformations. For example, depending upon the height (H) of the stack, smoke stacks should be designed with a view to the following tolerated settlements (S) and inclinations (f) of the foundations: for $H \leqslant 100$ m, $S = 20 - 30$ cm, $f = 0.004$; for $100 < H \leqslant 200$ m, $S = 15$ cm, $f = 0.003$; for $200 < H \leqslant 300$ m, S = 10 cm, $f = 0.002$. In many instances smoke stacks can be erected on ring foundations.

At present, in Moscow, a television tower over 500 m high is being erected on a ring foundation. Thus the question of the ring foundation under consideration is not only of theoretical, but also of practical interest.

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