Calculations for the Stability of a Sand Bed by a Solution Combining the Theories of Elasticity and Plasticity

Calculs de la stabilité des massifs de sable utilisant la théorie de l'élasticité et la théorie de la plasticité des sols

M. I. GORBUNOV-POSSADOV, PROFESSOR, DOCTOR OF TECHNICAL SCIENCES, *Scientific Research Institute of Foundations and Underground Structures of the USSR State Building Committee, Moscow, USSR.*

SUMMARY

The calculations of the stability of a sand bed under a rigid rough footing resting on the surface have been carried out with a view to the formation under the plate of a compacted core. The latter consists of an elastic part adjoining the footing and a lower plastic part. The stresses and strain in the elastic part of the core are determined by means of the theory of elasticity. The lower boundary of the elastic part is established assuming that all three components of the stresses at both sides of the boundary are equal, and that the lines of sliding are a smooth continuation of the trajectories of the particles in the elastic part of the core. The stresses in the plastic part of the compacted core and in all the remaining displaced soil are determined in accordance with the theory of critically stressed soil. The vertical and tangential reactions of the soil to the footing at the moment when the compacted core is formed have been established. The results agree with experimental data and indicate an increase in the value of the critical load.

SOMMAIRE

Le calcul de stabilité d'un massif de sable supportant une fondation rugueuse et rigide est effectué en tenant compte de la formation d'un noyau compact sous la semelle. Le noyau est constituté d'une zone élastique contiguë à la semelle et d'une zone inférieure plastique. Les contraintes et les déformations dans la zone élastique du noyau sont déterminées d'après la théorie de l'élasticité. La frontière inférieure de la partie élastique est établie en partant de la condition que les trois composantes des contraintes des deux côtés de la frontière sont égales et que les lignes de glissement sont la suite continue des trajectoires des particules de la partie élastique du noyau. Les contraintes dans la zone plastique du noyau compacté et dans tout le reste du terrain déplacé sont déterminées selon la théorie de l'équilibre limite. On a établi les réactions normales et tangentielles du terrain agissant sur la semelle lors de la formation du noyau compacté. Les résultats sont conformes aux données expérimentales et permettent d'augmenter la valeur de la charge critique dans les calculs.

IN A PREVIOUS REPORT (Gorbunov-Possadov, 1961), the prospects opening up in the field of calculating beds and foundations were considered in connection with the solution of the mixed problem of the theory of elasticity and the theory of plasticity of soils. These considerations have served as the basis of the solution of a particular two-dimensional problem regarding the stability of a dense sand bed under a rigid, rough, centrally loaded shallow foundation. The aim of this solution is to eliminate the gap between experimental results and theoretical data on the process of loss of stability and the magnitude of the critical load.

When formulating the problem, the following experimental data were used (Berezantsev, 1952; Malishev, 1953; Kananyan, 1954): (1) Registration with a fixed camera of the movements of sand grains under a load close to the critical one shows that the trajectories are smooth lines, with no sharp changes in direction (Fig. 1). (2) After rigid connection of the camera to the footing a fixed core of sand grains that do not move in respect to the footing is formed under it (Fig. 2a). (3) After performing experiments with painted layers of sand after removal of a load approaching the critical one a core will also be discovered in the bed; it will have greater dimensions and breaks in the displacements along its boundary with the remaining soil (Fig. 2b). (4) The reactive pressures under the same loads are distributed along the foot of the loading plate in a saddle-shaped stress pattern.

Besides these data, which were later confirmed by Zaharescu (1961) and Biarez, *et al.* (1961), the assumption was made that the trajectories of the sand grains in the plastic zones

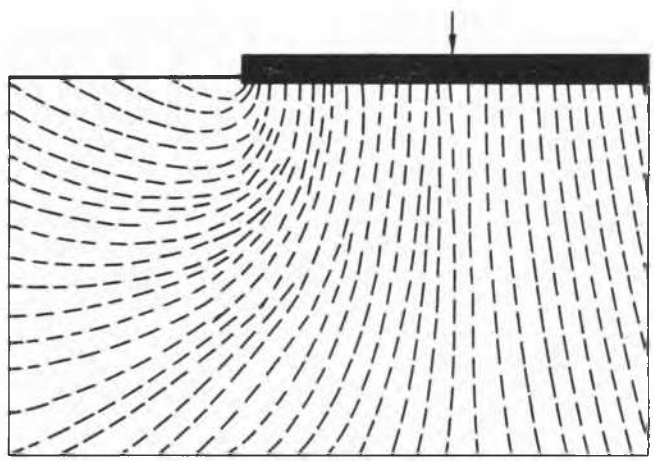

FIG. 1. Diagram showing results of recording sand grain movements with fixed camera.

coincide with the lines of sliding. (It should be noted in passing that the conclusion of Shield (1953), stating that the direction of the movements deviates from the line of sliding through the angle of internal friction ϕ, was not confirmed by these experiments.) From the experiments it follows that the trajectories come out to the surface at a sharp angle close to $(\pi/4 - \phi/2)$, as follows from the theory for lines of sliding.

On the basis of these propositions the process of failure

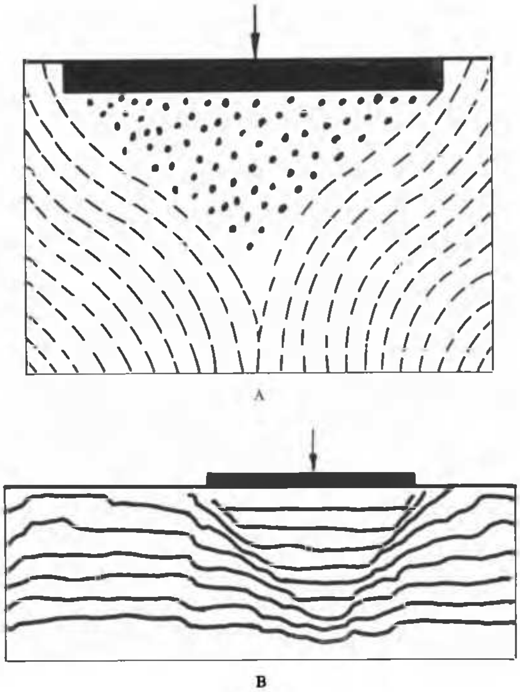

FIG. 2. Diagram showing results of experimental determination of compacted core boundaries: A, by means of camera rigidly secured to footing (the elastic part of the compacted core is photographed); B, with the aid of painted layers of sand (the compacted core as a whole is registered).

of a bed under an increasing load may be represented as follows (Gorbunov-Possadov, 1954).

1. After a certain initial compression of the bed a small core is formed directly under the footing (Fig. 2a, OA in Fig. 3). All the grains of this core move downward together with the footing. This core is in an elastic state, and the contact pressure stress pattern is uniform, as for a rigid body. The elastic movements in the small nucleus can be disregarded in comparison with the downward movement.

2. Upon an increase in the load under the elastic core, a large compacted core begins to form (Fig. 2b, OB in Fig. 3) at the expense of plastic shifts in the sand under the elastic core. Under a certain load P_1 formation of the core will be completed and the core will begin to sink as a unit moving apart and compacting the soil at the sides. When this occurs the movements at the edge of the core are interrupted (Fig. 2b).

3. Under the critical load of $P_f > P_1$ the soil at the sides of the large core becomes sufficiently compacted to pass into the critically stressed state and heaving takes place. The boundary of the compacted core remains the rupture line, but the lines of sliding of the active group (along which movement of the soil takes place) outside the core remain smooth continuations of the same lines in the core (Fig. 1). It is assumed that the form of the compacted core does not change with an increase of the load from P_1 to P_f. An increase in the stability safety factor would result if account were taken of the side load formed along the edges of the footing due to downward displacement of the compacted core at the time heaving occurs.

The first part of the theoretical solution consists in finding the lower boundary of the elastic part of the core. Brief information on this stage is contained in a previous report (Gorbunov-Possadov, 1961). The present report sets forth exhaustive information on the principal points of the mathematical solution, but the intermediate calculations published in a book by the author (Gorbunov-Possadov, 1962) are omitted.

The stresses and strain in the elastic part of the core are determined in accordance with the theory of elasticity. It is assumed that, along the contact of the footing with the core, there are no horizontal elastic movements, while the vertical ones are constant. Within the elastic part of the core the quantity $A = (\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2) < \sin \phi$, on the lower boundary $A = \sin \phi$. In the first approximation, it is assumed that the elastic part of the core reaches the edge of the footing. All three components of the stresses, at both sides of the boundary (in the elastic and the plastic zones), are correspondingly equal. The boundary of the elastic part is not a line of sliding or an intrinsic curve of the lines of sliding; it is like a retaining wall with a variable angle of friction δ between the wall and the plastic soil ($\delta \leqslant \phi$). The angle δ is determined from the condition that the lines of sliding approach the boundary vertically, as they are a smooth continuation of the trajectories of the forward motion of the grains in the elastic part of the core. In the upper corners of the elastic part all the components of the stresses are equal to zero due to the absence of a side load. In the lower corner they are also equal to zero, since the core wedges apart the lower sand, in which there can be no tensile stresses. The stresses caused by the weight of the sand in the elastic part of the core are not taken into account, as they are insignificant in comparison with the stresses caused by the external load.

On the basis of the theory of plasticity (critically stressed state) of soil (Sokolovski, 1954) and of the condition of

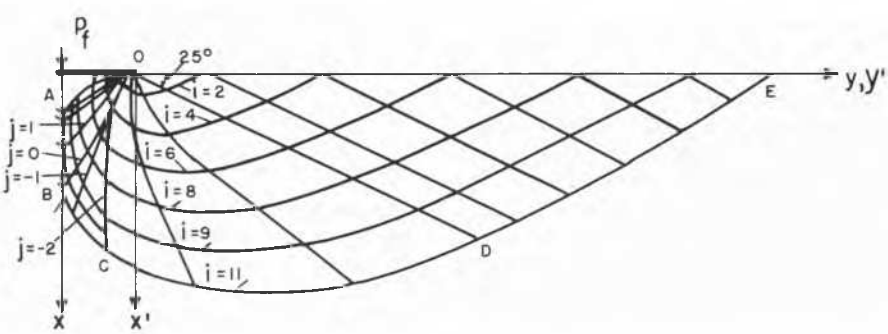

FIG. 3. Results of theoretical solution of problem.
OA—boundary of elastic part of compacted core, *OB*—boundary of compacted core as a whole, *OBC*—transition zone, *OCD*—zone calculated according to Karman's equations, *ODE*—Rankine's zone.

the vertical direction of the lines of sliding along the boundary there is established a relationship between the values of δ and the angle of inclination to a horizontal line α of a tangent at any point of the boundary:

$$\tan \delta = \frac{\sin \phi \cdot \cos(2\alpha - \phi)}{1 - \sin \phi \cdot \sin(2\alpha - \phi)}. \tag{1}$$

In accordance with the same condition on the axis of symmetry the value of the angle α will be

$$\alpha_0 = \frac{3\pi}{8} + \frac{\phi}{2} - \tfrac{1}{2} \operatorname{arc} \cos \left[\sin \phi \cdot \cos \left(\frac{\pi}{4} - \frac{\phi}{2} \right) \right]. \tag{2}$$

Assuming in the first approximation that $\tan \alpha$ changes linearly from $\tan \alpha_0$ at the apex to $\alpha = 0$ at the edge of footing, we can obtain the corresponding initial equation of the boundary in the form of two branches of a quadratic parabola:

$$x = \tfrac{1}{2} \operatorname{arc} \tan \alpha_0 (1 - |y|)^2. \tag{3}$$

A Cartesian system of co-ordinates reduced to half the thickness a of the footing has been selected with its origin at the centre of the plate, the x axis directed downward and the y axis to the right. At a value of $\phi = 40°$, for which the example given below has been solved, Eq 3 leads to a form of the elastic part of the nucleus that closely corresponds to experimental data. However, the given boundary conditions are not yet fulfilled on curve 3.

In order to find the actual boundary we shall introduce a function of the stresses:

$$\psi(x, y) = \psi^*(x, y) + \psi^{**}(x, y) \tag{4}$$

where $\psi^*(x,y)$ is a double exponential polynomial of the seventh degree with even numbers in respect to y:

$$\psi^*(x, y) = a_{20}x^2 + a_{02}y^2 + \ldots + a_{16}xy^6 \tag{5}$$

The relations deduced from the biharmonic property are imposed upon the coefficients of this polynomial. Further

$$\psi^{**}(x, y) = \sum_{i=0}^{i=3} C_i r_i^2 \ln r_i + \frac{D}{2} \sum_{j=1}^{j=2} r_j^2 \ln r_j \tag{6}$$

Biharmonic functions of the type $r_i^2 \ln r_i$ have their poles at points on the x axis, viz. $x_0 = 0.75$, $x_1 = 0.75$, $x_2 = 1$, $x_3 = 1.25$, while of the type $r_j^2 \ln r_j$ at the points $x = 0.595$, $y = \pm 0.01$; r is the reduced distance from points on the half-plane to the poles. In the first approximation it is taken that $C_0 = D = 0$. It should be understood that it would have been possible to solve the problem with the aid of other biharmonic functions having singularities outside the elastic part of the core. The question of the convergence of the solution is not considered.

To ensure fulfilment of the conditions imposed along the upper boundary on the movements:

$$(\partial u/\partial y)_{x=0} \equiv 0, \qquad (\partial v/\partial y)_{x=0} \equiv 0 \tag{7}$$

the components of these values, depending upon functions of Eq 6, are approximated by the method of least squares correspondingly with an odd (in respect to y) polynomial of the fifth degree and an even polynomial of the fourth degree. The components of functions of Eq 7, depending upon exponential function of Eq 5, are directly expressed by polynomials of the same kind. By summing up both kinds of components and by making them equal to zero on the basis of Eq 7 the coefficients of the terms of y of all degrees we obtain six equations. Six more such equations are obtained from the conditions $\sigma_x = \sigma_y = \tau_{xy} = 0$ at the apex of the elastic nucleus ($x = x_0$, $y = 0$) and at the edge of the foundation ($x = 0$, $y = \pm 1$). The remaining equations are obtained from Eq 1, which is used for several points on the lower boundary ($y = 0, 0.2, 0.4, 0.6, 0.8$). Besides, on the basis of the condition of equilibrium all the arbitrary constant values being sought are expressed through the load.

Thus when the lower boundary has been previously selected all the boundary conditions will be met precisely or approximately except for the condition $A = \sin \phi$ on the lower boundary. By calculating the actual value of A at separate points of this boundary and by analysing the magnitude and sign of the difference between A and $\sin \phi$ by trial and error the boundary is changed to reduce this difference as much as possible. In this instance all the remaining conditions for each following trial boundary should be complied with in the same way as for the boundary taken for the first approximation.

The co-ordinates of the elastic core boundary, which in our opinion gives satisfactory results, are contained in a previous report (Gorbunov-Possadov, 1961). Instead of the precise value of $A = \sin 40° = 0.643$ the values of A at separate points of the boundary change from 0.617 to 0.653. The stress patterns of the contact normal and tangential stresses expressed in parts of P_1/a are shown in Fig. 4.

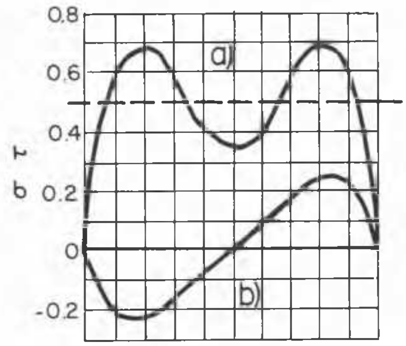

FIG. 4. Stress patterns of contact stresses in parts of the value P_1/a: (a) vertical stresses, (b) tangential stresses. The dotted line designates the average value of the vertical stresses.

The form of the elastic part of the core (OA in Fig. 3) was found to be very close to that proposed by Lundgren and Mortensen (1953) for the entire core. The difference consists in the fact that Lundgren did not suppose that there exists a plastic part of the compacted core, and therefore he obtained a lower value of the critical load.

The second part of the solution consists in plotting the lines of sliding and determining the stresses in the plastic region of the bed.

In accordance with the solution obtained the lower boundary of the elastic part of the core touches the base of the footing at its edge, while the state of the soil here is close to the critically stressed one. Without altering the first part of the solution, therefore, it can be accepted that the boundary of the core begins not at the very edge, but at a certain distance from it, as shown in experiments. Hence on a certain small section near the edge, the lines of sliding, in accordance with accepted principles, should approach the base of the footing vertically.

The stresses $\sigma = (\sigma_1 + \sigma_2)/2$ near the edge on a section with a width of $0.2a$ increase in direct proportion to the distance from the edge. To determine the small region near the edge the differential equations of Karman-Sokolovski can therefore be used:

$$\frac{d\psi}{d\theta} + 1 = \frac{\cos\theta - \sin\phi\cos(2\psi + \theta) - s\cdot\cos^2\phi}{2s\cdot\sin\phi\cdot(\cos 2\psi - \sin\phi)}, \quad (8)$$

$$\frac{ds}{d\theta} = \frac{s\cdot\sin 2\psi - \sin(2\psi + \theta)}{\cos 2\psi - \sin\phi}$$

which, as is known, can be used if the following ratio is observed:

$$\sigma = \gamma r s(\theta) \quad (9)$$

The equation of the lines of sliding in this instance will be

$$r = C\exp\int_0^\theta \cot\,(\psi \pm \mu)d\theta \quad (10)$$

In Eqs 8 to 10 r and θ are polar co-ordinates with their pole under the edge of the foundation, ψ is the angle of inclination between the major principal normal stress σ_1 and a radius vector passing through the point under consideration (Fig. 5), γ is the unit weight of the soil, $\mu = (\pi/4 - \phi/2)$, and C is an arbitrary constant.

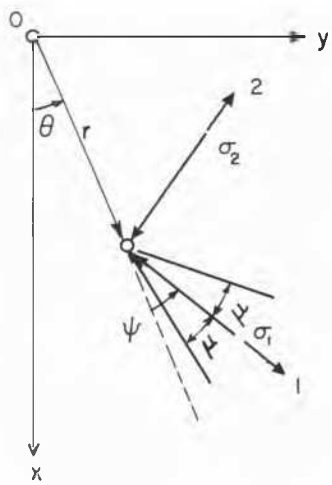

FIG. 5. Graphical definition of symbols used.

Proceeding from the given value of ψ on the y axis and the values of ψ and s on the boundary of Rankin's zone, integration is carried out within the finite differences of Eq 8. The value of $s = s_0$ on the y axis is found by trial and error. Omitting the details of integration, we note that, with $\phi = 40°$, the equation of the stresses along the axis y', which has the same direction as the y axis, but originates at the edge of the footing (Fig. 3), will be

$$\sigma = -396.4\gamma y'a \quad (11)$$

where a is the half-width of the loaded area.

Simultaneously on the basis of the solution of the problem for the elastic part of the core on the section near the edge:

$$\sigma = -2.1325y'P_1/a. \quad (12)$$

From Eqs 11 and 12 the values of the load P_1 will be

$$P_1 = 185.9\gamma a^2. \quad (13)$$

The network of lines of sliding and the value of the stresses in the plastic part of the consolidated core are determined as follows: Let us assume that the core is set back from the edge over a reduced value equal to 0.05. This value approaches the results of experiments. A more exact estimate would be of no importance; if 0.1 or 0.01 is taken instead of 0.05 an almost identical result is obtained.

Along the whole lower boundary OA of the elastic part of the compacted core the stresses and the direction (vertical) of the lines of sliding are known. This is sufficient to plot a network of lines of sliding and determine the stresses in most of the core. For these purposes Coche's problem is solved by employing the approximate methods of Sokolovski (1954). The solution of this problem, however, does not give the stresses or the position of the network of the lines of sliding in the narrow region adjacent to the axis of symmetry. Along this axis the lines of symmetry should be vertical (Fig. 1), that is the lines of sliding should touch this axis, while the axis itself should be the rupture line. By employing these considerations the problem is also solved for the remaining region near the axis of symmetry.

The boundary of the compacted core as a whole is shown in Fig. 3 (line OB).

It remains to find the network of the lines of sliding and the stresses in the plastic area outside the compacted core.

For this purpose we shall assume, in accordance with the results set forth above, that within the limits of the region below the line of sliding of the first (active) family originating from a point located on the y axis at a distance of $y' = -0.2$ from the edge (line $i = 2$ in Fig. 3) the condition of the soil is determined by solving Karman's problem. We assume as previously that the boundary of the core OB coincides with the line of sliding of the second family $j = 0$. This boundary up to its intersection with the line $i = 2$ coincides with the line of sliding of the second family $j = 2$, determined according to Karman and originating from a point located on the axis y' at a distance of 0.05 from the edge (Fig. 3). Beginning from this point of intersection the line of sliding $j = -2$ diverges from the boundary of the core $j = 0$, which is an envelope of the line of sliding of the second family. In order to plot the line $j = -2$ the value of C in Eq 10 should be determined. In accordance with the solution of Karman's problem for $C = 1$ the distance from the point on the axis y' where the line of sliding of the second family originates is equal to $d = 0.0107$. Assuming that all the linear dimensions have been reduced to half the width of the footing, the value of C for the line $j = -2$ is determined according to the equation $C = 0.05/0.0107 = 4.67$.

The solution of the problem for the area lying between the lines of sliding $j = 0$ and $j = -2$ is obtained from the condition that the line $j = 0$ is an intrinsic curve of the second family, and that the lines of the first family are a smooth continuation of the same lines inside the core. As the value of the stresses on the line $j = -2$ is known beforehand, the Sokolovski methods can be employed to establish easily the distribution of the stresses along the boundary of the compacted core, which will now be different from those under a load of P_1.

All the remaining lines of sliding between the line $j = -2$ and the boundary of Rankin's zone (region OCD in Fig. 3) are plotted according to the solution of Karman's problem. The values of C in Eq 10 are established for each of the lines of sliding of the first family in such a way as to ensure their being a continuation of the lines of the same family in the transition region. The lines of the second family are plotted for any values of C less than 4.67. In Fig. 3 they are given for $C = 1$ and $C = 0.01$.

Calculation of the critical load as the integral of the vertical components of the normal and tangential stresses taken along the lower boundary of the compacted core gives the value of the critical load (at $\phi = 40°$): $P_f = 383.5\gamma a^2$. This value of P_f is considerably higher than the values obtained according to other methods of calculation, including methods which approximately take into account the existence of a compacted core as a rigid body, the vertical section of which is usually taken in the form of a triangle: for example, Terzaghi (1943), employing interpolation for $\phi = 40°$, $P_f = 260\gamma a^2$; Caquot and Kérisel (1953) $P_f = 228\gamma a^2$; Berezantsev (1960) $P_f = 200\gamma a^2$. With the core in the form of the curvilinear triangle of Lundgren (1953) $P_f = 187\gamma a^2$ (approximately), and from the experiment of Kananyan (1954), with medium-grain sand and $\gamma = 1.7$ tons/cu.m., $P_f = 320\gamma a^2$.

REFERENCES

BERESANTSEV, V. G. (1952). *Axis of symmetry problem of the theory of critical equilibrium.* Moscow, State Technical-Theoretical Literature Publishing House.

——— (1960). *Calculation of strength of structure beds.* Moscow-Leningrad, Gosstroiizdat Publishing House.

BIAREZ, J., M. BUREL, and B. WACK (1961). Contribution to the study of the bearing capacity of foundations. *Proc. Fifth International Conference on Soil Mechanics and Foundation Engineering*, Vol. 1, p. 603.

CAQUOT, A., and J. KÉRISEL (1953). Sur le terme de surface dans le calcule des fondation en milieu pulverulent. *Proc. Third International Conference on Soil Mechanics and Foundation Engineering*, Vol. 1, p. 336.

GORBUNOV-POSSADOV, M. I. (1954). Calculation of beds as a mixed problem of the theory of elasticity and the theory of critically-stressed state of a soil medium. *Conference on Scientific-Research Work of the Institute of Beds and Foundations.* Moscow, Gosstroiizdat Publishing House.

——— (1961). Panel Discussion. *Proc. Fifth International Conference on Soil Mechanics and Foundation Engineering*, Vol. 3, p. 224.

——— (1962). *Stability of foundations on a sand bed.* Moscow, Gosstroiizdat Publishing House.

KANANYAN, A. S. (1954). Experimental investigation of the destruction of a sand bed by a vertical load. *Collected Articles of Scientific Research Institute of Beds and Foundations*, No. 24. Moscow, Gosstroiizdat Publishing House.

LUNDGREN, H., and K. MORTENSEN (1953). Determination by the theory of plasticity of the bearing capacity of continuous footings on sand. *Proc. Third International Conference on Soil Mechanics and Foundation Engineering*, Vol. 1, p. 409.

MALISHEV, M. V. (1953). *Theoretical and experimental investigations of bearing capacity of sand bed.* Moscow, All-Union Scientific Research Institute VODGEO.

SHIELD, R. T. (1953). Mixed boundary value problems in soil mechanics. *Quart. Applied Mathematics*, Vol. 11, No. 1.

SOKOLOVSKI, V. V. (1954). *Statics of a loose medium.* 2nd ed. Moscow, State Technical-Theoretical Literature Publishing House.

TERZAGHI, K. (1943). *Theoretical soil mechanics.* New York.

ZAHARESCU, E. (1961). *Contributii la studiul capacitatii portente a fundatiilor.* Buçuresti, Editura Academiei Republicii Populare Romine.