Consolidation of Clay Layers Based on Non-linear Stress-Strain

Consolidation de couches d'argile basée sur la relation contrainte-déformation non-linéaire

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SUMMARY

A differential equation for one-dimensional primary consolidation is derived on the basis of strain, instead of additional stress, and the equation is solved for some typical cases.

Applied to the case of no lateral yield one finds that a 10-m-thick clay layer carrying a uniformly distributed load consolidates, in the first half of the process, about twice as fast as found by the conventional method. This finding appears in principle to be in accordance with experimental evidence.

The strain procedure is also modified and applied to concentrated loads (strip and square) on deep clay layers, by means of simple formulae for estimating the effective drainage path, less than the total layer thickness. The effect of non-vertical drainage for such cases is not investigated.

The procedure derived is capable of taking the influence of the stress history into account, while this important effect is altogether neglected in the conventional method.

SOMMAIRE

Une équation différentielle pour la consolidation primaire unidimensionnelle est dérivée de la déformation au lieu de la contrainte additionnelle et l'équation est résolue dans quelques cas typiques.

Dans le cas sans déformation latérale, on trouve qu'une couche d'argile de 10 mètres d'épaisseur portant une charge uniformément distribuée se consolide, durant la première moitié du processus, environ deux fois plus vite que par la méthode conventionnelle. En principe ce résultat semble correspondre à l'expérience.

La méthode de déformation relative est aussi modifiée et appliquée à des charges concentrées (oblongues ou carrées) placées sur des couches d'argile profondes au moyen de formules simples pour calculer la distance du drainage qui est inférieure à l'épaisseur totale de la couche. L'effet du drainage non vertical dans pareils cas n'a pas été étudié.

La procédure dérivée peut prendre en considération l'influence de l'histoire des contraintes, tandis que cet effet important est négligé dans la méthode conventionnelle.

THE DIFFERENTIAL EQUATION for one-dimensional consolidation of a thin saturated clay layer expressed in terms of pore pressure reads, in general

$$m_v(\partial u/\partial t) = \partial/\partial z[(k/\gamma_w)(\partial u/\partial z)] \qquad (a)$$

where m_v = constant, implying linear stress-strain. In addition one has to assume k = constant, in order to arrive at the conventional equation

$$\partial u/\partial t = c_v(\partial^2 u/\partial z^2). \qquad (b)$$

This differential equation has been solved for c_v = constant, and initial u either constant or equal to a linear function of z. Corresponding degrees of consolidation as functions of the time factor are available.

However, experimental evidence, collected internationally over decades, has clearly demonstrated that (for $t = \infty$) the effective stress-strain relationship for clays is non-linear, implying directly that there is generally no proportionality between the additional stress diagram and the primary consolidation. Hence, the entire basis for the conventional method of obtaining the time rate of consolidation is questionable, except perhaps for very thin layers, such as in an oedometer. The most serious objection is probably that the conventional method is not capable of taking the effect of the stress history into account, an effect which is generally found to be of considerable magnitude.

In order to arrive at an improved method it is believed essential to base the entire analysis on strain instead of on additional pore pressure, and also to use the experimentally determined relationships between effective stress and strain and between velocity and gradient—both theoretically idealized.

DERIVATIONS

For the derivations reference is made to Fig. 1, which shows the soil profile, and the vertical stress and strain distributions as functions of depth and time. Idealized relationships between velocity and gradient (Hansbo, 1960) and between effective stress and strain, represented by the tangent modulus (Janbu, 1963), are included in the figure.

The case considered is one-dimensional (K_0/K_0' condition, no lateral yield) and drainage takes place in the vertical direction. For arbitrary values of z and t the straight-line portion of the velocity, presented diagramatically in Fig. 1e, may be expressed as

$$v = k(i - i_0). \qquad (1)$$

For Equation 1, when $v = 0$, additional consolidation must take place, corresponding to the shaded area in Fig. 1e. This additional settlement will give the appearance of being a secondary phenomenon. Herein, the straight-line portion for $i \geqq i_0$ is termed primary consolidation.

An expression for the gradient is obtained from the following relationship, which is presented diagramatically in Fig. 1b:

$$q = \sigma' + h\gamma_w. \qquad (2)$$

Using the fundamental definition of i, one finds

$$i = -\partial h/\partial z = (1/\gamma_w)[(\partial\sigma'/\partial z) - (\partial q/\partial z)].$$

For the purpose of expressing the gradient in terms of strain, ϵ, a stress-strain relationship is needed. The definition of the tangent modulus M, Fig. 1f, yields

$$\partial\sigma' = M\partial\epsilon. \qquad (3)$$

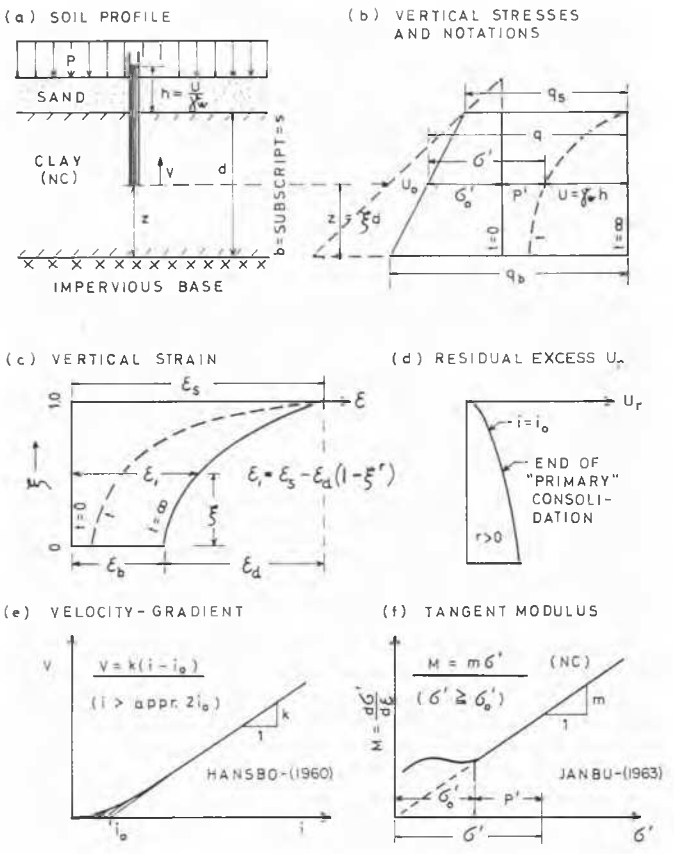

FIG. 1. Notations, definitions, and elementary relationships used for derivation of differential equation.

Moreover, it is advantageous to use the following abbreviation for effective stress variation, at $t = \infty$,

$$\gamma_q = \partial q/\partial z = dq/dz = (q_s - q_b)/d, \tag{4}$$

after which the gradient takes the form

$$i = (M/\gamma_w)(\partial\epsilon/\partial z) - \gamma_q/\gamma_w. \tag{5}$$

Inserting Equation 5 in Equation 1, one obtains the basic equation for vertical velocity

$$v = c_v(\partial\epsilon/\partial z) - v_0, \tag{6}$$

where the coefficient of consolidation c_v and the nominal velocity v_0 are given by the expressions

$$c_v = kM/\gamma_w \tag{7}$$

$$v_0 = k(i_0 + \gamma_q/\gamma_w). \tag{8}$$

The differential equation is now derived from the continuity equation: for saturated soils the volume of water leaving the element per unit volume and unit time is equal to the volume decrease per unit volume and unit time, hence

$$\partial\epsilon/\partial t = \partial v/\partial z. \tag{9}$$

Inserting Equation 6 in Equation 9 one obtains

$$\frac{\partial\epsilon}{\partial t} = \frac{\partial}{\partial z}\left(c_v\frac{\partial\epsilon}{\partial z}\right) - \frac{\partial v_0}{\partial z} \tag{10}$$

which is the basic differential equation for one-dimensional consolidation on a strain basis.

Using dimensionless variables

$$T = t(c_v/d^2), \quad \xi = z/d \tag{11}$$

the differential equation is written (c_v = constant),

$$\frac{\partial\epsilon}{\partial T} = \frac{\partial^2\epsilon}{\partial\xi^2} - \frac{d}{c_v}\frac{\partial v_0}{\partial\xi}. \tag{12}$$

For the following analysis Equations 6 and 12 are essential. The cases dealt with will be limited to $v_0 = f(\xi)$.

BOUNDARY CONDITIONS

The object of the analysis below is to find ϵ as a function of time and depth. In dimensionless scale $\epsilon = f(\xi,T)$. This unknown function must satisfy both the differential equation and the boundary conditions at the start and at the end of the primary consolidation process.

The starting conditions are

$$\epsilon = 0 \text{ for } T = 0 \text{ and } 1 > \xi \geqq 0 \tag{13a}$$

$$v = 0 \text{ for } T = 0 \text{ and } \xi = 0 \tag{13b}$$

$$\epsilon = \epsilon_s \text{ for } \xi = 1 \text{ and } \infty \geqq T > 0. \tag{13c}$$

The end conditions, corresponding to $t = \infty$, are

$$\epsilon = \epsilon_1 \text{ for } T = \infty \text{ and } 1 \geqq \xi \geqq 0 \tag{14a}$$

$$v = 0 \text{ for } T = \infty \text{ and } 1 \geqq \xi \geqq 0. \tag{14b}$$

In this paper several end strain distribution curves with depth will be studied (Fig. 1c). By introducing ϵ_1 for $T = \infty$ as follows

$$\epsilon_1 = \epsilon_s - \epsilon_d(1 - \xi^r), \tag{15}$$

it is seen that $r = 0$ corresponds to a constant ϵ_1, such as may be the case in an oedometer, while $r = 1$ and $r = 2$ correspond to a linear and parabolic strain distribution. Solutions of the differential equation will be presented for these three values of r.

For $t = \infty$, Equations 6, 14b, and 15 yield

$$v_0 = (c_v/d)(\partial\epsilon_1/\partial\xi) = (c_v/d)r\epsilon_d\xi^{r-1}. \tag{16}$$

The classical case, $r = 0$, leads to $v_0 = 0$ as would be expected. For linear ϵ_1, when $r = 1$, it is seen that v_0 = constant; and v_0 is linear for $r = 2$.

Interpreting v_0 as being a nominal velocity corresponding to a gradient caused by a nominal residual excess pore pressure u_r after the end of the primary consolidation, the distribution of this residual pore pressure for $r > 0$ must be in principle as shown in Fig. 1d.

Because the primary consolidation is connected with the straight-line portion of the $v - i$ curve, it is obvious that when $v = 0$ (i.e., $i = i_0$) a residual velocity and a corresponding residual pore pressure, u_r, must exist. The numerical values of u_r will not be considered herein.

SOLUTION OF DIFFERENTIAL EQUATION

The solution of the differential equation is, except for the influence of v_0, similar to the conventional procedure. Therefore the details are omitted.

When ϵ_1 is given the form of Equation 15, it is readily demonstrated that both the differential equation and the boundary conditions are satisfied by the expression

$$\epsilon = \epsilon_1 - \sum C_N \cos N\xi e^{-N^2 T} \quad (17)$$

$$N = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

where

$$C_N = \frac{2\epsilon_s}{N} \sin N - 2r\epsilon_d \frac{\sin^{r+1} N}{N^{r+1}}. \quad (17a)$$

The formulas are valid only for the following whole integers, $r = 0$, $r = 1$, and $r = 2$.

DEGREE OF CONSOLIDATION

The final object of the analysis is to find the degree of consolidation U as a function of the time factor T. By definition

$$U = \int_0^1 \epsilon d\xi \Big/ \int_0^1 \epsilon_1 d\xi. \quad (18)$$

Integrating Equations 17 and 15 and inserting for C_N according to Equation 17a one finds

$$U = \frac{U_0 - f_s F(T)}{1 - f_s} \quad (19)$$

in which f_s is a "shape factor" referring to the ϵ_1 distribution,

$$f_s = r\epsilon_d/(1 + r)\epsilon_s \quad (19a)$$

and the function $F(T)$ is

$$F(T) = 1 - 2(r + 1) \sum \frac{\sin^{2+r} N}{N^{2+r}} e^{-N^2 T}. \quad (19b)$$

The classical degree of consolidation U_0 corresponds to $r = 0$, whereof $U_0 = F_0(T)$.

The functions U_0, $F_1(T)$, and $F_2(T)$, corresponding to $r = 0$, $r = 1$, and $r = 2$ respectively, have been calculated by means of electronic computers, and the numerical values are given in Table I.

TABLE I. VALUES OF $F(T)$

T	$U_0 = F_0(T)$ $r=0$	$F_1(T)$ $r = 1$	$F_2(T)$ $r = 2$
0.000	0.0000	0.0000	0.0000
0.001	0.0356	0.0021	0.0029
0.002	0.0504	0.0041	0.0057
0.005	0.0797	0.0100	0.0141
0.01	0.1128	0.0199	0.0276
0.02	0.1595	0.0399	0.0535
0.05	0.2523	0.0999	0.1247
0.1	0.3568	0.1977	0.2285
0.2	0.5040	0.3703	0.3981
0.5	0.7639	0.6994	0.7129
1.0	0.9313	0.9125	0.9164
2.0	0.9942	0.9926	0.9929
∞	1.0000	1.0000	1.0000

By means of Equations 19 and 19a and Table I a large number of $U - T$ curves can be drawn corresponding to a similar number of linear and parabolic ϵ_1 distributions. For illustration purposes three such curves are drawn in Fig. 2, corresponding to: Case A: constant ϵ_1 (classical); case B: triangular ϵ_1; and case C: parabolic ϵ_1 with zero strain at the impervious base (real or fictitious).

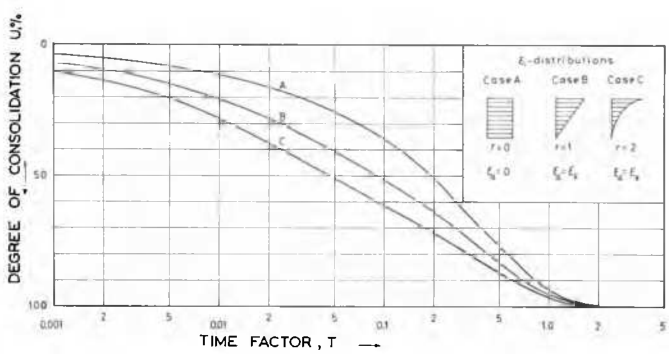

FIG. 2. Degree of consolidation as function of time factor for different end strain distributions.

Fig. 2 is capable of showing that the rate of consolidation increases the more ϵ_1 decreases with depth. Since ϵ_1 decreases with depth even when the additional stress is constant, it is evident that the rate of consolidation on the basis of ϵ is generally more rapid (particularly for the first phase of the consolidation) than that obtained from the conventional method (based on additional stress).

PROCEDURE OF APPLICATION

The first step of the analysis for a given case is to obtain the ϵ_1 distribution with depth, and therefrom the settlement δ_c and shape factor f_s. The time rate is then estimated from tables or graphs in the conventional manner.

Based on a linear tangent modulus, $M = m\sigma'$, the unit strain ϵ_1, at $t = \infty$ is calculated for any depth by the equation (Janbu, 1963),

$$\epsilon_1 = \frac{1}{m} \ln \frac{\sigma_0' + p'}{\sigma_0'} \quad (20)$$

and the settlement is by definition

$$\delta_c = \int_0^d \epsilon_1 dz \quad (21)$$

which is represented by the area of the $\epsilon_1 - z$ diagram.

Having obtained δ_c the shape factor f_s in Equation 19a is found by integrating Equation 15. Hence

$$\delta_c = \left(\epsilon_s - \frac{r}{1 + r}\epsilon_d\right) d = (1 - f_s)\epsilon_s d$$

and therefore,

$$f_s = 1 - \delta_c/\epsilon_s d. \quad (22)$$

In Equation 22 the value to be used for δ_c is the area of $\epsilon_1 - z$ as defined by Equation 21.

Equation 22 is valid only up to the maximum f_s corresponding to $\epsilon_d = \epsilon_s$ when, according to Equation 19a,

$$f_{s\,max} = r/(1 + r). \quad (23)$$

When the end strain for all practical purposes approaches zero for a depth smaller than the actual layer thickness, the effective length, d_e, of the drainage path is smaller than the layer thickness itself. In such cases the effective drainage path, d_e, is determined by substituting Equation 23 in Equation 22 and using d_e instead of d. Then

$$d_e = (1 + r)\delta_c/\epsilon_s. \quad (24)$$

It is important to notice that d_e (and not d) should be used in calculating the time

$$t = T(d_e^2/c_v). \tag{25}$$

Equations 24 and 25 must now be applied together with the maximum shape factor, Equation 23, i.e., curves B and C in Fig. 2 are directly applicable for $r = 1$ and $r = 2$ respectively.

EXAMPLES

The first example deals with the time rate of consolidation for a fill of large horizontal extensions placed on a drainage blanket of sand on top of a 10-m-thick, normally consolidated clay layer resting on an impervious base. The required data are presented in the profiles in Fig. 3.

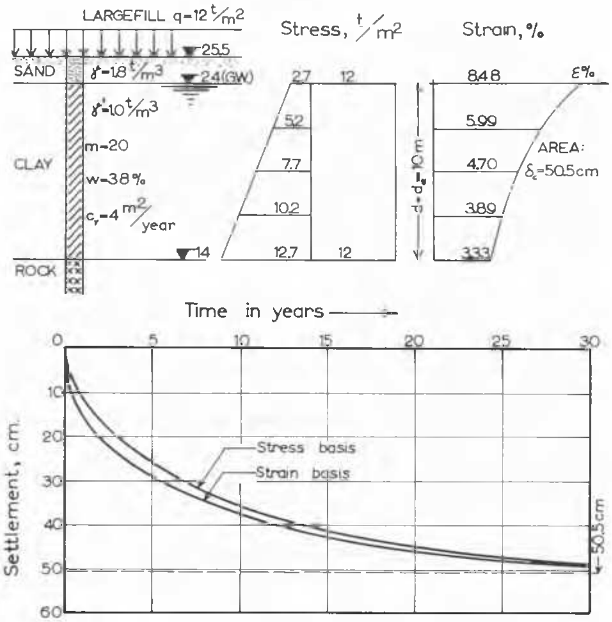

FIG. 3. Example I. Time rate of consolidation for a large fill, estimated on strain and stress basis.

The area of the $\epsilon_1 - z$ diagram equals $\delta_c = 50.5$ cm and $\epsilon_s = 0.0848$ while $d = 10$ m giving $f_s = 0.405$. Using this shape factor in Equation 19 and applying U_0 and $F(T) = F_2(T)$ from Table I, a $U - T$ curve is obtained. The rest of the analysis is the same as in the conventional approach, and the numerical $\delta - t$ diagram is shown in Fig. 3. For comparison the $\delta - t$ curve, interpreted on additional stress basis by the conventional method, is included in Fig. 3.

Even though both curves in Fig. 3 deal with an idealized case of no lateral yield, constant additional stress, and vertical drainage, it is seen that ϵ_1 decreases with depth due to the influence of effective overburden on the compressibility, and as a consequence the consolidation takes place more rapidly than it would in a solution obtained on the basis of constant additional stress.

For concentrated loads (i.e., strip footings) on clay, lateral yield may take place, and the drainage path is not generally vertical in the zone of consolidation. Lateral yield must in practice be treated separately, and even though the drainage path, in average, deviates from the vertical, the above-presented procedure can be adopted to such cases, at least for practical purposes.

In Fig. 4 the time rates of consolidation for a strip footing and a square footing are analysed for the same soil profile as in example I. The stress distributions with depth are

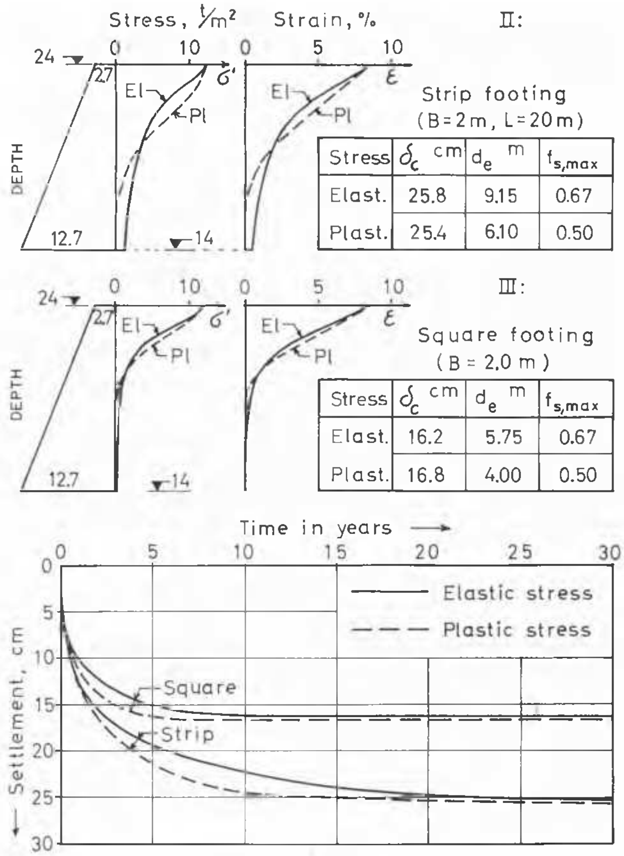

FIG. 4. Examples II and III. Time rate of consolidation for a strip and square footing.

calculated from both the theory of elasticity and the theory of plasticity (unpublished report). The required data are given in Fig. 4. The time is now estimated from Equation 25 using the calculated d_e and obtaining $T - U$ from curves C and B in Fig. 2 for elastic ($r = 2$) and plastic ($r = 1$) distribution, respectively. The results are shown in Fig. 4.

As will be noted, the difference in settlement between elastic and plastic stress distribution is insignificant. However, the plastic distribution appears to lead to faster consolidation particularly in the last half of the process. As would be expected the square footing settles considerably less (65 per cent) and more rapidly than the strip footing, and as a consequence the differential settlement between two such footings may tend to develop in an unfavourable way as time passes.

Since the three examples dealt with consider the same soil profile and unit load, some values for comparison are assembled in Table II.

TABLE II. SUMMARY OF DATA FROM EXAMPLES I, II, AND III (FOR THE SAME SOIL PROFILE)

		Time of 50 per cent consolidation	
Example	δ_c (cm)	Strain basis	Stress basis
I. Extensive fill	50.5	3.2 years	5.0 years
II. Strip footing	25.6	ca. 0.9 years	No suitable
III. Square footing	16.5	ca. 0.4 years	procedure available

DISCUSSION

In the numerical analysis presented herein, it is assumed that c_v = constant, and moreover the $U - T$ curves are obtained for $v_0 = f(z)$ only. In reality the problem is complicated by the fact that c_v is not constant and v_0 may also be a function of time and stress.

For some Norwegian marine clays it has been found that c_v increases for increasing stresses above the preconsolidation load, in approximately the following manner

$$c_v = c_{v0}(\sigma'/\sigma_0')^\alpha \qquad (26)$$

where the pure number α is of the order of magnitude of 0.4 to 0.6.

Substituting for k from $\gamma_w c_v = kM$ into Equation 8 and using $M = m\sigma'$ the differential equation 10 takes the form

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial}{\partial z}\left(c_v \frac{\partial \epsilon}{\partial z}\right) + \frac{\gamma_r c_v}{\sigma'} \frac{\partial \epsilon}{\partial z} \qquad (27)$$

where

$$\gamma_r = (1 - \alpha)(\gamma_w i_0 + \gamma_q). \qquad (28)$$

Equation 27 is so complicated to solve, however, that only one solution (for a special case) has so far been obtained, but not evaluated (Mortensen, 1962).

The analysis (for primary consolidation) was concerned with the straight-line portion of the $v - i$ curve, Fig. 1. For small gradients the velocity varies in a non-linear way, say $v = Ci^2$. In the last phase of the consolidation process, the velocity represented by the shaded area in Fig. 1 will lead to a settlement in addition to the calculated δ_c for linear v. This additional settlement will in principle be similar to a secondary consolidation. Moreover, if the load increment is small so that the gradient at time $t = 0$ is less than about 2 i_0, this phenomenon will start at the very beginning ($t = 0$) indicating that the secondary consolidation may be distributed over the entire consolidation process (Leonards and Girault, 1961).

The nominal velocity v_0, Equation 16, is indicative of a residual pore pressure at the end of the primary consolidation ($i = i_0$). Attempts have been made to establish mathematical formulae for the distribution and magnitude of this residual pore pressure. Because of uncertainties about the correct boundary conditions, however, this analysis is not yet completed.

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